Hauptseminar Geometrie
Graduate Seminar on Differential Geometry

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Summary

A closed surface S of genus $g \geq 1$ can be obtained as the connected sum of g two-tori. If $g \geq 2$ then it is known that such a surface admits a hyperbolic metric. Equivalently, it can be obtained as a quotient of the upper half-space $\{z \in \mathbb{C} \mid \Im(z) > 0\}$ by a subgroup of $PSL(2, \mathbb{R})$ acting by linear fractional transformations. In fact, there is an entire moduli space of such hyperbolic structures.

For each such hyperbolic structure h on S, one can associate naturally defined numerical invariants. One such invariant is the smallest positive eigenvalue of the Laplacian Δ of h. Understanding this invariant turns out to be surprisingly difficult, and in fact, in spite of dramatic progress during the last 5 years, many basic questions are still open.

The goal of this seminar is to give an introduction to this beautiful subject which remains as elementary as possible while presenting some fundamental results at the basis of current developments.

1. Talk 1: The hyperbolic plane, isometries, the Gauss Bonnet theorem for hyperbolic triangles.
   Bergeron Chapter 1.2, Katok Chapters 1.1–1.4

2. Talk 2: Hyperbolic surfaces, Fuchsian groups, fundamental domains..
   Bergeron Chapter 1.3.1, Katok Chapter 2.2, 3.1–3.2.

3. Talk 3: Examples of hyperbolic surfaces, Fricke space.
   Buser Chapter 6.8 (and additional literature)

4. Talk 4: Hyperbolic Laplacian, eigenfunctions, the Green's function.
   Bergeron Chapters 1.3.2, 3.2.1.

5. Talk 5: Invariant integral operators, the heat kernel.
   Bergeron Chapter 3.3, 3.5.

6. Talk 6: The Laplacian on a hyperbolic surface.
   Bergeron Chapter 3.6.

7. Talk 7: The spectral theorem.
   Bergeron Chapter 3.8–3.9.

8. Talk 8: The minmax principle and small eigenvalues, the Cheeger constant, examples.
   Bergeron Chapter 3.10, Buser Chapter 8.2–8.3.

9. Talk 9: The trace formula
   Bergeron Chapter 5.1–5.2, Buser Chapter 9.5, Burrin.

10. Talk 10: Weyl's law and the prime geodesic theorem.
    Bergeron Chapter 5.4, Buser Chapter 9.6, Burrin.

There are four additional talks available which are more involved. The goal of these talks is to relate the material in the first 10 talks to current developments in geometry, arithmetic and probability.

11 Talk 11: Expanders and property (τ).
Lubotzky Chapter 4.1-4.3, in particular Theorem 4.3.2, Theorem 4.4.2 and Example 4.3.3 C, Brooks Theorem 1.

12 Talk 12: Selberg's $\frac{3}{16}$ theorem and expanders.
Bergeron Chapter 2.3.2, Chapter 3.10.2, Lubotzky Chapter 4.4, Chapter 4.5.

13 Talk 13: Random covers and fixed points of random permuations.
Magee, Naud, Puder Theorem 1.11, mostly presenting the ideas of random permutations and fixed point expectations.

14 Talk 14: Spectral gap of random covers of surfaces.
Magee, Naud, Puder Theorem 1.5.

Literature for the class:

1. N. Bergeron, *The spectrum of hyperbolic surfaces*, Springer Universitext, Springer 2016.
2. R. Brooks, *The spectral geometry of a tower of coverings*, J. Diff. Geom. 23 (1986), 97–107.
3. P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkhäuser Progress in Math. 106, 1992.
4. C. Burrin, *Spectral theory of hyperbolic surfaces*, Lecture notes, Univ. Zürich 2021.
5. S. Katok, *Fuchsian groups*, Chicago Univ. Press 1992.
6. A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Modern Birkhäuser Classics, Birkhäuser 1994/2010.
7. M. Magee, F. Naud, D. Puder, *A random cover of a compact hyperbolic surface has relative spectral gap* $\frac{3}{16} - \epsilon$, Geom. Funct. Anal. 32 (2022), 595–661.