Oberwolfach Arbeitsgemeinschaft on Quantum Signal processing and Nonlinear Fourier Analysis

András Gilyén, Lin Lin, Christoph Thiele

October 6-11 2024

Contents

1 Introduction

The recent study of and progress on quantum signal processing has created renewed interest in nonlinear Fourier analysis, which is related to important algorithms in quantum signal processing.

These algorithms of quantum signal processing represent functions on a real variable using a product of unitary matrices depending on this variable. This relates to the nonlinear Fourier series (NLFT), which is a multiplicative version of the classical additive Fourier series. Prominent examples of NLFT live on low dimensional Lie groups. The unitary group $SU(2)$ provides a natural link to quantum signal processing, while historically the group $SU(1,1)$ has played a prominent role.

In the following sections, we list topics meant for a 45 min presentation at Oberwolfach.

In Section 2, we give an introduction to quantum computing in general and to quantum signal processing in particular.

In Section 3, we give an introduction to nonlinear Fourier series and explain the connection to quantum signal processing.

In Section 4, we elaborate more on quantum signal processing.

In Section 5, we elaborate more on the NLFT and related classical results. The literature is vast and the themes abundant. We focus on some specially selected material that is most adapt for our purpose.

2 Basics of quantum computing

2.1 The postulates of quantum computing

Present the basic principles of quantum computing underlying the computational model and illustrate them through the example of teleportation. Follow Chapter 1 of R. de Wolf's lecture notes `https://arxiv.org/abs/1907.09415`

2.2 The circuit model of quantum computing

The circuit model is a theoretical model for a universal quantum computer. Present this model. As an example discuss the Deutsch-Józsa algorithm. Follow

Chapter 2 of R. de Wolf's lecture notes https://arxiv.org/abs/1907.09415

2.3 Simon's algorithm

Present Simon's algorithm. Follow Chapter 3 of R. de Wolf's lecture notes https://arxiv.org/abs/1907.09415 Possibly start with a review of Deutsch-Józsa and Bernstein-Varizani of the previous chapter.

2.4 The fast and the faster quantum Fourier transform

Discuss the FFT and its quantum variant. A classical computer takes $O(N\log(N))$ time to compute the full output vector using FFT, while a quantum computer outputs a quantum state proportional to the output vector represented as a superposition of states, remarkably in time $O(\log(N)\log\log(N))$. Follow Chapter 4 of R. de Wolf's lecture notes https://arxiv.org/abs/1907.09415

See also András Gilyén's PCMI summer school lectures http://gilyen.hu/teaching/PCMI_2023_QFT_prez_day_2.pdf

2.5 Quantum period finding and Shor's algorithm

Shor's algorithm is a fast quantum computing algorithm that factors a natural number into a nontrivial product of two natural numbers. If ever built to factor sufficiently large numbers, it would compromise many existing cryptographic protocols. Follow Chapter 5 of R. de Wolf's lecture notes https://arxiv.org/abs/1907.09415

2.6 Quantum phase estimation and gradient computation

Follow Chapter 4.6 of R. de Wolf's lecture notes https://arxiv.org/abs/1907.09415 and Sections 5.1 and 5.2 of https://arxiv.org/pdf/1711.00465.pdf. See also parts of Day 2 and Day 3 of András Gilyén's PCMI summer school lectures http://gilyen.hu/teaching/PCMI_2023_QFT_prez_day_3.pdf (AG: Will have some proper notes on this by the end of May.)

2.7 Quantum signal processing

Sections 2 and 3 of https://arxiv.org/pdf/1806.10236.pdf Also in 1 qubit case under change of basis show equivalent formulations outlined in Section 7.6 of https://arxiv.org/pdf/2201.08309.pdf. See also Footnote 2 of https://arxiv.org/pdf/2312.09072.pdf.

3 Nonlinear Fourier series

3.1 Nonlinear Fourier series for better than square summable

This is the first of a string of presentations on the $SU(1,1)$ model of nonlinear Fourier series. Discuss lecture 1 of the PCMI lecture series by T. Tao and C. Thiele. Focus on $p = 0, 1$ and do $1 < p < 2$ as time allows.

`https://arxiv.org/abs/1201.5129`

3.2 Nonlinear Fourier series, square summable, half line

Lecture 2 of the PCMI lecture series by T. Tao and C. Thiele. The point is, that one has a homeomorphism from a space of square summable sequences on the half line to a space that one can explicitly describe.

`https://arxiv.org/abs/1201.5129`

3.3 Nonlinear Fourier series, square summable, full line

Lecture 3 of the PCMI lecture series by T. Tao and C. Thiele. Thanks to the half line theory, the full line can be reduced to the half line modulo a Riemann Hilbert problem, which is much more subtle than the half line problem.

`https://arxiv.org/abs/1201.5129`

3.4 The Riemann Hilbert problem for rational functions

Lecture 4 of the PCMI lecture series by T. Tao and C. Thiele. The rational case sheds some partial light on the issues discussed in the previous lecture.

`https://arxiv.org/abs/1201.5129`

3.5 Nonlinear Fourier series, unitary case, half line

We now turn to the $SU(2)$ model of nonlinear Fourier series. It is largely analogous to $SU(1,1)$ but at some point bifurcates into a different behaviour. Highlight the differences and parallel features compared to the $SU(1,1)$ case discussed in earlier presentations. Follow Section 6 in the paper by M. Alexis, G. Mnatsakanyan, and C. Thiele

`https://arxiv.org/abs/2310.12683`

See also Section 3.3. (Chapter 3 has excerpts form Ya.Ju Tsai's thesis) in

`https://www.math.uni-bonn.de/people/thiele/teaching/2012NLFT/chapter.pdf`

3.6 Nonlinear Fourier series, unitary case, full line

Highlight the differences and parallel features compared to the $SU(1,1)$ case discussed in previous presentations. Follow Section 7 in the paper by Michel Alexis, Gevorg Mnatsakanyan, and Christoph Thiele

`https://arxiv.org/abs/2310.12683`

At the time of writing of this topics list, a paper is upcoming by M. Alexis, L. Lin, G. Mnatsakanyan, J. Wang. Present the improved analoguous result in that paper instead if it exists by the time of your preparation.

3.7 QSP and NLFT

We are finally ready to compare QSP and NLFT. Summarize from the paper by M. Alexis, G. Mnatsakanyan, and C. Thiele

`https://arxiv.org/abs/2310.12683`, how the results of Sections 6,7 discussed above provide solutions to the infinite quantum signal processing problem. Highlights are the permutation of Pauli matrices and the construction of the outer a. As time allows, compare with the other quantum signal processing papers of this Arbeitsgemeinschaft.

4 Diving deep into QSP

4.1 Block-encoding & Qubitization

Eventually, QSP is used in performing singular value transformations of a matrix that is block encoded by a unitary. Read Section 6 for block encoding, and 7.1 of `https://arxiv.org/pdf/2201.08309.pdf` for qubitization (or "lifting" process).

4.2 Quantum Singular Value Transformation

A simple matrix-algebra-based treatment:

`http://gilyen.hu/teaching/AdditionalMaterial.pdf` See also Sections 3.1-3.3 of `https://arxiv.org/pdf/1806.01838.pdf`

4.3 Direct methods for finding QSP angles

Root finding based method for computing phase factors, and understand why high precision arithmetic operations is needed. `http://arxiv.org/abs/1806.10236` Read Section 4 of `https://arxiv.org/pdf/2110.04993.pdf` for the factorization with symmetric phase factors and the concept of maximal solution.

Read this paper for an contour integral approach of finding the complementary polynomial without root finding `https://arxiv.org/abs/2202.02671`

4.4 Iterative methods for finding QSP angles and infinite quantum signal processing

Read Section 5 for the fixed point iteration algorithm (perhaps the simplest algorithm for finding phase factors) `https://arxiv.org/pdf/2209.10162.pdf`

Then read the construction of Newton's algorithm which (numerically) significantly improves the convergence of iterative methods `https://arxiv.org/pdf/2307.12468.pdf`

4.5 Quantum signal processing with continuous variables

SU(1,1) aspect of QSP: https://arxiv.org/abs/2304.14383

4.6 Alternative and Multivariable quantum signal processing (M-QSP)

Two papers in this direction for QSP with more variables, which may be useful for treating certain functions of commuting matrices: https://arxiv.org/pdf/2308.01501.pdf and https://arxiv.org/abs/2205.06261. A third paper provides a unifying perspective and might be more pedagogical: https://arxiv.org/abs/2312.09072

5 More on NLFT

5.1 Variational non-linear Hausdorff Young

By Terry Lion's theory, some sufficently strong quantitative estimates in the linear setting transfer to the nonlinear setting by a generic machine. Present the main result of the paper "A variational non-linear Hausdorff Young inequality in the discrete setting" by D. Oliveira e Silva in Mathematical Research Letters, Volume 25 (2018), Number 6, see also

https://arxiv.org/abs/1704.00688

5.2 Cantor group nonlinear Hausdorff Young

While uniform bounds for Hausdorff Young remain a open problem for the real model of NLFT, it can be proven in a Cantor group model by a Bellmann function technique. Present the main result of the paper "Uniform constants in Hausdorff-Young inequalities for the Cantor group model of the scattering transform" by V. Kovac in Proceedings of the American Mathematical Society, Vol. 140, No. 3 (MARCH 2012), pp. 915-926, see also

https://arxiv.org/abs/1012.3146

5.3 Schur's algorithm

Schur's algorithm generalizes the layer stripping of the nonlinear Fourier series. It appears in Schur's 1917 paper. We present the main theorem in Boyd's paper "Schur's algorithm for bounded holomorphic functions" (Bull. London Math Society 11 (1979) 145-150,

https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/blms/11.2.145

which compares Schur's algorithm for a function f with the NLFT (in slightly modified coordinates) precisely in case f is not an extreme point of the unit ball of the bounded holomorphic functions.

5.4 Orthogonal polynomials, Geronimus' Theorem

Schur's algorithm relates further to orthogonal polynomials on the unit circle, as is witnessed by Geronimus' theorem. We follow the source "Schur functions, Schur Parameters and Orthogonal Polynomials on the Unit Circle" by L.B. Golinskii in Zeitschrift für Analysis und ihre Anwendungen Vol 12 (1993) 457-469.

`https://ems.press/journals/zaa/articles/12279`

As time allows we include a sample of results relating properties of the measure to properties of the Schur parameters such as Theorem 1 in Section 4 or Theorem 5 in Section 5.

5.5 Jacobi matrices and Schrödinger operators

We are interested in the relation of Schur's algorithm and orthogonal polynomials to Jacobi matrices. This is described in the paper "Half line Schrödinger operators with no bound states by D. Damanik and R. Killip in Acta Math 193 (2004) no. 1 31–72, see also

`https://arxiv.org/abs/math-ph/0303001`

This paper is too long to be discusse din full, we first restrict attention to the discrete case. Even this is too much, so we make sure we present Theorem 5 in Section 2, which explains the desired connections, and then discuss whatever time allows.

5.6 The commutation method

A generalization of the theory of Schur function allows poles in the disc. This has many interesting applications. One prominent example is bound states of of Jacobi matrices. The commutation mathod allows to add and subtract such states. We discuss this following the paper "Commutation method for Jacobi matrices" by F. Gesztesy and G. Teschl in Journal of Differential equations 128, 252–299 (1996). As the paper is too long, we restrict attention to a single step of the the single commutation method

5.7 Modified Korteweg de Vries by inverse scattering, solitons

We discuss the inverse scattering method to the mKdV equation, a nonlinear perturbation of te Airy equation. We compare to the linear Fourier method to solving the Airy equation. Sources include the scetch in Lecture 6 of the PCMI lecture series ($SU(1,1)$) by T. Tao and C. Thiele as well as the paper "Some remarks on the modified Korteweg de Vries equations" by S. Tanaka in Publ. RIMS, Kyoto Univ. 8 (1972/73) 429–437, which also contains a construction of soliton solutions in the focusing (SU(2)) case. If time is short, we shall be satisfied with understanding the N=1 case.

5.8 The discrete NLS, Ablowitz-Ladik

We discuss the inverse scattering method for the Ablowitz-Ladik equation, a discrete model for the nonlinear Schrödinger equation. We follow the preprint "Stability of Schur's iterates and fast solutions of the discrete integrable NLS" by R. Bessenov and P. Gubkin in

`https://arxiv.org/abs/2402.02434`,

In so far as there is overlap with the PCMI lecture series ($SU(1,1)$) by T. Tao and C. Thiele that is presented in previous lectures, we try to save time in the exposition.