# Basic Notions in Algebra – Exercise No. 14

1. Let $G$ be a finite group, $\rho_1 : G \to GL(V_1)$, $\rho_2 : G \to GL(V_2)$ two finite dimensional representations over a field $k$. Consider the vector space of $k$-linear transformations $\mathrm{Hom}_k(V_1, V_2)$.

   (a) Define, for every $g \in G$ and $T \in \mathrm{Hom}_k(V_1, V_2)$,
   $$\rho(g)(T)(v_1) = \rho_2(g) T \rho_1(g)^{-1}(v_1)$$
   Prove that $\rho : G \to GL(\mathrm{Hom}_k(V_1, V_2))$ is a representation. $\rho$ is denoted by $\mathrm{Hom}(\rho_1, \rho_2)$.

   (b) Prove that the subspace of $\rho$-invariant vectors in $\mathrm{Hom}_k(V_1, V_2)$ is equal to $\mathrm{Hom}_G(V_1, V_2)$, the space of homomorphisms from $\rho_1$ to $\rho_2$, i.e. $T \in \mathrm{Hom}_k(V_1, V_2)$ s.t. $\rho_2(g)T = T\rho_1(g)$ for all $g \in G$.

   (c) Deduce that if $\rho_1, \rho_2$ are irreducible and $k$ is algebraically closed then $\mathrm{Hom}_k(\rho_1, \rho_2)$ contains the trivial representation with multiplicity 1.

   (d) Prove that $V_1^\vee \otimes V_2 \simeq \mathrm{Hom}_k(V_1, V_2)$ by considering the bi-linear (prove!) mapping $(\varphi_1, v_2) \mapsto (v_1 \mapsto \varphi_1(v_1)v_2)$ for $\varphi_1 \in V_1^\vee, v_2 \in V_2$.

   (e) Show that $\mathrm{Hom}(\rho_1, \rho_2)$ and $\rho_1^\vee \otimes \rho_2$ are equivalent.

2. Let $G$ be a finite group, $N \lhd G$ a normal subgroup.

   (a) Show that any representation $\pi : G/N \to GL(V)$ lifts to a representation $\tilde{\pi} : G \to G/N \to GL(V)$.

   (b) $\tilde{\pi}$ is irreducible if and only if $\pi$ is irreducible.

   (c) The character table of $G/N$ is a submatrix of the character table of $G$.

In the next questions, we always work over the field $\mathbb{C}$.

3. Let $G$ be a finite group, $\rho : G \to GL(V)$ a representation, $\chi_\rho$ its character. Let $\rho_1, \ldots, \rho_r$ be the irreducible representations of $G$.

   (a) Show that $|\chi_\rho(g)| \leq \chi_\rho(1)$ for all $g \in G$.

   (b) $\ker \rho = \{g \in G : \chi_\rho(g) = \chi_\rho(1)\}$

   (c) If $\rho$ decomposes as $\rho = \sum m_j \rho_j$, then $\ker \rho = \bigcap \ker \rho_j$.

   (d) If $N \lhd G$, there exist $\rho_{j_1}, \ldots, \rho_{j_s}$ such that $N = \bigcap_{i=1}^{s} \ker \rho_{j_i}$.

4. Let $G$ be a finite group, $\rho : G \to GL(V)$ an irreducible representation, $\chi$ its character.

   (a) If $z \in Z(G)$, then $\rho(z)$ is a scalar matrix.
   *Hint.* Use Schur's lemma.

   (b) For any $g \in G$, $\rho(g)$ is a scalar matrix if and only if $|\chi(g)| = \chi(1)$.

5. Let $G$ be a finite group. Suppose we know just the character table of $G$, that is, we know that there are $r$ conjugacy classes $c_1, \ldots, c_r$ and $r$ characters $\chi_1, \ldots, \chi_r$, and all the values $\chi_i(c_j) \in \mathbb{C}$ are known.

   In particular, we *do not* know the multiplication rule in $G$. Let's see what can be deduced from the character table.

(a) If $g \in c_j$, show that $|C_G(g)| = \sum_{i=1}^{r} |\chi_i(c_j)|^2$, hence the size of $c_j$ can be deduced from the table.

(b) Show how to determine if $G$ is Abelian.

(c) Show how to find the lattice of normal subgroups of $G$ and their orders.

*Remark.* Here and in the next questions, by finding a normal subgroup we mean to describe it as a union of conjugacy classes.

(d) Show how to find the commutator subgroup $G'$.

(e) Show how to find the center of $G$.

(f) The *upper central series* of $G$ is defined by induction by $Z_0 = 1$, $Z_1 = Z(G)$ and $Z_{i+1}$ is the subgroup corresponding to $Z(G/Z_i)$ under the correspondence theorem, i.e. $Z_{i+1}/Z_i = Z(G/Z_i)$.

$G$ is *nilpotent* if there is $c \geq 1$ such that $Z_c = G$.

Show how to determine (using the character table) if $G$ is nilpotent.

6. Use character degree counting to prove the following assertions.

(a) Let $G$ be a group of order $p^n$ where $p$ is a prime and $n \geq 2$. Show that $[G : G'] \geq p^2$. In particular, show that a group of order $p^2$ is Abelian.

(b) Show that a group $G$ of order $pq$ (where $p < q$ are primes) is not simple.

(c) Let $G$ be a group of order $pq$ where $p < q$ and $p \nmid q - 1$. Show that $G$ is cyclic.