BOUNDS ON THE GLOBAL DIMENSION OF CERTAIN PIECEWISE HEREDITARY CATEGORIES

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Abstract. We give bounds on the global dimension of a finite length, piecewise hereditary category in terms of quantitative connectivity properties of its graph of indecomposables.

We use this to show that the global dimension of a finite dimensional, piecewise hereditary algebra A cannot exceed 3 if A is an incidence algebra of a finite poset or more generally, a sincere algebra. This bound is tight.

1. Introduction

Let $\mathcal{A}$ be an abelian category and denote by $\mathcal{D}^b(\mathcal{A})$ its bounded derived category. $\mathcal{A}$ is called *piecewise hereditary* if there exist an abelian hereditary category $\mathcal{H}$ and a triangulated equivalence $\mathcal{D}^b(\mathcal{A}) \simeq \mathcal{D}^b(\mathcal{H})$. Piecewise hereditary categories of modules over finite dimensional algebras have been studied in the past, especially in the context of tilting theory, see [1, 2, 3].

It is known [2, (1.2)] that if $\mathcal{A}$ is a finite length, piecewise hereditary category with n non-isomorphic simple objects, then its global dimension satisfies $\operatorname{gl.dim} \mathcal{A} \leq n$. Moreover, this bound is almost sharp, as there are examples [5] where $\mathcal{A}$ has n simples and $\operatorname{gl.dim} \mathcal{A} = n-1$.

In this note we show how rather simple arguments can yield effective bounds on the global dimension of such a category $\mathcal{A}$, in terms of quantitative connectivity conditions on the graph of its indecomposables, regardless of the number of simple objects.

Let $G(\mathcal{A})$ be the directed graph whose vertices are the isomorphism classes of indecomposables of $\mathcal{A}$, where two vertices Q, Q' are joined by an edge $Q \to Q'$ if $\operatorname{Hom}_{\mathcal{A}}(Q, Q') \neq 0$.

Let $r \geq 1$ and let $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{r-1})$ be a sequence in $\{+1, -1\}^r$. An *ε-path* from Q to Q' is a sequence of vertices $Q_0 = Q, Q_1, \dots, Q_r = Q'$ such that $Q_i \to Q_{i+1}$ in $G(\mathcal{A})$ if $\varepsilon_i = +1$ and $Q_{i+1} \to Q_i$ if $\varepsilon_i = -1$.

For an object Q of $\mathcal{A}$, let $\operatorname{pd}_{\mathcal{A}} Q = \sup\{d : \operatorname{Ext}^d_{\mathcal{A}}(Q, Q') \neq 0 \text{ for some } Q'\}$ and $\operatorname{id}_{\mathcal{A}} Q = \sup\{d : \operatorname{Ext}^d_{\mathcal{A}}(Q', Q) \neq 0 \text{ for some } Q'\}$ be the projective and injective dimensions of Q, so that $\operatorname{gl.dim} \mathcal{A} = \sup_Q \operatorname{pd}_{\mathcal{A}} Q$.

Theorem 1.1. *Let $\mathcal{A}$ be a finite length, piecewise hereditary category. Assume that there exist $r \geq 1$, $\varepsilon \in \{1, -1\}^r$ and an indecomposable Q_0 such that for any indecomposable Q there exists an ε-path from Q_0 to Q.*

Then $\operatorname{gl.dim} \mathcal{A} \leq r+1$ *and* $\operatorname{pd}_{\mathcal{A}} Q + \operatorname{id}_{\mathcal{A}} Q \leq r+2$ *for any indecomposable Q.*

We give two applications of this result for finite dimensional algebras.

Let A be a finite dimensional algebra over a field k, and denote by $\operatorname{mod} A$ the category of finite dimensional right A-modules. Recall that a module M in $\operatorname{mod} A$ is *sincere* if all the simple modules occur as composition factors of M. The algebra A is called sincere if there exists a sincere indecomposable module.

Corollary 1.2. *Let A be a finite dimensional, piecewise hereditary, sincere algebra. Then* $\operatorname{gl.dim} A \leq 3$ *and* $\operatorname{pd} Q + \operatorname{id} Q \leq 4$ *for any indecomposable module Q in* $\operatorname{mod} A$.

Let X be a finite partially ordered set (*poset*) and let k be a field. The *incidence algebra* kX is the k-algebra spanned by the elements e_{xy} for the pairs $x \leq y$ in X, with the multiplication defined by setting $e_{xy}e_{y'z} = e_{xz}$ when $y = y'$ and zero otherwise.

Corollary 1.3. *Let X be a finite poset. If the incidence algebra kX is piecewise hereditary, then* $\operatorname{gl.dim} kX \leq 3$ *and* $\operatorname{pd} Q + \operatorname{id} Q \leq 4$ *for any indecomposable kX-module Q.*

The bounds in Corollaries 1.2 and 1.3 are sharp, see Examples 3.2 and 3.3.

The paper is organized as follows. In Section 2 we give the proofs of the above results. Examples demonstrating various aspects of these results are given in Section 3.

2. The proofs

2.1. **Preliminaries.** Let $\mathcal{A}$ be an abelian category. If X is an object of $\mathcal{A}$, denote by $X[n]$ the complex in $\mathcal{D}^b(\mathcal{A})$ with X at position $-n$ and 0 elsewhere. Denote by $\operatorname{ind} \mathcal{A}$, $\operatorname{ind} \mathcal{D}^b(\mathcal{A})$ the sets of isomorphism classes of indecomposable objects of $\mathcal{A}$ and $\mathcal{D}^b(\mathcal{A})$, respectively. The map $X \mapsto X[0]$ is a fully faithful functor $\mathcal{A} \to \mathcal{D}^b(\mathcal{A})$ which induces an embedding $\operatorname{ind} \mathcal{A} \hookrightarrow \operatorname{ind} \mathcal{D}^b(\mathcal{A})$.

Assume that there exists a triangulated equivalence $F : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{H})$ with $\mathcal{H}$ hereditary. Then F induces a bijection $\operatorname{ind} \mathcal{D}^b(\mathcal{A}) \simeq \operatorname{ind} \mathcal{D}^b(\mathcal{H})$, and we denote by $\varphi_F : \operatorname{ind} \mathcal{A} \to \operatorname{ind} \mathcal{H} \times \mathbb{Z}$ the composition

$$\operatorname{ind} \mathcal{A} \hookrightarrow \operatorname{ind} \mathcal{D}^b(\mathcal{A}) \xrightarrow{\sim} \operatorname{ind} \mathcal{D}^b(\mathcal{H}) = \operatorname{ind} \mathcal{H} \times \mathbb{Z}$$

where the last equality follows from [4, (2.5)].

If Q is an indecomposable of $\mathcal{A}$, write $\varphi_F(Q) = (f_F(Q), n_F(Q))$ where $f_F(Q) \in \operatorname{ind} \mathcal{H}$ and $n_F(Q) \in \mathbb{Z}$, so that $F(Q[0]) \simeq f_F(Q)[n_F(Q)]$ in $\mathcal{D}^b(\mathcal{H})$. From now on we fix the equivalence F, and omit the subscript F.

Lemma 2.1. *The map $f : \operatorname{ind} \mathcal{A} \to \operatorname{ind} \mathcal{H}$ is one-to-one.*

Proof. If Q, Q' are two indecomposables of $\mathcal{A}$ such that $f(Q), f(Q')$ are isomorphic in $\mathcal{H}$, then $Q[n(Q') - n(Q)] \simeq Q'[0]$ in $\mathcal{D}^b(\mathcal{A})$, hence $n(Q) = n(Q')$, and $Q \simeq Q'$ in $\mathcal{A}$. □

As a corollary, note that if A and H are two finite dimensional algebras such that $\mathcal{D}^b(\operatorname{mod} A) \simeq \mathcal{D}^b(\operatorname{mod} H)$ and H is hereditary, then the representation type of H dominates that of A.

We recall the following three results, which were introduced in [1, (IV,1)] when $\mathcal{H}$ is the category of representations of a quiver.

Lemma 2.2. *Let Q, Q' be two indecomposables of $\mathcal{A}$, Then*

$$\mathrm{Ext}^i_{\mathcal{A}}(Q, Q') \simeq \mathrm{Ext}^{i+n(Q')-n(Q)}_{\mathcal{H}}(f(Q), f(Q'))$$

Corollary 2.3. *Let Q, Q' be two indecomposables of $\mathcal{A}$ with* $\mathrm{Hom}_{\mathcal{A}}(Q, Q') \neq 0$*. Then $n(Q') - n(Q) \in \{0, 1\}$.*

Lemma 2.4. *Assume that $\mathcal{A}$ is of finite length and there exist integers n_0, d such that $n_0 \leq n(P) < n_0 + d$ for every indecomposable P of $\mathcal{A}$.*

If Q is indecomposable, then $\mathrm{pd}_{\mathcal{A}} Q \leq n(Q) - n_0 + 1$ *and* $\mathrm{id}_{\mathcal{A}} Q \leq n_0 + d - n(Q)$*. In particular,* $\mathrm{gl.dim}\, \mathcal{A} \leq d$*.*

Proof. See [1, IV, p.158] or [2, (1.2)]. □

2.2. **Proof of Theorem 1.1.** Let $r \geq 1$, $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{r-1})$ and Q_0 be as in the Theorem. Denote by r_+ the number of positive ε_i, and by r_- the number of negative ones. Let $F : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{H})$ be a triangulated equivalence and write $f = f_F$, $n = n_F$.

Let Q be any indecomposable of $\mathcal{A}$. By assumption, there exists an ε-path $Q_0, Q_1, \ldots, Q_r = Q$, so by Corollary 2.3, $n(Q_{i+1}) - n(Q_i) \in \{0, \varepsilon_i\}$ for all $0 \leq i < r$. It follows that $n(Q) - n(Q_0) = \sum_{i=0}^{r-1} \alpha_i \varepsilon_i$ for some $\alpha_i \in \{0, 1\}$, hence

$$n(Q_0) - r_- \leq n(Q) \leq n(Q_0) + r_+$$

and the result follows from Lemma 2.4 with $d = r + 1$ and $n_0 = n(Q_0) - r_-$.

2.3. **Variations and comments.**

Remark 2.5. The assumption in Theorem 1.1 that any indecomposable Q is the end of an ε-path from Q_0 can replaced by the weaker assumption that any *simple* object is the end of such a path.

Proof. Assume that $\varepsilon_{r-1} = 1$ and let Q be indecomposable. Since Q has finite length, we can find a simple object S with $g : S \hookrightarrow Q$. Let $Q_0, Q_1, \ldots, Q_{r-1}, S$ be an ε-path from Q_0 to S with $f_{r-1} : Q_{r-1} \twoheadrightarrow S$. Replacing S by Q and f_{r-1} by $g f_{r-1} \neq 0$ gives an ε-path from Q_0 to Q.

The case $\varepsilon_{r-1} = -1$ is similar. □

Remark 2.6. Let $\widetilde{G}(\mathcal{A})$ be the undirected graph obtained from $G(\mathcal{A})$ by forgetting the directions of the edges. The *distance* between two indecomposables Q and Q', denoted $d(Q, Q')$, is defined as the length of the shortest path in $\widetilde{G}(\mathcal{A})$ between them (or $+\infty$ if there is no such path).

The same proof gives that $|n(Q) - n(Q')| \leq d(Q, Q')$ for any two indecomposables Q and Q'. Let $d = \sup_{Q,Q'} d(Q, Q')$ be the *diameter* of $\widetilde{G}(\mathcal{A})$. When $d < \infty$, $\inf_Q n(Q)$ and $\sup_Q n(Q)$ are finite, and by Lemma 2.4 $\mathrm{gl.dim}\, \mathcal{A} \leq d + 1$ and $\mathrm{pd}_{\mathcal{A}} Q + \mathrm{id}_{\mathcal{A}} Q \leq d + 2$ for any indecomposable Q.

Remark 2.7. The conclusion of Theorem 1.1 (or Remark 2.6) is still true under the slightly weaker assumption that $\mathcal{A}$ is a finite length, piecewise hereditary category and $\mathcal{A} = \oplus_{i=1}^r \mathcal{A}_i$ is a direct sum of abelian full subcategories such that each graph $G(\mathcal{A}_i)$ satisfies the corresponding connectivity condition.

2.4. **Proof of Corollary 1.2.** Let A be sincere, and let $S_1, \dots, S_n$ be the representatives of the isomorphism classes of simple modules in $\operatorname{mod} A$. Let $P_1, \dots, P_n$ be the corresponding indecomposable projectives and finally let M be an indecomposable, sincere module.

Take $r = 2$ and $\varepsilon = (-1, +1)$. Now observe that any simple S_i is the end of an ε-path from M, as we have a path of nonzero morphisms $M \leftarrow P_i \twoheadrightarrow S_i$ since M is sincere. The result now follows by Theorem 1.1 and Remark 2.5.

2.5. **Proof of Corollary 1.3.** Let X be a poset and k a field. A *k-diagram* $\mathcal{F}$ is the data consisting of finite dimensional k-vector spaces $\mathcal{F}(x)$ for $x \in X$, together with linear transformations $r_{xx'} : \mathcal{F}(x) \to \mathcal{F}(x')$ for all $x \le x'$, satisfying the conditions $r_{xx} = 1_{\mathcal{F}(x)}$ and $r_{xx''} = r_{x'x''} r_{xx'}$ for all $x \le x' \le x''$.

The category of finite dimensional right modules over kX can be identified with the category of k-diagrams over X, see [6]. A complete set of representatives of isomorphism classes of simple modules over kX is given by the diagrams S_x for $x \in X$, defined by

$$S_x(y) = \begin{cases} k & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

with $r_{yy'} = 0$ for all $y < y'$. A module $\mathcal{F}$ is sincere if and only if $\mathcal{F}(x) \neq 0$ for all $x \in X$.

The poset X is *connected* if for any $x, y \in X$ there exists a sequence $x = x_0, x_1, \dots, x_n = y$ such that for all $0 \le i < n$ either $x_i \le x_{i+1}$ or $x_i \ge x_{i+1}$.

Lemma 2.8. *If X is connected then the incidence algebra kX is sincere.*

Proof. Let k_X be the diagram defined by $k_X(x) = k$ for all $x \in X$ and $r_{xx'} = 1_k$ for all $x \le x'$. Obviously k_X is sincere. Moreover, k_X is indecomposable by a standard connectivity argument; if $k_X = \mathcal{F} \oplus \mathcal{F}'$, write $V = \{x \in X : \mathcal{F}(x) \neq 0\}$ and assume that V not empty. If $x \in V$ and $x < y$, then $y \in V$, otherwise we would get a zero map $k \oplus 0 \to 0 \oplus k$ and not an identity map. Similarly, if $y < x$ then $y \in V$. By connectivity, $V = X$ and $\mathcal{F} = k_X$. □

If X is connected, Corollary 1.3 now follows from Corollary 1.2 and Lemma 2.8. For general X, observe that if $\{X_i\}_{i=1}^r$ are the connected components of X, then the category $\operatorname{mod} kX$ decomposes as the direct sum of the categories $\operatorname{mod} kX_i$, and the result follows from Remark 2.7.

Corollary 2.9. *Let X and Y be posets such that $\mathcal{D}^b(kX) \simeq \mathcal{D}^b(kY)$ and* $\operatorname{gl.dim} kY > 3$. *Then kX is not piecewise hereditary.*

3. Examples

We give a few examples that demonstrate various aspects of global dimensions of piecewise hereditary algebras. In these examples, k denotes a field and all posets are represented by their Hasse diagrams.

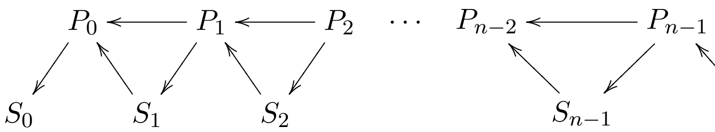
Example 3.1 ([5])**.** Let $n \ge 2$, $Q^{(n)}$ the quiver

$$0 \xrightarrow{\alpha_1} 1 \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_n} n$$

and $I^{(n)}$ be the ideal (in the path algebra $kQ^{(n)}$) generated by the paths $\alpha_i\alpha_{i+1}$ for $1 \leq i < n$. By [1, (IV, 6.7)], the algebra $A^{(n)} = kQ^{(n)}/I^{(n)}$ is piecewise hereditary of Dynkin type A_{n+1}.

For a vertex $0 \leq i \leq n$, let S_i, P_i, I_i be the simple, indecomposable projective and indecomposable injective corresponding to i. Then one has $P_n = S_n$, $I_0 = S_0$ and for $0 \leq i < n$, $P_i = I_{i+1}$ with a short exact sequence $0 \to S_{i+1} \to P_i \to S_i \to 0$.

The graph $G(\operatorname{mod} A^{(n)})$ is shown below (ignoring the self loops around each vertex).



Regarding dimensions, we have $\operatorname{pd} S_i = n - i$, $\operatorname{id} S_i = i$ for $0 \leq i \leq n$, and $\operatorname{pd} P_i = \operatorname{id} P_i = 0$ for $0 \leq i < n$, so that $\operatorname{gl.dim} A^{(n)} = n$ and $\operatorname{pd} Q + \operatorname{id} Q \leq n$ for every indecomposable Q. The diameter of $\widetilde{G}(\operatorname{mod} A^{(n)})$ is $n + 1$.

The following two examples show that the bounds given in Corollary 1.3 are sharp.

Example 3.2. *A poset X with kX piecewise hereditary and* $\operatorname{gl.dim} kX = 3$.

Let X, Y be the two posets:

X $\qquad\qquad$ Y

Then $\mathcal{D}^b(kX) \simeq \mathcal{D}^b(kY)$, $\operatorname{gl.dim} kX = 3$, $\operatorname{gl.dim} kY = 1$.

Example 3.3. *A poset X with kX piecewise hereditary and an indecomposable $\mathcal{F}$ such that* $\operatorname{pd}_{kX} \mathcal{F} + \operatorname{id}_{kX} \mathcal{F} = 4$.

Let X, Y be the following two posets:

X $\qquad\qquad$ Y

Then $\mathcal{D}^b(kX) \simeq \mathcal{D}^b(kY)$, $\operatorname{gl.dim} kX = 2$, $\operatorname{gl.dim} kY = 1$ and for the simple S_x we have $\operatorname{pd}_{kX} S_x = \operatorname{id}_{kX} S_x = 2$.

We conclude by giving two examples of posets whose incidence algebras are not piecewise hereditary.

Example 3.4. *A product of two trees whose incidence algebra is not piecewise hereditary.*

By specifying an orientation ω on the edges of a (finite) tree T, one gets a finite quiver without oriented cycles whose path algebra is isomorphic to the incidence algebra of the poset $X_{T,\omega}$ defined on the set of vertices of T by saying that $x \leq y$ for two vertices x and y if there is an oriented path from x to y.

A poset of the form $X_{T,\omega}$ is called a *tree*. Equivalently, a poset is a tree if and only if the underlying graph of its Hasse diagram is a tree. Obviously, $\operatorname{gl.dim} kX_{T,\omega} = 1$, so that $kX_{T,\omega}$ is trivially piecewise hereditary. Moreover, while the poset $X_{T,\omega}$ may depend on the orientation ω chosen, its derived equivalence class depends only on T.

Given two posets X and Y, their *product*, denoted $X \times Y$, is the poset whose underlying set is $X \times Y$ and $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$ where $x, x' \in X$ and $y, y' \in Y$. It may happen that the incidence algebra of a product of two trees, although not being hereditary, is piecewise hereditary. Two notable examples are the product of the Dynkin types $A_2 \times A_2$, which is piecewise hereditary of type D_4, and the product $A_2 \times A_3$ which is piecewise hereditary of type E_6.

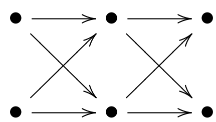
Consider $X = A_2 \times A_2$ and $Y = D_4$ with the orientations given below.

X Y

Then $\operatorname{gl.dim} kX = 2$, $\operatorname{gl.dim} kY = 1$ and $\mathcal{D}^b(kX) \simeq \mathcal{D}^b(kY)$, hence $\mathcal{D}^b(k(X \times X)) \simeq \mathcal{D}^b(k(Y \times Y))$. But $\operatorname{gl.dim} k(X \times X) = 4$, so by Corollary 2.9, $Y \times Y$ is a product of two trees of type D_4 whose incidence algebra is not piecewise hereditary.

Example 3.5. *The converse to Corollary 1.3 is false.*

Let X be the poset



Then $\operatorname{gl.dim} kX = 2$, hence $\operatorname{pd}_{kX} \mathcal{F} \leq 2$, $\operatorname{id}_{kX} \mathcal{F} \leq 2$ for any indecomposable $\mathcal{F}$, so that X satisfies the conclusion of Corollary 1.3. However, kX is not piecewise hereditary since $\operatorname{Ext}^2_X(k_X, k_X) = k$ does not vanish (see [1, (IV, 1.9)]). Note that X is the smallest poset whose incidence algebra is not piecewise hereditary.

References

[1] Happel, D. *Triangulated categories in the representation theory of finite-dimensional algebras*, vol. 119 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[2] Happel, D., Reiten, I., and Smalø, S. Piecewise hereditary algebras. *Arch. Math. (Basel) 66*, 3 (1996), 182–186.

[3] Happel, D., Rickard, J., and Schofield, A. Piecewise hereditary algebras. *Bull. London Math. Soc. 20*, 1 (1988), 23–28.

[4] Keller, B. Derived categories and tilting. In *Handbook of tilting theory*, L. Angeleri Hügel, D. Happel, and H. Krause, Eds., vol. 332 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2007, pp. 49–104.

[5] Kerner, O., Skowroński, A., Yamagata, K., and Zacharia, D. Finiteness of the strong global dimension of radical square zero algebras. *Cent. Eur. J. Math. 2*, 1 (2004), 103–111 (electronic).

[6] Ladkani, S. On derived equivalences of categories of sheaves over finite posets. *J. Pure Appl. Algebra 212*, 2 (2008), 435–451.

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