Atkin-Lehner operators for GL(n)*

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Abstract

We prove that the normaliser of the congruence subgroup $\Gamma_0(N)$ inside $\mathrm{GL}_n(\mathbb{Q})$ is trivial for $n>2$. Since this normaliser was the source of Atkin-Lehner operators for subgroups of $\mathrm{SL}_2(\mathbb{R})$, we give a different perspective in order to obtain generalisations of Atkin-Lehner operators in higher rank. Under this perspective, the only non-trivial operator is the generalised Fricke involution, which provides the dual form in L-function theory.

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1 Introduction

Let $\Gamma_0^n(N)$ be the subgroup of $\mathrm{SL}_n(\mathbb{Z})$ consisting of matrices with last row of the form

$$(0,\ldots,0,*) \pmod N,$$

where $*$ is a unit modulo N. This family of arithmetic subgroups is of great importance in number theory, lying at the basis of the theory of newforms (s. section 13.8 in [3]). For $n=2$, the theory of newforms is intimately related to the theory of Atkin-Lehner operators. Yet for $n>2$, definitions for Atkin-Lehner operators do not seem to be in print, at least not in classical language, i.e. not representation-theoretic.

It is the purpose of this note to present a possible generalisation of Atkin-Lehner operators to the Hecke congruence subgroups of $\mathrm{SL}_n(\mathbb{R})$ with $n>2$. This generalisation yields the corresponding Fricke involutions, which sends an automorphic form to the dual form appearing in the functional equation of its L-function.

According to our definition, there are no other Atkin-Lehner operators apart from the Fricke involution (and the trivial identity) in the case $n>2$. This apparent shortcoming of the definition is, in fact, a meaningful phenomenon. The proof of this negative result applied to the case $n=2$ shows that this scarcity is an exacerbation of the shortage of Atkin-Lehner operators for powerful levels, which is a well-known technical difficulty in applications.

2 The normaliser of $\Gamma_0(N)$

In the theory of autormophic forms on $\mathrm{SL}_2(\mathbb{R})$, an Atkin-Lehner operator S is obtained by setting $Sf(z)=f(gz)$ for all $z\in\mathbb{H}$, where g lies in the normaliser of $\Gamma_0^2(N)$ inside $\mathrm{SL}_2(\mathbb{R})$. This is a natural method of producing automorphisms of spaces of automorphic forms, since the invariance of $f(z)$ under a group Γ is equivalent to the invariance of $f(gz)$ under $g^{-1}\Gamma g$. The normaliser has been computed by Atkin and Lehner in [1] and an example of a non-trivial normalising element is

$$g=\begin{pmatrix} & -1\\ N & \end{pmatrix},$$

which induces the so-called *Fricke involution*.

Thus, searching for symmetries of automorphic forms in higher rank should involve computing the normalisers of $\Gamma_0^n(N)$ for $n>2$. Unfortunately, this method can only produce the identity operator, since these normalisers, in contrast to the case $n=2$, are trivial. In the following we denote by $\mathrm{GL}_n^+(\mathbb{Q})$ the subgroup of invertible matrices with positive determinant.

Theorem 1. *For $n>2$, the normaliser of $\Gamma_0^n(N)$ inside $\mathrm{GL}_n^+(\mathbb{Q})$ is trivial, that is, equal to $\mathbb{Q}_{>0}\cdot\Gamma_0^n(N)$.*

*. This article has been written using GNU TeXmacs [4].

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For simplicity, we prove the theorem in the case of $n=3$. Consider the action of $G:=\mathrm{GL}_3^+(\mathbb{Q})$ on full $\mathbb{Z}$-lattices in $\mathbb{R}^3$. Let $L_1=\langle e_1,e_2,e_3\rangle$ be the standard lattice for a basis (e_1,e_2,e_3) of $\mathbb{R}^3$ and consider $\mathcal{L}=G\cdot L_1$, the orbit of L_1 under the action of G.[1]

Note that the stabiliser of L_1 under this action is the group $\mathrm{SL}_3(\mathbb{Z})$. More generally, for $M\in\mathbb{N}$, let $L_M=\langle e_1,e_2,M e_3\rangle$, or in other words,

$$L_M=\begin{pmatrix}1&&\\&1&\\&&M\end{pmatrix}\cdot L_1.$$

The stabiliser of L_M is

$$\mathrm{Stab}(L_M)=\begin{pmatrix}1&&\\&1&\\&&M\end{pmatrix}\mathrm{Stab}(L_1)\begin{pmatrix}1&&\\&1&\\&&M\end{pmatrix}^{-1}=\left\{\begin{pmatrix}a_{11}&a_{12}&\frac{a_{13}}{M}\\a_{21}&a_{22}&\frac{a_{23}}{M}\\Ma_{31}&Ma_{32}&a_{33}\end{pmatrix}:(a_{ij})\in\mathrm{SL}_3(\mathbb{Z})\right\}.$$

It follows that $\mathrm{Stab}(L_1)\cap\mathrm{Stab}(L_M)=\Gamma_0^3(M)$. Since $\Gamma_0^3(N)\subset\Gamma_0^3(M)$ for all $M|N$, we deduce that

$$\bigcap_{M|N}\mathrm{Stab}(L_M)=\Gamma_0^3(N).$$

The following lemma asserts that these lattices are essentially all the lattices fixed by $\Gamma_0^3(N)$. Intuitively, this means that $\Gamma_0^3(N)$ is quite large and therefore cannot have a much larger normaliser.

Lemma 2. *The set of lattices fixed by* $\Gamma_0^3(N)$ *is*

$$\bigcup_{M|N}\{q\,L_M: q\in\mathbb{Q}_{>0}\}.$$

Proof. Let $L=g\cdot L_1\in\mathcal{L}$, where $g\in\mathrm{GL}_3^+(\mathbb{Q})$, and assume that $\Gamma_0^3(N)$ fixes L. Then $g^{-1}\Gamma_0^3(N)g$ fixes L_1, so we must have $g^{-1}\Gamma_0^3(N)g\subset\mathrm{SL}_3(\mathbb{Z})$.

Without loss of generality, that is, by scaling g by a positive rational number, we may assume that $g\in\mathcal{M}_{3\times3}(\mathbb{Z})$. Let then H be the Hermite normal form of g, so that

$$H=g\,U,$$

with $U\in\mathrm{SL}_3(\mathbb{Z})$ and H lower triangular. We have $H\,L_1=g\,U\,L_1=g\,L_1=L$. So we may further assume that $g=H$ is lower triangular. More explicitely, write

$$H=\begin{pmatrix}\alpha_1&0&0\\\beta_1&\beta_2&0\\\gamma_1&\gamma_2&\gamma_3\end{pmatrix}\in\mathcal{M}_{3\times3}(\mathbb{Z}).$$

We test the inclusion $H^{-1}\gamma\,H\in\mathrm{SL}_3(\mathbb{Z})$ with various matrices $\gamma\in\Gamma_0(N)$.

$$H^{-1}\begin{pmatrix}1&1&\\&1&\\&&1\end{pmatrix}H\in\mathrm{SL}_3(\mathbb{Z})\ \text{ implies that }\ \frac{\beta_1}{\alpha_1},\frac{\beta_2}{\alpha_1},\frac{\beta_1\gamma_2-\gamma_1\beta_2}{\alpha_1\gamma_3}\in\mathbb{Z};$$

$$H^{-1}\begin{pmatrix}1&&1\\&1&\\&&1\end{pmatrix}H\in\mathrm{SL}_3(\mathbb{Z})\ \text{ implies that }\ \frac{\gamma_1}{\alpha_1},\frac{\gamma_2}{\alpha_1},\frac{\gamma_3}{\alpha_1}\in\mathbb{Z};$$

$$H^{-1}\begin{pmatrix}1&&\\1&1&\\&&1\end{pmatrix}H\in\mathrm{SL}_3(\mathbb{Z})\ \text{ implies that }\ \frac{\alpha_1}{\beta_2},\frac{\alpha_1}{\beta_2}\cdot\frac{\gamma_2}{\gamma_3}\in\mathbb{Z};$$

$$H^{-1}\begin{pmatrix}1&&\\&1&\\N&&1\end{pmatrix}H\in\mathrm{SL}_3(\mathbb{Z})\ \text{ implies that }\ N\frac{\alpha_1}{\gamma_3}\in\mathbb{Z}.$$

1. The orbit $\mathcal{L}$ is essentially the set of lattices commensurable to L_1.

Since $\frac{\beta_2}{\alpha_1}, \frac{\alpha_1}{\beta_2} \in \mathbb{Z}$, we must have $\frac{\beta_2}{\alpha_1} = \pm 1$. Since $\frac{\gamma_3}{\alpha_1}, N\frac{\alpha_1}{\gamma_3} \in \mathbb{Z}$, we must have $\frac{\gamma_3}{\alpha_1} = \pm M | N$. Using the rest of the findings above, we may do column manipulations and obtain

$$H = \alpha_1 \begin{pmatrix} 1 & 0 & 0 \\ \frac{\beta_1}{\alpha_1} & \frac{\beta_2}{\alpha_1} & 0 \\ \frac{\gamma_1}{\alpha_1} & \frac{\gamma_2}{\alpha_1} & \frac{\gamma_3}{\alpha_1} \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & & \\ & 1 & \\ & & M \end{pmatrix} U',$$

with $U' \in \mathrm{SL}_3(\mathbb{Z})$. Thus $L = H L_1 = L_M$ up to $\mathbb{Q}_{>0}$ scalars. □

Proof of Theorem 1. Let $g \in \mathrm{GL}_3^+(\mathbb{Q})$ such that $g^{-1}\Gamma_0^3(N) g = \Gamma_0^3(N)$. Since $\Gamma_0^3(N)$ fixes the lattices L_M for all divisors M of N, we find that $\Gamma_0^3(N)$ must also fix the lattices $g L_M$ for $M | N$. By the previous lemma, for each divisor M of N there is a rational number q_M and a divisor $f(M) | N$ such that

$$g L_M = q_M L_{f(M)}, \quad \text{for all } M | N.$$

By the definition of L_M and using the fact that $\mathrm{Stab}(L_1) = \mathrm{SL}_3(\mathbb{Z})$, we can deduce that

$$q_M^{-1} \begin{pmatrix} 1 & & \\ & 1 & \\ & & f(M)^{-1} \end{pmatrix} \cdot g \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & M \end{pmatrix} \in \mathrm{SL}_3(\mathbb{Z}), \tag{1}$$

for all $M | N$.

Rescaling g by $q_1 \in \mathbb{Q}$ we may assume that $q_1 = 1$. Taking $M = 1$ in (1) and applying determinants, we deduce that $\det(g) = f(M)$. Applying determinants to all other equations, we find that

$$q_M^3 = \frac{f(1) M}{f(M)}.$$

In particular, for $M = N$, we have $q_N^3 f(N) = N f(1)$. Since $f(N) | N$, we must have $q_N \in \mathbb{Z}$.

Let us make (1) more explicit. Taking $M = 1$, we have

$$g = \begin{pmatrix} * & * & * \\ * & * & * \\ f(1)* & f(1)* & f(1)* \end{pmatrix},$$

where $*$ denotes unknown integers. In particular, the last column of g is integral. If we now take $M = N$, we have

$$g = \begin{pmatrix} q_N * & q_N * & * \\ q_N * & q_N * & * \\ q_N f(N)* & q_N f(N)* & * \end{pmatrix}.$$

Using the properties of the determinant and that $*$ denotes integers, we deduce that $q_N^2 | \det(g) = f(1)$.

Let $f(1) = q_N^2 k$ for some $k \in \mathbb{Z}$. Now the last row of g is divisible by $q_N^2 k$ and the first two columns are divisible by q_N, so by the same method we infer that $q_N k \cdot q_N \cdot q_N = q_N^3 k$ divides $\det(g) = f(1) = q_N^2 k$. Therefore $q_N = 1$, which implies that $f(N) = N f(1)$. Since $f(N) | N$, we have $f(1) = 1$ and $f(N) = N$. Putting everything together, it follows that $g \in \Gamma_0(N)$. □

Remark 3. The case $n > 3$ can be done similarly. In essence, what makes the case $n = 2$ differ from the rest is the imbalance between the number of columns with divisibility conditions and the number of rows wich such conditions. This leads to the different exponents of q_N in the proof and ultimately to the triviality of the solutions to our equations.

Remark 4. We believe that a similar proof with slight adjustments can show that the normaliser inside $\mathrm{GL}_n(\mathbb{R})$ is also trivial.

3 The Atkin-Lehner operators

3.1 A different perspective

We have seen in the last section that $n=2$ is singular in the sequence of families $\Gamma_0^n(N)$ of congruence subgroups. To arrive at a general definition of Atkin-Lehner operators, it is useful to note another way in which the group $\mathrm{SL}_2(\mathbb{R})$ is distinguished, as described below.

A very important automorphism of matrices in $\mathrm{SL}_n(\mathbb{R})$ is the map $g \mapsto g^{-T}$, sending a matrix to its inverse transpose. In number theory, this map is used to define the dual form of an automorphic form for $\mathrm{SL}_n(\mathbb{Z})$ (s. section 9.2 in [2]), or also to define the contragredient representation. In the theory of automorphic forms for $\mathrm{SL}_2(\mathbb{Z})$, dual forms are not usually mentioned because dualising turns out to be trivial. Indeed, if we take $w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ to be the long Weyl element, then we easily compute that

$$w\, g^{-T} w^{-1} = -\frac{1}{\det(g)}\, g. \tag{2}$$

In particular, the map $z \mapsto z^{-T}$ induces the identity under the projection $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}(2)$.

We can artificially introduce the dual map into the theory of Atkin-Lehner operators. For instance, one could write the Fricke involution W_N as

$$W_N f(z) = f\left(\begin{pmatrix} & -1 \\ N & \end{pmatrix} z\right) = f\left(\begin{pmatrix} & -1 \\ N & \end{pmatrix} w\, z^{-T} w\right) = f\left(\begin{pmatrix} 1 & \\ & N \end{pmatrix} z^{-T}\right).$$

Though slightly cumbersome in rank 1, this approach leads to the right definition of Atkin-Lehner operators for $n > 2$.

Let $g \in \mathrm{GL}_n(\mathbb{R})$ such that

$$g^{-1} \Gamma_0^n(N) g = \Gamma_0^n(N)^T. \tag{3}$$

Then the map $f(z) \mapsto f(g\, z^{-T})$ is an operator on the space of automorphic forms for $\Gamma_0^n(N)$, which we call by definition an *Atkin-Lehner operator*. As in the previous example, all Atkin-Lehner operators for $n=2$ can be construed as above. More precisely, taking a matrix in the normaliser of $\Gamma_0^2(N)$ and multiplying from the right by the long Weyl element gives a matrix g satisfying (3).

In this interpretation, the group structure coming from the normaliser is not obvious any more. Indeed, using this definition, we cannot recover the identity for $n > 2$, though this is still possible for $n=2$ through the special property (2). Finding an ever more general definition proves difficult, since the available types of automorphisms of invertible matrices are scarce. As explained in [5], all automorphisms in the case $n > 2$ are constructed out of inner automorphisms, radial automorphisms, and the inverse-transpose automorphism. Inner automorphisms cannot contribute, since we have proved that the normaliser of $\Gamma_0^n(N)$ is trivial; radial automorphisms are trivial in our context, since automorphic forms are invariant under the center of $\mathrm{GL}_n(\mathbb{R})$; and the inverse-transpose automorphism is precisely the basis for the definition given in this note.

3.2 The Fricke involution

Nevertheless, this definition does yield an important operator. We define the *Fricke involution* (of level N) to be the Atkin-Lehner operator given by the matrix

$$W_N = \mathrm{diag}(1, \ldots, 1, N),$$

which is easily seen to satisfy (3). By slight abuse of notation, we denote this operator by W_N as well. It can be checked that the Fricke involution is indeed an involution. If we consider the space of automorphic form for $\Gamma_0^n(N)$ with character χ, then the image of the operator is the space of automorphic forms with character $\bar{\chi}$, as expected.

Another expected property of the Fricke involution is that it essentially commutes with Hecke operators. In fact, not taking characters into consideration, the precise formulation is that $T_m W_N = W_N T_m^*$, where T_m is the m-th Hecke operator. It is another special feature of $\mathrm{SL}_2(\mathbb{R})$ that $T_m = T_m^*$, but this is no longer true in higher rank (s. Theorem 9.3.6 and its proof in [2]).

Lemma 5. *For $g \in \mathcal{M}_n(\mathbb{Z})$ with $\det(g) = m$ and last row of the form $(0, \ldots 0, *) \bmod N$, let T_g denote the Hecke operator on automorphic forms for $\Gamma_0^n(N)$ corresponding to the double coset $\Gamma_0^n(N) g \Gamma_0^n(N)$. Then $T_g W_N = W_N T_g^*$.*

Proof. By a variant of the Smith normal form, we may assume that g is diagonal and by a variant of the transposition anti-automorphism for $\Gamma_0^n(N)$ (generalising Lemma 4.5.2 and Theorem 4.5.3 in [6], we may assume that there are matrices α_i, $i = 1, \ldots, k$, for some k, such that

$$\Gamma_0^n(N)\, g\, \Gamma_0^n(N) = \bigcup_i \Gamma_0^n(N)\, \alpha_i = \bigcup_i \alpha_i\, \Gamma_0^n(N).$$

Then by definition we have

$$T_g W_N f(z) = \sum_i W_N f(\alpha_i z) = \sum_i f(W_N \cdot \alpha_i^{-T} z^{-T}) = \sum_i f(\beta_i \cdot W_N \cdot z^{-T}) = W_N \sum_i f(\beta_i z),$$

where $\beta_i = W_N \alpha_i^{-T} W_N^{-1}$.The proof is finished by showing that $\bigcup_i \Gamma_0^n(N) \beta_i = \Gamma_0^n(N)\, g^{-1}\, \Gamma_0^n(N)$, since this double coset corresponds to T_g^* (s. [Goldfeld 6.4.10]). Indeed,

$$\begin{aligned}
\bigcup_i \Gamma_0^n(N) \beta_i &= \bigcup_i \Gamma_0^n(N) W_N \alpha_i^{-T} W_N^{-1} \\
&= \bigcup_i W_N\, \Gamma_0^n(N)^T W_N^{-1} W_N \alpha_i^{-T} W_N^{-1} \\
&= W_N \Big[\bigcup_i \Gamma_0^n(N) \alpha_i \Big]^{-T} W_N^{-1} \\
&= W_N\, \Gamma_0^n(N)^T g^{-1}\, \Gamma_0^n(N)^T W_N^{-1} \\
&= \Gamma_0^n(N)\, g^{-1}\, \Gamma_0^n(N).
\end{aligned}$$

Here we made use of fundamental property (3) of W_N and of the fact that g is diagonal. ☐

Another property of the Fricke involution that we may expect is self-adjointness. This can easily be seen by using a known fact about the dual map for $\mathrm{SL}_n(\mathbb{Z})$. Namely, the map $f(z) \mapsto f(w\, z^{-T} w^{-1})$, where w is the long Weyl element, is self-adjoint (one can compute directly in explicit coordinates given in [2], Proposition 9.2.1 or Proposition 6.3.1). We can interpret the Fricke involution as

$$W_N f(z) = f(m\, w\, z^{-T} w^{-1}),$$

where $m = W_N w^{-1}$, that is, as the composition of the dual map with the left-action of m. Since the measure on $\mathbb{H}^n$ is $\mathrm{GL}_n(\mathbb{R})$-invariant, we can make the same explicit computations and change of coordinates as for the dual map. Since W_N as a matrix is diagonal, we easily deduce the conclusion

$$W_N^* = W_N$$

as operators.

As in the case of $n = 2$, the properties given above should lead to the appearence of the Fricke involution in the functional equation of automorphic L-functions. From this point of view, our definition does not give a properly new operator, but rather provides the classical formulation of an important concept that is already implicitly present in the representation-theoretic language.

3.3 A negative result

The theory of Atkin-Lehner operators for $\Gamma_0^n(N)$ shows some weaknesses already in the well-understood case $n = 2$. Indeed, one can only define Atkin-Lehner operators for divisors M of the level N, such that M and N/M are coprime. More precisely, there are no operators induced by matrices with determinant equal $M|N$, such that $(M, N/M) \neq 1$ (s. [1], p. 138). This phenomenon creates difficulties in applications when considering powerful levels. In this section, we see that these difficulties only get more problematic in higher rank. In fact, the only Atkin-Lehner operator for $n > 2$, according to our definition, is the Fricke involution.

Proposition 6. *Let $g\in \mathrm{GL}_n^+(\mathbb{Q})$ satisfy $g^{-1}\Gamma_0^n(N)g=\Gamma_0^n(N)^T$. Then, after scaling by a suitable rational number, g is integral, the last row and the last column of g are divisible by N, and $\det(g)=N$. Equivalently, $g\in\mathbb{Q}_{>0}\cdot\Gamma_0^n(N)\,W_N$.*

Proof. We apply the same ideas as in the proof of Theorem 1. Again the proof is done for $n=3$, merely for simplicity. One can check that $\Gamma_0(N)^T$ stabilises the lattices

$$L_{M^{-1}}=\langle e_1,e_2,M^{-1}e_3\rangle=\mathrm{diag}(1,1,M^{-1})L_1$$

for all divisors $M|N$. It follows that $\Gamma_0(N)$ must stabilise (up to scalars) the lattices $g\,L_{M^{-1}}$. By Lemma 2 determining the fixed points of $\Gamma_0(N)$, we have

$$g\,L_{M^{-1}}=q_M L_{f(M)},$$

with $f(M)|N$. We normalise g by a rational number so that $q_1=1$. The equations above imply that

$$g\in q_M\,\mathrm{diag}(1,1,f(M))\,\mathrm{SL}_3(\mathbb{Z})\,\mathrm{diag}(1,1,M), \tag{4}$$

using that the stabiliser of L_1 is $\mathrm{SL}_3(\mathbb{Z})$. Let us take determinants and deduce that

$$\det g=q_M^3\cdot f(M)\cdot M. \tag{5}$$

By our assumption, $\det g=f(1)$.

Take $M=N$ in (5) and note that

$$q_N^{-3}=\frac{f(N)N}{f(1)}.$$

Since $f(1)|N$, we deduce that $q_N^{-3}\in\mathbb{Z}$, so $d:=q_N^{-1}\in\mathbb{Z}$. Using this notation we have $d^3f(1)=f(N)N$.

Now we use the matrix equation for $M=1$ and $M=N$ to find that

$$g=\begin{pmatrix} & & \\ & & \\ f(1)* & f(1)* & f(1)*\end{pmatrix}\quad\text{and}\quad g=\begin{pmatrix} & & \frac{N}{d}*\\ & & \frac{N}{d}*\\ \frac{f(N)}{d}* & \frac{f(N)}{d}* & \frac{f(N)N}{d}*\end{pmatrix}, \tag{6}$$

where the $*$'s stand for integers and the rest of the matrices are filled by integers.

Notice that $d|N$. Indeed, say there is a prime p such that $p^k|d$, but $p^k\nmid N$. Then $p^k\nmid f(N)$ since $f(N)|N$, and thus $p^{2k}\nmid Nf(N)$. But we know that $d^3f(1)=Nf(N)$, so we must have $p^{3k}|Nf(N)$, which is a contradiction unless $k=0$.

Now suppose p is a prime dividing d such that $p^k\|\,d$ is the maximal power of p dividing d, with $k\geqslant 1$. As in the last paragraph, it would follow that $p^{3k}|f(N)N$ and $p^k|N$. Since $f(N)|N$, we deduce that p divides N/d. We now use the divisibility conditions from the right of (6) for the last column of g and the divisibility conditions from the left of (6) for the first two entries of the last row of g, so that putting everything together we obtain

$$g=\begin{pmatrix} & & p*\\ & & p*\\ f(1)* & f(1)* & f(1)p^2*\end{pmatrix}.$$

It would follow that $\det g=f(1)\cdot p$, but this is a contradiction. Therefore $d=1$.

We infer that $f(1)=Nf(N)$, so considering divisibility we must have $f(1)=N$ and $f(N)=1$. This implies that $\det g=N$ and that the last row and column of g are divisible by N.

Thus g is of the form

$$g=\begin{pmatrix}\alpha_1 & \alpha_2 & N\alpha_3\\ \beta_1 & \beta_2 & N\beta_3\\ N\gamma_1 & N\gamma_2 & N\gamma_3\end{pmatrix},$$

with $\alpha_i,\beta_i,\gamma_i\in\mathbb{Z}$. Since $\det(g)=N$, it must be that γ_3 is coprime to N and that $(\alpha_3,\beta_3,\gamma_3)=1$. In fact, put these together to have $(N\alpha_3,N\beta_3,\gamma_3)=1$. Now take $x,y,z\in\mathbb{Z}$ such that

$$x\,N\alpha_3+y\,N\beta_3+z\,\gamma_3=1.$$

Then $(x N, y N, z) = 1$, so we can find a matrix $u \in \Gamma_0^3(N)$ with last row equal to $(x N, y N, z)$. It follows from the above that the entry in the lower right corner of $u \cdot g$ is equal to N. By doing row manipulations we can find $u' \in \Gamma_0^3(N)$ such that

$$u' g = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ N* & N* & N \end{pmatrix}.$$

In this form, it is obvious that we can find another $u'' \in \Gamma_0^3(N)$ so that $u'' g = W_N$. □

Remark 7. Let us note what changes in the proof in the case $n = 2$ and how this leads to the lack of Atkin-Lehner operators for powerful level. In the notation above, we would have the equation $d^2 f(1) = f(N)N$, where the exponent of d is equal to n in general. We can still prove that $d|N$, yet the next paragraph differs slightly.

We suppose p is a prime dividing d such that $p^k || d$ is the maximal power of p dividing d, with $k \geqslant 1$. As in the proof above, we deduce that $p^{2k} | f(N)N$ and $p^k|N$. To continue the proof and deduce that $d = 1$, we need the step showing that p divides N / d. This is not true in general any more. For example, if N is square free, then $k \leqslant 1$ and this may not hold for certain choices of $f(N)$. In fact, solving the matrix equations eventually leads to the matrices found by Atkin and Lehner (after suitably multiplying by the long Weyl element).

If N is powerful, then we could have that a higher power of p divides N, so that, for certain choices of d, we can indeed deduce that $p|N / d$ and produce a contradiction. These choices of d correspond to divisors M of N, such that $(M, N / M) \neq 1$. Indeed, suppose that $\det(g) = f(1) =: M$, $p|M$ and $p|N / M$. Then p divides $d = f(N)N / M$. If $p^k \| d$, then applying the p-adic valuation to $d^2 M = f(N)N$ and recalling that $f(N)|N$ shows that $p|N / d$. We proceed as in the proof above and derive a contradiction. This shows that there are no Atkin-Lehner operators for such divisors M as above.

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