Fundamental Notions in Algebra – Exercise No. 3

1. Give an example of two abelian groups A and B and a short exact sequence $0 \to A \to A \oplus B \to B \to 0$ which does not split.

2. (a) Let M be a module of finite length. Show that every submodule and factor module of M have finite length.

   (b) Conversely, assume that $N \subset M$ and M/N have finite length. Show that M has finite length, and that $l(M) = l(N) + l(M/N)$.

   (c) Assume that R has finite length as an R-module. Show that every finitely generated R-module has finite length.

3. Let $0 \to M_1 \to \ldots \to M_n \to 0$ be an exact sequence of modules of finite length. Show that $\sum_{i=1}^{n}(-1)^i l(M_i) = 0$.

4. Let M be a module such that each of its submodules is a direct summand. Show that M is semi-simple as follows:

   (a) Show that every submodule M' of M inherits the property that each submodule (of M') is a direct summand (of M').

   (b) Show that M contains two submodules $N_1 \subset N_2$ such that N_2/N_1 is simple (use question 1(a) of Ex. 1). Deduce that M contains a simple submodule.

   (c) Let M' be the sum of all simple submodules of M. Deduce from (a) and (b) that $M' = M$, hence M is semisimple.

5. (a) Let R be a commutative ring. Assume that the free R-modules R^n and R^m are isomorphic. Show that $m = n$.

   Hint: Show first that if $M = R^n$ and $I \subset R$ is a maximal ideal, then the quotient M/IM is an n-dimensional R/I-vector space.

   (b) Show that the assertion of (a) holds if instead of assuming that R is commutative, we assume that the ring R is semisimple.

6. Let V be an infinite-dimensional vector space over a field F with a countable basis $\{x_n\}_{i=1}^{\infty}$, and let $R = \mathrm{End}_F(V)$ be the set of linear endomorphisms of V. Show that R-modules R and R^2 are isomorphic. (In particular, the conclusion of Question 5 is false in this case).

   Hint: Set $I := \{r \in R : r(x_{2n}) = 0 \text{ for all } n \geq 0\}$ and $J := \{r \in R : r(x_{2n+1}) = 0 \text{ for all } n \geq 0\}$. Show that

   (a) I and J are left ideals of R

   (b) $R = I \oplus J$, that is, $R = I + J$ and $I \cap J = 0$

   (c) $I \cong J \cong R$ as R-modules.