“SHARP INEQUALITIES IN HARMONIC ANALYSIS” KOPP SUMMER SCHOOL (AUG 30 – SEP 4, 2015)

1. Introduction

This summer school aims at a better understanding of the ideas behind the methods and techniques developed in recent years in order to establish sharp forms of several crucial inequalities in Euclidean harmonic analysis. In particular, we will focus our attention on sharp forms of restriction inequalities for the Fourier transform, classical inequalities of convolution type, geometric inequalities of different flavors, and optimal estimates for oscillatory integrals.

In more detail, we address: (i) Extremizers in the context of Fourier restriction theorems on spheres (9, 13), paraboloids and cones (12), and other hypersurfaces (11). Heat flow methods in this context are the subject of (3). (ii) Classical inequalities in “hard” analysis, including Hausdorff-Young (2, 8, 18), Young’s convolution (1, 2, 18) and its converse (1), Hardy (16), and Hardy-Littlewood-Sobolev (15). (iii) Geometric inequalities including rearrangement and isoperimetric inequalities (14), Brunn–Minkowski inequalities **??** and Sobolev inequalities (4, 10), and Radon transforms (7). The method of competing symmetries was initiated in (6). (iv) Sharp estimates related to multidimensional oscillatory integrals (5).

2. List of topics

(1) F. Barthe, *Optimal Young’s inequality and its converse: a simple proof.* Geom. Funct. Anal. **8** (1998), no. 2, 234–242.

(2) W. Beckner, *Inequalities in Fourier analysis.* Ann. of Math. (2) **102** (1975), no. 1, 159–182. (Focus on Section II, the sharp Hausdorff-Young inequality.)

(3) J. Bennett, N. Bez, A. Carbery and D. Hundertmark, *Heat-flow monotonicity of Strichartz norms.* Anal. PDE **2** (2009), no. 2, 147–158.

(4) G. Bianchi and H. Egnell, *A note on the Sobolev inequality.* J. Funct. Anal. **100** (1991), no. 1, 18–24. (Section 2 (Inequality (0.2)) may be skipped. The computations in the appendix can be complemented by the corresponding part of Section 2 in:)
S. Chen, R. Frank and T. Weth, *Remainder terms in the fractional Sobolev inequality.* Indiana Univ. Math. J. **62** (2013), 1381–1397.

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(5) A. Carbery, M. Christ and J. Wright, *Multidimensional van der Corput and sublevel set estimates.* J. Amer. Math. Soc. **12** (1999), no. 4, 981–1015. (Focus on Sections 2 and 3. Time permitting, cover Section 4 and some applications from Section 8.)

(6) E. Carlen and M. Loss, *Extremals of functionals with competing symmetries.* J. Funct. Anal. **88** (1990), no. 2, 437–456. (You may ignore Section 4.)

(7) M. Christ, *Extremizers of a Radon transform inequality.* Preprint, arXiv: 1106.0719 (2011). (Focus on Sections 2, 4 and 5.)

(8) M. Christ, *A sharpened Hausdorff-Young inequality.* Preprint, arXiv: 1406.1210 (2014). (Theorem 1.1 is the main result but its proof is long. Follow the outline of Section 2 and prove Proposition 1.4.)

(9) M. Christ and S. Shao, *Existence of extremals for a Fourier restriction inequality.* Anal. PDE. **5** (2012), no. 2, 261–312. (Theorem 1.4 is the main result but its proof is long. Follow the outline of Section 2 and focus on the details from Sections 3,4 and 5. Time permitting, discuss Section 16.)

(10) D. Cordero-Erausquin, B. Nazaret and C. Villani, *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities.* Adv. Math. **182** (2004), no. 2, 307–332. (The focus should be on Sections 1 and 2.)

(11) L. Fanelli, L. Vega and N. Visciglia, *On the existence of maximizers for a family of restriction theorems.* Bull. London Math. Soc. **43** (2011), no. 4, 811–817. (Complement with notes by Frank-Sabin on existence of optimizers for the Strichartz inequality on the conference homepage.)

(12) D. Foschi, *Maximizers for the Strichartz inequality.* J. Eur. Math. Soc. (JEMS) **9** (2007), no. 4, 739–774. (Focus on Sections 3 and 6 or Sections 4 and 5, but not both. If time permits, discuss Section 7.)

(13) D. Foschi, *Global Maximizers for the Sphere Adjoint Fourier Restriction Inequality.* Preprint, arXiv: 1310.2510 (2013). (To highlight the fact that existence of extremizers is a rather subtle phenomenon, this can be complemented by a discussion of the main result in:)
R. Quilodrán, *Nonexistence of extremals for the adjoint restriction inequality on the hyperboloid.* J. Anal. Math **125** (2015), no. 1, 37–70.

(14) R. Frank, *Schwarz symmetrization and applications.* Lecture notes. (The focus should be on the proof of the isoperimetric inequality, Section 3. If time permits discuss Sections 1 and 2. The material on symmetrization can be complemented by subchapters 3.6–3.9 in:)
E. Lieb and M. Loss, *Analysis.* Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 1997.

(15) R. Frank and E. H. Lieb, *A new, rearrangement-free proof of the sharp Hardy-Littlewood-Sobolev inequality.* Spectral theory, function spaces and inequalities,

55–67, Oper. Theory Adv. Appl., 219, Birkhäuser/Springer Basel AG, Basel, 2012.

(16) R. Frank and R. Seiringer, *Non-linear ground state representations and sharp Hardy inequalities.* J. Funct. Anal. **255** (2008), no. 12, 3407–3430. (The classical method (Section 2.3) should be discussed in detail. Remainder terms (Theorem 1.2) and the appendix may be skipped.)

(17) E. Lieb, *Proof of an entropy conjecture of Wehrl.* Comm. Math. Phys. **62** (1978), no. 1, 35–41.

(18) E. H. Lieb, *Gaussian kernels have only Gaussian maximizers.* Invent. Math. **102** (1990), no. 1, 179–208. (Focus on Sections 2 and 3. This can additionally be complemented by a discussion of the main result in:)
M. Christ and R. Quilodrán, *Gaussians rarely extremize adjoint Fourier restriction inequalities for paraboloids.* Proc. Amer. Math. Soc. **142** (2014), no. 3, 887–896.

(19) B. Simon, *Convexity.* Cambridge Univ. Press, Cambridge, 2011. (Discuss Chapter 13 on 'Brunn-Minkowski inequalities and log concave functions'.)

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