Hybrid sup-norm bounds for automorphic forms in higher rank

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The sup-norm problem

- $X = \Gamma \backslash G / K$: compact locally symmetric space of rank r
- $\mathcal{Z}(U(\mathfrak{g}))$: algebra of bi-invariant differential operators
- Automorphic form ϕ: $\phi \in L^2(X)$, $\|\phi\|_2 = 1$, eigenfunction of $\mathcal{Z}(U(\mathfrak{g}))$
- $\Delta\phi = \lambda\phi$, where Δ is the Laplacian on X

Question

How large is $\|\phi\|_\infty$ as $\lambda \to \infty$?

Motivation, local bounds and conjectures

- Related to the theory of quantum chaos and quantum unique ergodicity
- Related to the multiplicity problem, study of nodal domains, subconvexity problem
- Local/generic bound

$$\|\phi\|_\infty \ll \lambda^{(\dim X - r)/4}$$

- Example of conjecture: for $G = \mathrm{SL}_2(\mathbb{R})$, we expect $\|\phi\|_\infty \ll_\varepsilon \lambda^\varepsilon$

Problem

Show that $\|\phi\|_\infty \ll \lambda^{(\dim X - r)/4 - \delta}$ *for some* $\delta > 0$.

A prototype result

- The general problem seems out of reach in general, so we restrict to X arithmetic and ϕ a joint eigenfunction of the Hecke algebra

- Iwaniec-Sarnak 1995: for ϕ Hecke-Maaß form on arithmetic compact hyperbolic surface X (also non-compact modular curves)

$$\|\phi\|_\infty \ll_\varepsilon \lambda^{1/4-1/24+\varepsilon}$$

- Many more results in higher rank, over number fields, etc.

The volume aspect

- E.g. $X = \Gamma_0(N)\backslash\mathbb{H}$ with $\mathrm{vol}(X) = N^{1+o(1)}$
- Inspired by the level aspect in the subconvexity problem

Question

How large is $\|\phi\|_\infty$ as $N \to \infty$?

- "Local bound": $\|\phi\|_\infty \ll_\lambda N^\varepsilon$
- Work of Harcos and Templier (2013, N square-free):

$$\|\phi\|_\infty \ll_\varepsilon \lambda^{1/4-1/24} N^{-1/6} \lambda^\varepsilon N^\varepsilon$$

- More precise version by Saha 2017 for general N

The volume aspect: compact case

- A: indefinite division quaternion algebra over $\mathbb{Q}$
- If $\mathcal{O}$ is an order in A (subring, full $\mathbb{Z}$-lattice), then $\mathcal{O}^1$ is a cocompact subgroup of $\mathrm{SL}_2(\mathbb{R})$
- Let $N = [\mathcal{O}_m : \mathcal{O}]$ for some maximal order $\mathcal{O}_m$
- $\mathrm{vol}(\mathcal{O}^1 \backslash \mathbb{H}) = \mathrm{disc}(\mathcal{O})^{1/2+o(1)} = \mathrm{disc}(A)^{1/2+o(1)} \cdot N^{1+o(1)}$
- Work of Templier (2010) and Saha, Saha-Hu (2020):

$$\|\phi\|_\infty \ll_{A,\lambda,\varepsilon} N^{-1/24+\varepsilon}$$

Questions

- Uniformity in disc(A)?

- Uniformity in both λ and volume (i.e. hybrid bounds)?

- Volume aspect in higher rank? (only result is due to Hu, 2018, in the depth aspect, non-compact case)

Main theorem

Let $\mathfrak{h}^n = \mathrm{SL}_n(\mathbb{R}) / \mathrm{SO}(n)$.

Theorem (T. 2022)

Let p be a prime and A a central division algebra of degree p over $\mathbb{Q}$ that is split over $\mathbb{R}$. Let $\mathcal{O} \subset A$ be an order of covolume $V := \mathrm{vol}\, \mathcal{O}^1 \backslash \mathfrak{h}^p$. If ϕ is an L^2-normalised Hecke-Maaß form on $\mathcal{O}^1 \backslash \mathfrak{h}^p$ with large eigenvalue λ, then

$$\|\phi\|_\infty \ll \lambda^{\frac{p(p-1)}{8} - \delta_1 + \varepsilon} V^{-\delta_2 + \varepsilon},$$

where the savings can be taken to be $\delta_1 = (16p^3)^{-1}$ and $\delta_2 = (8p^3(p-1))^{-1}$, and the implied constant depends on p and ε.

A similar theorem is given for quaternion algebras over totally real number fields.

Beyond prime degree

Orders of $\mathcal{O}_0(N)$-type: locally isomorphic to

$$\mathcal{O}_0(N)_p = \{\gamma \in M_n(\mathbb{Z}_p) \mid \text{last row of } \gamma \equiv (0, \dots, 0, *) \bmod N\mathbb{Z}_p\}$$

Theorem (T. 2022)

Let $n \geq 3$ be an odd integer and A a central division algebra of degree n over $\mathbb{Q}$ that is split over $\mathbb{R}$. Let $\mathcal{O} \subset A$ be an order of type $\mathcal{O}_0(N)$ and let $V := \operatorname{vol} \mathcal{O}^1 \backslash \mathfrak{h}^n$ be its covolume. If ϕ is an L^2-normalised Hecke-Maaß form on $\mathcal{O}^1 \backslash \mathfrak{h}^n$ with large eigenvalue λ, then

$$\|\phi\|_\infty \ll_A \lambda^{\frac{n(n-1)}{8} - \delta_1 + \varepsilon} V^{-\delta_2 + \varepsilon},$$

where the savings can be taken to be $\delta_1 = (8n^3)^{-1}$ and $\delta_2 = (4n^3(n-1))^{-1}$, and the implied constant depends on n, ε, and the discriminant of A.

Pre-trace formula

Pre-trace formula

$$\sum_{j\in\mathbb{N}} \tilde{f}(\mu_j)|\phi_j(z)|^2 = \sum_{\gamma\in\mathcal{O}^1} f(z^{-1}\gamma z),$$

By the Jacquet-Langlands correspondence (Bădulescu et al. in higher rank), we can use the test function f_μ of Blomer-Maga (for GL(n)) and obtain

$$\begin{aligned} |\phi(z)|^2 &\leq \sum_{\gamma\in\mathcal{O}^1} f_\mu(z^{-1}\gamma z) \\ &\ll B(\lambda)^2 \cdot \#\{\gamma \in \mathcal{O}^1 : z^{-1}\gamma z = k + O(\rho), \text{ for some } k \in \mathrm{SO}(n)\}. \end{aligned}$$

Here ρ is small enough in terms of n.

The baseline bound

Remark

For n odd, every matrix in $\mathrm{SO}(n)$ has eigenvalue 1.

If $\gamma \in \mathcal{O}^1$ and $z^{-1}\gamma z = k + O(\rho)$, then

$$\mathrm{nr}(\gamma - 1) = \det(k - 1 + O(\rho)) = O_n(\rho).$$

Since ρ is small enough and $\gamma - 1$ is an integral element, it follows that $\mathrm{nr}(\gamma - 1) = 0$ and so $\gamma = 1$.
Therefore,

$$|\phi(z)|^2 \leq \sum_{\gamma \in \mathcal{O}^1} f_\mu(z^{-1}\gamma z) \ll B(\lambda)^2 = \lambda^{n(n-1)/4}.$$

The amplified pre-trace formula

Amplifier: introduce a shorter average over Hecke eigenvalues

$$\sum_{j\in\mathbb{N}} \tilde{f}(\mu_j)|\phi_j(z)|^2 \cdot |\sum_{l\asymp L} a_j(l)|^2,$$

by applying Hecke operators on the pre-trace formula.
As in Blomer-Maga (2016), we have

$$L^2 \cdot |\phi(z)|^2 \ll L^\varepsilon B(\lambda)^2 \Big(L + \sum_{\nu=1}^{n} \sum_{l_1,l_2\asymp L} \frac{1}{L^{(n-1)\nu}} \#\mathcal{O}(l_1^\nu l_2^{(n-1)\nu}; z, \delta)$$
$$+ B(\lambda)^{\frac{-2}{n(n-1)}} \cdot \delta^{-\frac{1}{2}} \sum_{\nu=1}^{n} \sum_{l_1,l_2\asymp L} \frac{1}{L^{(n-1)\nu}} \#\mathcal{O}(l_1^\nu l_2^{(n-1)\nu}; z, \rho) \Big),$$

for parameters δ, L.

The counting problem

Here,

$$\mathcal{O}(m; z, \delta) = \{\gamma \in \mathcal{O} : \mathrm{nr}(\gamma) = m, \quad z^{-1}\gamma z = m^{1/n}(k + O(\delta))\}.$$

For $n = p$ prime, we count

- in the discriminant aspect, taking L small enough in terms of the discriminant
- in the spectral aspect, taking δ small enough in terms of L

In special cases (e.g. $n = 2$), we can do both simultaneously.

The discriminant aspect

- Let A_L be the $\mathbb{Q}$-algebra generated by $\bigcup_{1\leq m\leq L}\mathcal{O}(m;z,\delta)$.
- If $n=p$ is prime, then $A_L\neq A$ implies A_L is a field. (*rigidity*)
- For linear algebra argument, note that A_L is contained in the $\mathbb{Q}$-vector space spanned by $\bigcup_{1\leq m\leq L^{2p-2}}\mathcal{O}(m;z,(2p-2)\delta)$.
- Let $D=\mathrm{disc}(\mathcal{O})$.

Properness

Lemma

The $\mathbb{Q}$-vector space spanned by $\bigcup_{1 \leq m \leq L^{2p-2}} \mathcal{O}(m; z, (2p-2)\delta)$ is proper, i.e. not equal to A, if $L \ll D^{1/4p(p-1)-\varepsilon}$, where the implicit constant depends only on p and δ.

Let $\gamma_1, \ldots, \gamma_{p^2} \in \bigcup_{1 \leq m \leq L^{2p-2}} \mathcal{O}(m; z, \delta)$. Then $\mathrm{nr}(\gamma_i \gamma_j) \ll L^{4(p-1)}$ and $z^{-1}\gamma_i\gamma_j z = \mathrm{nr}(\gamma_i\gamma_j)^{1/p}(k + O(\delta))$. In particular

$$\mathrm{tr}(\gamma_i \gamma_j) \ll_{\delta,p} L^{4(p-1)/p},$$

by applying the trace.

Properness

Lemma

The $\mathbb{Q}$-vector space spanned by $\bigcup_{1 \le m \le L^{2p-2}} \mathcal{O}(m; z, (2p-2)\delta)$ is proper, i.e. not equal to A, if $L \ll D^{1/4p(p-1)-\varepsilon}$, where the implicit constant depends only on p and δ.

Consider now

$$s = \det(\mathrm{tr}(\gamma_i \gamma_j)_{i,j}).$$

Then $D \mid s$.

On the other hand, $s \ll L^{4p(p-1)}$. Thus if $L \ll D^{1/4p(p-1)-\varepsilon}$, then $s = 0$.

Then $\gamma_1, \ldots, \gamma_{p^2}$ are *not* linearly independent, by the non-degeneracy of $\mathrm{tr}(.,.)$.

Counting in fields

- So A_L is a field for L small enough.
- We count in the ring of integers of A_L.
- $\#\mathcal{O}(m; z, \delta)$ is bounded by number of ideals of norm m times number of suitable units

Counting in fields

- Number of ideals of norm m is bounded by m^{ε}.
- Let $\gamma \in \mathcal{O}^{\times}$ with $z^{-1}\gamma z = k + O(\delta)$. Then $z^{-1}\gamma^{j} z = k_j + O_j(\delta)$.
- Thus $\mathrm{tr}\left(\gamma^{j}\right) \ll_j 1$.
- The coefficients $\chi_\gamma \in \mathbb{Z}[X]$ can be given in terms of $\mathrm{tr}\left(\gamma^{j}\right)$ for $j = 1, \ldots, p$.
- There are only $\ll_p 1$ many possibilities for the χ_γ, so only $\ll_p 1$ many possibilities for γ (A_L is a field!)

The spectral aspect

Let now $p \geq 3$ be any odd integer.

Lemma

The $\mathbb{Q}$-algebra generated by $\bigcup_{1 \leq m \leq L} \mathcal{O}(m; z, \delta)$ is commutative, i.e. a field, if $\delta \ll L^{-2-\varepsilon}$, where the implicit constant depends only on p.

Let $\gamma_1, \gamma_2 \in \bigcup_{1 \leq m \leq L} \mathcal{O}(m; z, \delta)$. Then

$$z^{-1}\gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2 z = k + O(\delta).$$

As before, subtracting 1 we get $\mathrm{nr}(\gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2 - 1) = O_p(\delta)$. Thus,

$$\mathrm{nr}(\gamma_1\gamma_2 - \gamma_2\gamma_1) = \mathrm{nr}(\gamma_2\gamma_1) \cdot \mathrm{nr}(\gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2 - 1) = O_p(\delta L^2).$$

If $\delta \ll L^{-2-\varepsilon}$, then $\gamma_1\gamma_2 = \gamma_2\gamma_1$.

Odds and ends

- Now counting in fields is done as before.
- Counting $\#\mathcal{O}(m; z, \rho)$ is done trivially, by counting ideals and units. We count units again by assuming that $\rho \ll_p 1$ is small enough, so that we are counting in a field by a similar argument.
- Discriminant aspect: $\phi(z) \ll B(\lambda) \cdot D^{\frac{-1}{4n^3(n-1)}+\varepsilon}$
- Spectral aspect: $\phi(z) \ll B(\lambda) \cdot \lambda^{\frac{-1}{2n^4(n-1)}+\varepsilon} \cdot D^{\varepsilon}$
- We interpolate for the hybrid bound, thanks to uniformity.

Special orders

For $\mathcal{O}$ of $\mathcal{O}_0(N)$-type, we have

$$N \mid \mathrm{nr}(\gamma_1\gamma_2 - \gamma_2\gamma_1).$$

- By the argument in the spectral aspect, we are counting in a field as soon as $\delta \ll L^2 N^{-1}$.
- This gives a simultaneous treatment and better bounds.
- If $p \mid \mathrm{disc}(A)$ and A_p is division, then $p \mid \mathrm{nr}(\gamma_1\gamma_2 - \gamma_2\gamma_1)$. But this is not the case in general for composite degree.

The quaternion algebra case

For quaternion algebras, we note that SO(2) only generates a 2-dimensional vector space.
We have

$$\frac{1}{\prod_i \mathrm{nr}(\gamma_i)} \det\Big(\mathrm{tr}(\gamma_i \gamma_j)_{i,j}\Big) = \det\Big(\mathrm{tr}(k_i k_j)_{i,j}\Big) + O(\delta),$$

for certain $k_i \in \mathrm{SO}(2)$. By linear dependence, $\det\Big(\mathrm{tr}(k_i k_j)_{i,j}\Big) = 0$.