RESEARCH ARTICLE

Splittings of global Mackey functors and regularity of equivariant Euler classes

Stefan Schwede

Mathematisches Institut, Universität
Bonn, Bonn 53115, Germany

Correspondence
Stefan Schwede, Mathematisches Institut,
Universität Bonn, Endenicher Allee 60,
53115 Bonn, Germany.
Email: schwede@math.uni-bonn.de

Abstract
We establish natural splittings for the values of global
Mackey functors at orthogonal, unitary and symplec-
tic groups. In particular, the restriction homomor-
phisms between the orthogonal, unitary and symplec-
tic groups of adjacent dimensions are naturally split epi-
morphisms. The interest in the splitting comes from
equivariant stable homotopy theory. The equivariant sta-
ble homotopy groups of every global spectrum form a
global Mackey functor, so the splittings imply that cer-
tain long exact homotopy group sequences separate into
short exact sequences. For the real and complex global
Thom spectra **MO** and **MU**, the splittings imply the reg-
ularity of various Euler classes related to the tautological
representations of $O(n)$ and $U(n)$.

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INTRODUCTION

The purpose of this paper is to establish a splitting result for the values of global Mackey functors at orthogonal, unitary and symplectic groups. As a corollary, we derive regularity properties of equivariant Euler classes related to the tautological representations of $O(n)$ and $U(n)$. For this introduction, I will concentrate on the unitary case, where the splitting includes the following statement:

Theorem. *For every global functor F and every $n \geqslant 1$, the restriction homomorphism*

$$\mathrm{res}^{U(n)}_{U(n-1)} \ : \ F(U(n)) \longrightarrow F(U(n-1))$$

is a naturally split epimorphism.

The group $F(U(n))$ then naturally splits as the direct sum of the kernels of the restriction homomorphisms $\mathrm{res}^{U(k)}_{U(k-1)} : F(U(k)) \longrightarrow F(U(k-1))$ for $k = 0, \dots, n$, by induction. The above theorem is included in Theorem 1.4, where we exhibit a specific natural splitting. Besides the orthogonal, unitary and symplectic groups, the splitting also has an analog for symmetric groups, see Remark 1.5; this case is a direct generalization of Dold's arguments [7] from group cohomology to global functors. The splittings do *not* have analogs for alternating, special orthogonal and special unitary groups, see Section 4.

The interest in the splitting comes from equivariant stable homotopy theory. Indeed, for every global equivariant spectrum X, that is, an object of the global stable homotopy category [18, Section 4], the collection of equivariant homotopy groups $\pi^G_*(X)$ naturally forms a $\mathbb{Z}$-graded global functor as G varies over all compact Lie groups. Hence the restriction homomorphism $\mathrm{res}^{U(n)}_{U(n-1)} : \pi^{U(n)}_*(X) \longrightarrow \pi^{U(n-1)}_*(X)$ is a naturally split epimorphism. This is a genuinely global phenomenon: As we illustrate in Example 1.7, this restriction homomorphism is not surjective for general $U(n)$-spectra.

An interesting special case is the global Thom spectrum **MU** defined in [18, Example 6.1.53]. For every compact Lie group G, the underlying G-homotopy type of **MU** is that of tom Dieck's homotopical equivariant bordism [22]. This equivariant version of the complex bordism spectrum has been the object of much study, as it is related to equivariant bordism of stably almost complex manifolds [15, 22], equivariant complex-oriented cohomology theories [4, 9] and equivariant formal groups laws [3, 9, 10]. Our second main result is as follows:

Theorem. *For all $k_1, \dots, k_m \geqslant 1$, the Euler class of the tautological representation of $U(k_1) \times \cdots \times U(k_m)$ on $\mathbb{C}^{k_1 + \cdots + k_m}$ is a non zero-divisor in the graded ring $\mathbf{MU}^*_{U(k_1) \times \cdots \times U(k_m)}$.*

This result will be proved in Corollary 3.3. As we explain in Section 3, the regularity property is a relatively direct consequence of the surjectivity of the restriction homomorphisms $\mathrm{res}^{U(n)}_{U(n-1)}$: $\mathbf{MU}^*_{U(n)} \longrightarrow \mathbf{MU}^*_{U(n-1)}$. For $n = 1$ and $n = 2$, the surjectivity of the restriction homomorphisms, and hence the regularity of the Euler classes, were previously known and have a more direct proof. Indeed, the standard embeddings $U(0) \longrightarrow U(1)$ and $U(1) \longrightarrow U(2)$ admit retractions by continuous group homomorphisms; inflation along such a retraction then provides a splitting to the restriction homomorphism. For $n \geqslant 3$, however, the embedding $U(n-1) \longrightarrow U(n)$ does not admit such a retraction, and the splitting only exists after passage to unreduced suspension spectra of the global classifying spaces. To the best of the author's knowledge, ours is the first general regularity result for unitary groups of arbitrary rank.

Many of the major structural results about equivariant homotopical bordism are so far only known for *abelian* compact Lie groups, and regularity properties of Euler classes play an important role. Examples of such results include the following:

- For abelian compact Lie groups, the ring $\mathbf{MU}^*_G$ is a free module over the non-equivariant homotopy ring $\mathbf{MU}^*$ and concentrated in even degrees [6, Theorem 5.3], [14].

- The MU-cohomology of the non-equivariant classifying space of an abelian compact Lie group is the completion of $\mathbf{MU}_G^*$ at the augmentation ideal [5, Theorem 1.1], [15].
- For abelian compact Lie groups, $\mathbf{MU}_G^*$ carries the universal G-equivariant formal group law [10, Theorem A].
- The collection of rings $\mathbf{MU}_G^*$ for all abelian compact Lie groups carries the universal global formal group [10, Theorem C].

I hope that our regularity results might be useful to understand if and how the above results generalize from abelian compact Lie groups to general compact Lie groups.

Our splittings for global functors translate directly into stable global splittings of the global classifying spaces $B_{\mathrm{gl}}O(n)$, $B_{\mathrm{gl}}U(n)$ and $B_{\mathrm{gl}}Sp(n)$ of the orthogonal, unitary and symplectic groups, see Corollary 2.5. On underlying non-equivariant homotopy types, the stable splittings of $BU(n)$ and $BSp(n)$ are due to Snaith, see [19, Theorem 4.2], [20, Theorem 2.2] and the stable splitting of $BO(n)$ was constructed by Mitchell and Priddy [16, Theorem 4.1]. If G is a compact Lie group, we can apply the forgetful functor from the global to the genuine G-equivariant stable homotopy category, compare [18, Theorem 4.5.24]. We obtain G-equivariant stable splittings of the classifying G-spaces for G-equivariant real, complex and quaternionic vector bundles; as far as I know, these splittings are new.

1 | THE SPLITTING

In this section we formulate and prove our first main result, Theorem 1.4, the natural splitting of the values of a global functor at orthogonal, unitary and symplectic groups. We recall that a *global functor* in the sense of [18, Definition 4.2.2] is an additive functor from the global Burnside category of [18, Construction 4.2.1] to the category of abelian groups. In more explicit terms, a global functor specifies values on all compact Lie groups, restriction homomorphisms along continuous group homomorphisms and transfers along inclusions of closed subgroups; these data have to satisfy a short list of explicit relations that can be found after Theorem 4.2.6 of [18]. The data of such a global functor are equivalent to that of a *'functor with regular Mackey structure'* in the sense of Symonds [21, section 3, p. 177].

The proof of our splitting in Theorem 1.4 is inspired by Dold's elegant proof [7] of Nakaoka's splitting [17] of the cohomology of symmetric groups. We generalize Dold's strategy in three ways:

- from symmetric groups to orthogonal, unitary and symplectic groups;
- from the non-equivariant to the global context and
- from group cohomology to general global functors.

The proof of our splittings relies on the full global structure, as the splitting maps involve restriction, inflation and transfers.

We let $\mathbb{K}$ be one of the real division algebras $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We denote by $G(n)$ the compact Lie group of $(n \times n)$-matrices A over $\mathbb{K}$ that satisfy $A \cdot \bar{A}^t = \bar{A}^t \cdot A = E_n$, where $\bar{A}^t$ is the conjugate transpose matrix (for $\mathbb{K} = \mathbb{R}$, conjugation is the identity). So $G(n)$ is the orthogonal group $O(n)$ for $\mathbb{K} = \mathbb{R}$, it is the unitary group $U(n)$ for $\mathbb{K} = \mathbb{C}$, and it is the symplectic group $Sp(n)$ for $\mathbb{K} = \mathbb{H}$.

We let F be a global functor. We write

$$i_n \; : \; G(n-1) \longrightarrow G(n), \quad A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

for the standard embedding. This continuous monomorphism induces a restriction operation i_n^*, which is a morphism from $G(n)$ to $G(n-1)$ in the global Burnside category. The global functor sends it to a restriction homomorphism

$$F(i_n^*) \; : \; F(G(n)) \longrightarrow F(G(n-1)).$$

We write

$$\text{tr}_{m,n} \; : \; F(G(m) \times G(n)) \longrightarrow F(G(m+n))$$

for the transfer homomorphism associated to the continuous monomorphism

$$\mu_{m,n} \; : \; G(m) \times G(n) \longrightarrow G(m+n), \quad (A,B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

We recall the double coset formula for the subgroups $G(n-1)$ and $G(k) \times G(n-k)$ inside $G(n)$. The following result ought to be well-known to experts: In the unitary situation, the second summand in the following double coset formula actually vanishes, and similar double coset formulae were established in [8, Example IV.9] and [21, Lemma 4.2].

Proposition 1.1. *Let $\mathbb{K}$ be one of the real division algebras $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and let F be a global functor. Then for every $1 \leqslant k \leqslant n-1$, the relation*

$$F(i_n^*) \circ \text{tr}_{k,n-k} = \text{tr}_{k,n-k-1} \circ F((G(k) \times i_{n-k})^*) - \text{tr}_\Delta \circ F((i_k \times i_{n-k})^*)$$

$$+ \text{tr}_{k-1,n-k} \circ F((i_k \times G(n-k))^*)$$

holds as homomorphisms $F(G(k) \times G(n-k)) \longrightarrow F(G(n-1))$, where tr_Δ denotes the transfer along the closed embedding

$$\Delta \; : \; G(k-1) \times G(n-k-1) \longrightarrow G(n-1), \quad (A,B) \longmapsto \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B \end{pmatrix}.$$

Proof. We write $G(n-1, \sharp)$ for the image of the embedding $i_n : G(n-1) \longrightarrow G(n)$, and we write $G(k, n-k)$ for the closed subgroup of those block matrices of $G(n)$ of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ for $(A,B) \in G(k) \times G(n-k)$. The double coset space $G(n-1, \sharp) \backslash G(n) / G(k, n-k)$ is a closed interval; more precisely, the map

$$[0,\pi] \longrightarrow G(n-1, \sharp) \backslash G(n) / G(k, n-k), \quad t \mapsto G(n-1, \sharp) \cdot \gamma(t) \cdot G(k, n-k)$$

with

$$\gamma(t) = \begin{pmatrix} E_{k-1} & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & -\sin(t) \\ 0 & 0 & E_{n-k-1} & 0 \\ 0 & \sin(t) & 0 & \cos(t) \end{pmatrix}$$

is a homeomorphism.

For $t = 0$, the stabilizer of the right coset $\gamma(0) \cdot G(k, n - k)$ under the left $G(n - 1, \sharp)$-action is the subgroup $G(k, n - k - 1, \sharp)$ consisting of all matrices of the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $(A, B) \in G(k) \times G(n - k - 1)$. For $t = \pi$, the left stabilizer of the right coset $\gamma(\pi) \cdot G(k, n - k)$ is the subgroup $G(k - 1, \sharp, n - k)$ consisting of all matrices of the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B \end{pmatrix}$$

with $(A, B) \in G(k - 1) \times G(n - k)$. For $t \in (0, \pi)$, the left stabilizer of the right coset $\gamma(t) \cdot G(k, n - k)$ is the subgroup

$$G(k - 1, \sharp, n - k - 1, \sharp) \ = \ G(k, n - k - 1, \sharp) \cap G(k - 1, \sharp, n - k).$$

This shows that the orbit-type decomposition is as $\{0\} \cup (0, \pi) \cup \{\pi\}$.

The double coset formula [12, IV section 6], [18, Theorem 3.4.9] thus has three summands. The first summand indexed by $\gamma(0) = E_n$ contributes

$$\operatorname{tr}_{G(k,n-k-1,\sharp)}^{G(n-1,\sharp)} \ \circ \ \operatorname{res}_{G(k,n-k-1,\sharp)}^{G(k,n-k)} \ .$$

Under the identification of $G(k) \times G(n - k - 1)$ with $G(k, n - k - 1, \sharp)$, this becomes the term

$$\operatorname{tr}_{k,n-k-1} \circ F((G(k) \times i_{n-k})^*).$$

The summand indexed by $(0, \pi)$ occurs with coefficient -1, the internal Euler characteristic of the open interval. For $t \in (0, \pi)$, the matrix $\gamma(t)$ centralizes the subgroup $G(k - 1, \sharp, n - k - 1, \sharp)$; so in every global functor, the corresponding conjugation homomorphism $c_{\gamma(t)}^*$ is the identity. The second contribution to the double coset formula is thus

$$- \ \operatorname{tr}_{G(k-1,\sharp,n-k-1,\sharp)}^{G(n-1,\sharp)} \ \circ \ \operatorname{res}_{G(k-1,\sharp,n-k-1,\sharp)}^{G(k,n-k)} \ .$$

In the notation of the theorem, this becomes the term $- \operatorname{tr}_\Delta \circ F((i_k \times i_{n-k})^*)$. The third summand indexed by $\gamma(\pi)$ contributes

$$\operatorname{tr}_{G(k-1,n-k,\sharp)}^{G(n-1,\sharp)} \ \circ \ c_{\gamma(\pi)}^* \ \circ \ \operatorname{res}_{G(k-1,\sharp,n-k)}^{G(k,n-k)} \ .$$

The following square of group homomorphisms commutes:

$$
\begin{array}{ccc}
G(k-1) \times G(n-k) & \xrightarrow{\quad i_k \times G(n-k) \quad} & G(k) \times G(n-k) \\
\Big\downarrow {\scriptstyle \mu_{k-1,n-k}} & & \Big\downarrow {\scriptstyle \mu_{k,n-k}} \\
G(n-1) \xrightarrow{\ i_n^*\ } G(n) & \xrightarrow{\quad c_{\gamma(\pi)} \quad} & G(n)
\end{array}
$$

So under the identification of $G(k-1) \times G(n-k)$ with $G(k-1, n-k, \sharp)$, the third summand becomes the term $\mathrm{tr}_{k-1, n-k} \circ F((i_k \times G(n-k))^*)$. $\qquad\square$

Remark 1.2. In the unitary and symplectic case, that is, when the skew field is $\mathbb{C}$ or $\mathbb{H}$, the transfer $\mathrm{tr}_\Delta : F(G(k-1) \times G(n-k-1)) \longrightarrow F(G(n-1))$ that occurs in the double coset formula of Proposition 1.1 is actually zero. Indeed, the image of the embedding $\Delta : G(k-1) \times G(n-k-1) \longrightarrow G(n-1)$ is centralized by the subgroup of matrices of the form

$$\begin{pmatrix} E_{k-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & E_{n-k-1} \end{pmatrix},$$

a group isomorphic to $G(1)$. Since the groups $U(1)$ and $Sp(1)$ have positive dimension, the Weyl group of the image of Δ has positive dimension. The transfer tr_Δ is thus trivial. In the case of the orthogonal groups, the second summand in the double coset formula of Proposition 1.1 is generically non-zero.

Construction 1.3. As before we let F be a global functor in the sense of [18, Definition 4.2.2]. We now formulate the splittings of $F(O(n))$, $F(U(n))$ and $F(Sp(n))$. We write

$$F(O; k) = \ker(F(i_k^*) : F(O(k)) \longrightarrow F(O(k-1))),$$

$$F(U; k) = \ker(F(i_k^*) : F(U(k)) \longrightarrow F(U(k-1))) \quad \text{and}$$

$$F(Sp; k) = \ker(F(i_k^*) : F(Sp(k)) \longrightarrow F(Sp(k-1)))$$

for the kernels of the restriction homomorphism along i_k. For $k = 0$, we interpret this as $F(O; 0) = F(U; 0) = F(Sp; 0) = F(e)$, the value of F at the trivial group. For $0 \leqslant k \leqslant n$, we write

$$p_{k, n-k}^* : F(O(k)) \longrightarrow F(O(k) \times O(n-k))$$

for the inflation homomorphism associated to the projection to the first factor. We define a natural homomorphism

$$\psi_{k,n} : F(O; k) \longrightarrow F(O(n))$$

as the following composite:

$$F(O; k) \xrightarrow{\text{inclusion}} F(O(k)) \xrightarrow{p_{k, n-k}^*} F(O(k) \times O(n-k)) \xrightarrow{\mathrm{tr}_{k, n-k}} F(O(n)),$$

and similarly for

$$\psi_{k,n} : F(U; k) \longrightarrow F(U(n)) \quad \text{and} \quad \psi_{k,n} : F(Sp; k) \longrightarrow F(Sp(n)).$$

Because the group $O(0)$ is trivial, the map $\psi_{0,n}$ specializes to inflation along the unique homomorphism $O(n) \longrightarrow O(0)$, and $\psi_{n,n}$ is the inclusion $F(O; n) \longrightarrow F(O(n))$.

Theorem 1.4. *For every global functor F, and every $n \geqslant 1$, the maps*

$$\sum \psi_{k,n} : \bigoplus_{k=0}^{n} F(O;k) \longrightarrow F(O(n)),$$

$$\sum \psi_{k,n} : \bigoplus_{k=0}^{n} F(U;k) \longrightarrow F(U(n)) \quad and$$

$$\sum \psi_{k,n} : \bigoplus_{k=0}^{n} F(Sp;k) \longrightarrow F(Sp(n))$$

are isomorphisms of abelian groups, and the restriction homomorphisms

$$F(i_n^*) : F(O(n)) \longrightarrow F(O(n-1)),$$

$$F(i_n^*) : F(U(n)) \longrightarrow F(U(n-1)) \qquad and$$

$$F(i_n^*) : F(Sp(n)) \longrightarrow F(Sp(n-1))$$

are naturally split epimorphism.

Proof. We give the argument in the orthogonal case; the unitary and symplectic cases are analogous. We suppose that $1 \leqslant k \leqslant n-1$. We precompose the double coset formula of Proposition 1.1 with the homomorphism $p_{k,n-k}^* \circ \mathrm{incl} : F(O;k) \longrightarrow F(O(k) \times O(n-k))$. We observe that the last two of the three summands on the right-hand side compose trivially. Indeed,

$$\mathrm{tr}_\Delta \circ F((i_k \times i_{n-k})^*) \circ p_{k,n-k}^* \circ \mathrm{incl} = \mathrm{tr}_\Delta \circ p_{k-1,n-k-1}^* \circ F(i_k^*) \circ \mathrm{incl} = 0$$

and

$$\mathrm{tr}_{k-1,n-k} \circ F((i_k \times O(n-k))^*) \circ p_{k,n-k}^* \circ \mathrm{incl} = \mathrm{tr}_{k-1,n-k} \circ p_{k-1,n-k}^* \circ F(i_k^*) \circ \mathrm{incl} = 0.$$

So the double coset formula implies the relation

$$F(i_n^*) \circ \psi_{k,n} = F(i_n^*) \circ \mathrm{tr}_{k,n-k} \circ p_{k,n-k}^* \circ \mathrm{incl}$$

$$= \mathrm{tr}_{k,n-k-1} \circ F((O(k) \times i_{n-k})^*) \circ p_{k,n-k}^* \circ \mathrm{incl}$$

$$= \mathrm{tr}_{k,n-k-1} \circ p_{k,n-k-1}^* \circ \mathrm{incl} = \psi_{k,n-1}.$$

This final relation also holds for $k = 0$ by direct inspection, that is, $F(i_n^*) \circ \psi_{0,n} = \psi_{0,n-1}$.

Now we can prove the claim by induction on n. The induction starts with $n = 0$, where there is nothing to show. For $n \geqslant 1$ we obtain a commutative diagram

The upper row is exact by definition, and the lower row is exact at $F(O; n)$ and $F(O(n))$, also by definition. The right vertical map is an isomorphism by the inductive hypothesis; so the restriction map $F(i_n^*) : F(O(n)) \longrightarrow F(O(n-1))$ is in fact surjective, and the lower row is also exact. Since both rows are exact and the right vertical map is an isomorphism, the middle map is an isomorphism, too, and $F(i_n^*)$ is a naturally split epimorphism. $\qquad\square$

Remark 1.5 (Splitting for symmetric groups). The splittings of Theorem 1.4 have an analog for symmetric groups as well. This case is a direct generalization of Dold's arguments [7] from group cohomology to global functors. The argument for symmetric groups is significantly simpler than for orthogonal, unitary and symplectic groups because the analog of the double coset formula in Proposition 1.1 is easier to derive. Indeed, the relevant double coset space $\Sigma_{n-1}\backslash\Sigma_n/\Sigma_{k,n-k}$ is discrete with two points, there is no need for an analysis of the orbit-type stratification, and the relevant double coset formula is

$$\operatorname{res}^{\Sigma_n}_{\Sigma_{n-1}} \circ \operatorname{tr}^{\Sigma_n}_{\Sigma_{k,n-k}} = \operatorname{tr}^{\Sigma_{n-1}}_{\Sigma_{k,n-k-1}} \circ \operatorname{res}^{\Sigma_{k,n-k}}_{\Sigma_{k,n-k-1}} + \operatorname{tr}^{\Sigma_{n-1}}_{\Sigma_{k-1,n-k}} \circ c^*_{(k,n)} \circ \operatorname{res}^{\Sigma_{k,n-k}}_{(\Sigma_{k-1,n-k})^{(k,n)}} .$$

Here Σ_n is the n-th symmetric group, and we write $\Sigma_{k,n-k}$ for its subgroup consisting of those permutations that leave the subsets $\{1, \dots, k\}$ and $\{k+1, \dots, n\}$ invariant. We abuse notation by identifying Σ_{n-1} with the subgroup of Σ_n of permutations that fix the last element n; finally (k, n) is the transposition that interchanges k and n. With this double coset formula at hand, the same argument as in the proof of Theorem 1.4 shows that for every global functor F and every $n \geqslant 1$, the analogously defined map

$$\sum \psi_{k,n} : \bigoplus_{k=0}^{n} F(\Sigma; k) \longrightarrow F(\Sigma_n) \tag{1.6}$$

is an isomorphism, and the restriction homomorphism $F(i_n^*) : F(\Sigma_n) \longrightarrow F(\Sigma_{n-1})$ is a naturally split epimorphism.

In the case of symmetric groups, it suffices to consider a $\mathcal{F}in$-*global functor*, that is, the analog of a global functor defined only on finite groups. These $\mathcal{F}in$-global functors have been studied under different names in the algebraic literature, for example, as 'inflation functors' in [23, p.271], or as 'global $(\emptyset, \infty)$-Mackey functors' in [13]. The $\mathcal{F}in$-global functors are a special case of the more general class of 'biset functors' [1]. We refer the reader to [18, Remark 4.2.16] for more details on the comparison. I would not be surprised if the splitting (1.6) was already known, possibly in different language, and published somewhere in the algebraic literature on the subject. However, I am not aware of a reference.

Equivariant stable homotopy theory provides examples of global functors. Indeed, for every global equivariant spectrum X, that is, an object of the global stable homotopy category [18, section 4], and every integer m, the collection of m-th equivariant stable homotopy groups $\pi_m^G(X)$ naturally forms a global functor as G varies over all compact Lie groups. Moreover, the preferred t-structure on the global stable homotopy category shows that every global functor arises in this way, see [18, Theorem 4.4.9]. The splittings of Theorem 1.4 show that for every global equivariant spectrum X, the restriction homomorphism $\operatorname{res}^{O(n)}_{O(n-1)} : \pi_*^{O(n)}(X) \longrightarrow \pi_*^{O(n-1)}(X)$ is a naturally split epimorphism, and for every $0 \leqslant k \leqslant n$ the graded abelian group $\pi_*^{O(k)}(X)$ is a natural direct summand of $\pi_*^{O(n)}(X)$. And the analogous statements hold for unitary and symplectic groups.

Example 1.7. The surjectivity of the restriction homomorphism $\mathrm{res}_{O(n-1)}^{O(n)} : \pi_*^{O(n)}(X) \longrightarrow \pi_*^{O(n-1)}(X)$ is a special feature of global stable homotopy types, and it does not hold for general genuine $O(n)$-spectra. An easy example for $O(1) \cong \Sigma_2$ is given by the Eilenberg–MacLane spectrum HM for the Σ_2-Mackey functor M with $M(\Sigma_2/e) = \mathbb{Z}/2$ and $M(\Sigma_2/\Sigma_2) = 0$. The Σ_2-equivariant stable homotopy groups of HM vanish, while the 0-th non-equivariant stable homotopy group of HM is non-trivial. In particular, restriction from Σ_2 to Σ_1, or from $O(1)$ to $O(0)$, is not surjective.

A different kind of example for unitary groups is the unreduced suspension spectrum of the free and transitive $U(1)$-space. The Wirthmüller isomorphism shows that the group $\pi_0^{U(1)}(\Sigma_+^\infty U(1))$ vanishes. The group $\pi_0^e(\Sigma_+^\infty U(1))$ is isomorphic to $\mathbb{Z}$, so restriction from $U(1)$ to $U(0)$ is not surjective on 0-th equivariant homotopy groups.

2 | STABLE SPLITTINGS OF GLOBAL CLASSIFYING SPACES

Snaith has shown that the unreduced suspension spectra of the classifying spaces $BU(n)$ and $BSp(n)$ stably split into wedges of certain Thom spaces, see [19, Theorem 4.2] and [20, Theorem 2.2]. Mitchell and Priddy obtained such splittings by a different method in [16, Theorem 4.1], and their proof also applies to stably split the classifying spaces $BO(n)$ and $B\Sigma_n$ of the orthogonal and the symmetric groups. Corollary 2.5 below is a global refinement of this splitting, referring to the unreduced suspension spectra of the global classifying spaces $B_{\mathrm{gl}}O(n)$, $B_{\mathrm{gl}}U(n)$ and $B_{\mathrm{gl}}Sp(n)$ as defined in [18, Definition 1.1.27]. The splitting takes place in the global stable homotopy category $\mathcal{GH}$, that is, the localization of the category of orthogonal spectra at the class of global equivalences [18, Definition 4.1.3]. The global stable homotopy category is a compactly generated tensor triangulated category, see [18, Section 4.4].

Construction 2.1 (Global classifying spaces). In the model of [18], unstable global homotopy types are represented by orthogonal spaces. Orthogonal spaces are continuous functors to spaces from the category $\mathbf{L}$ of finite-dimensional inner product spaces and linear isometric embeddings, compare [18, Definition 1.1.1]. The category $\mathbf{L}$ is also denoted $\mathscr{I}$ or $\mathcal{I}$ by other authors, and orthogonal spaces are also known as $\mathscr{I}$-functors, $\mathscr{I}$-spaces or $\mathcal{I}$-spaces.

An important example is the *global classifying space* of a compact Lie group G, see [18, Definition 1.1.27]. The construction involves a choice of faithful G-representation V, and then

$$B_{\mathrm{gl}}G \ = \ \mathbf{L}(V, -)/G$$

is the orthogonal G-orbit space of the represented orthogonal space. The unstable global homotopy type of $B_{\mathrm{gl}}G$ is independent of the choice of faithful representation, and $B_{\mathrm{gl}}G$ 'globally represents' principal G-bundles over equivariant spaces, see [18, Proposition 1.1.30]. In particular, the underlying non-equivariant homotopy type of $B_{\mathrm{gl}}G$ is a classifying space for the Lie group G.

Every orthogonal space has an unreduced suspension spectrum, compare [18, Construction 4.1.7]. The suspension spectrum of $B_{\mathrm{gl}}G$ comes with a preferred G-equivariant homotopy class

$$e_G \ \in \ \pi_0^G(\Sigma_+^\infty B_{\mathrm{gl}}G) \,,$$

the *stable tautological class*, defined in [18, (4.1.12)]. By [18, Theorem 4.4.3], the pair $(\Sigma_+^\infty B_{\mathrm{gl}}G, e_G)$ represents the functor $\pi_0^G : \mathcal{GH} \longrightarrow \mathcal{A}b$.

Proposition 2.2. *Let $i : L \longrightarrow K$ be a continuous homomorphism between compact Lie groups such that for every global functor F, the restriction homomorphism $F(i^*) : F(K) \longrightarrow F(L)$ is surjective.*

  (i) *The morphism $i^* : K \longrightarrow L$ has a section in the global Burnside category.*
 (ii) *The morphism $\Sigma_+^\infty i : \Sigma_+^\infty B_{\mathrm{gl}} L \longrightarrow \Sigma_+^\infty B_{\mathrm{gl}} K$ has a retraction in the global stable homotopy category.*
(iii) *For every compact Lie group G and every global functor F, the restriction homomorphism $F((G \times i)^*) : F(G \times K) \longrightarrow F(G \times L)$ is a naturally split epimorphism.*

Proof.

  (i) Global functors are defined as additive functors from the global Burnside category $\mathbf{A}$ to the category of abelian groups. For the represented global functor $\mathbf{A}(L, -)$, the hypothesis shows that the restriction homomorphism

$$i^* \circ - \; : \; \mathbf{A}(L, K) \longrightarrow \mathbf{A}(L, L)$$

is surjective. Any preimage of the identity is a section to i^*.
 (ii) We let $\sigma \in \mathbf{A}(L, K)$ be a section to i^* as provided by part (i). The representability property of the pair $(\Sigma_+^\infty B_{\mathrm{gl}} K, e_K)$ provides a unique morphism $\rho : \Sigma_+^\infty B_{\mathrm{gl}} K \longrightarrow \Sigma_+^\infty B_{\mathrm{gl}} L$ in $\mathcal{GH}$ such that

$$\pi_0^K(\rho)(e_K) \; = \; \sigma(e_L)$$

in the group $\pi_0^K(\Sigma_+^\infty B_{\mathrm{gl}} L)$. Then

$$\pi_0^L(\rho \circ \Sigma_+^\infty B_{\mathrm{gl}} i)(e_L) \; = \; \pi_0^L(\rho)(i^*(e_K)) \; = \; i^*(\pi_0^K(\rho)(e_K)) \; = \; i^*(\sigma(e_L)) \; = \; e_L \,.$$

The representability property of the pair $(\Sigma_+^\infty B_{\mathrm{gl}} L, e_L)$ thus shows that $\rho \circ \Sigma_+^\infty B_{\mathrm{gl}} i$ is the identity of $\Sigma_+^\infty B_{\mathrm{gl}} L$. So ρ is the desired retraction.
(iii) The global Burnside category admits a biadditive symmetric monoidal structure that is given by the product of Lie groups on objects, see [18, Theorem 4.2.15]. Moreover, $G \times i^* = (G \times i)^*$ by (4.2.14) of [18]. Part (i) provides a section $\sigma : L \longrightarrow K$ to i^* in the global Burnside category $\mathbf{A}$. So the morphism

$$G \times \sigma \; : \; G \times L \longrightarrow G \times K$$

is a section to $(G \times i)^*$. Hence for every global functor F, the homomorphism $F(G \times \sigma)$ is a section to $F((G \times i)^*)$. $\qquad\square$

Theorem 1.4 and Proposition 2.2 together show that the global classifying space of $O(n-1)$ is globally stably a direct summand of the global classifying space of $O(n)$, and similarly for the unitary and symplectic groups. The next corollary refines this splitting and also identifies the summands as the suspension spectra of a global Thom spaces.

Construction 2.3 (Global Thom spaces). We let $\nu_{n, \mathbb{R}}$ denote the tautological real $O(n)$-representation on $\mathbb{R}^n$. For every $n \geqslant 0$, we define a based orthogonal space $M(n, \mathbb{R})$ by

$$M(n, \mathbb{R})(V) \; = \; \mathbf{L}(\nu_{n, \mathbb{R}}, V)_+ \wedge_{O(n)} S^{\nu_{n, \mathbb{R}}} \,.$$

So $M(n, \mathbb{R})$ is the global Thom space over $B_{\mathrm{gl}}O(n)$ of the global vector bundle associated to the tautological real $O(n)$-representation. We will show in Corollary 2.4 that the suspension spectrum of $M(n, \mathbb{R})$ represents the kernel of the restriction homomorphism from $O(n)$ to $O(n-1)$.

The inclusion $S^0 \longrightarrow S^{\nu_{n,\mathbb{R}}}$ of the $O(n)$-fixed points induces a morphism of based orthogonal spaces

$$j : B_{\mathrm{gl}}O(n)_+ = \mathbf{L}(\nu_{n,\mathbb{R}}, -)_+ \wedge_{O(n)} S^0 \longrightarrow \mathbf{L}(\nu_{n,\mathbb{R}}, -)_+ \wedge_{O(n)} S^{\nu_{n,\mathbb{R}}} = M(n, \mathbb{R}).$$

We pass to suspension spectra to obtain a morphism of orthogonal spectra

$$\Sigma^\infty j : \Sigma_+^\infty B_{\mathrm{gl}}O(n) \longrightarrow \Sigma^\infty M(n, \mathbb{R}).$$

We write

$$w_{n,\mathbb{R}} = \pi_0^{O(n)}(\Sigma^\infty j)(e_{O(n)}) \in \pi_0^{O(n)}(\Sigma^\infty M(n, \mathbb{R}))$$

for the image of the stable tautological class $e_{O(n)} \in \pi_0^{O(n)}(\Sigma_+^\infty B_{\mathrm{gl}}O(n))$.

Similarly, we define based orthogonal spaces $M(n, \mathbb{C})$ and $M(n, \mathbb{H})$ by

$$M(n, \mathbb{C})(V) = \mathbf{L}^\mathbb{C}(\nu_{n,\mathbb{C}}, \mathbb{C} \otimes_\mathbb{R} V)_+ \wedge_{U(n)} S^{\nu_{n,\mathbb{C}}} \quad \text{and}$$

$$M(n, \mathbb{H})(V) = \mathbf{L}^\mathbb{H}(\nu_{n,\mathbb{H}}, \mathbb{H} \otimes_\mathbb{R} V)_+ \wedge_{Sp(n)} S^{\nu_{n,\mathbb{H}}}.$$

Here $\nu_{n,\mathbb{C}}$ is the tautological complex $U(n)$-representation on $\mathbb{C}^n$, and $\nu_{n,\mathbb{H}}$ is the tautological quaternionic $Sp(n)$-representation on $\mathbb{H}^n$, and $\mathbf{L}^\mathbb{C}(-, -)$ and $\mathbf{L}^\mathbb{H}(-, -)$ denote the spaces of $\mathbb{C}$-linear and $\mathbb{H}$-linear isometric embeddings, respectively. The analogous construction as for $M(n, \mathbb{R})$ provides us with classes

$$w_{n,\mathbb{C}} \in \pi_0^{U(n)}(\Sigma^\infty M(n, \mathbb{C})) \quad \text{and} \quad w_{n,\mathbb{H}} \in \pi_0^{Sp(n)}(\Sigma^\infty M(n, \mathbb{H})).$$

Corollary 2.4. *The pair* $(\Sigma^\infty M(n, \mathbb{R}), w_{n,\mathbb{R}})$ *represents the functor*

$$\ker\left(\mathrm{res}^{O(n)}_{O(n-1)} : \pi_0^{O(n)} \longrightarrow \pi_0^{O(n-1)}\right) : \mathcal{GH} \longrightarrow \mathcal{Ab}.$$

The pair $(\Sigma^\infty M(n, \mathbb{C}), w_{n,\mathbb{C}})$ *represents the functor*

$$\ker\left(\mathrm{res}^{U(n)}_{U(n-1)} : \pi_0^{U(n)} \longrightarrow \pi_0^{U(n-1)}\right) : \mathcal{GH} \longrightarrow \mathcal{Ab}.$$

The pair $(\Sigma^\infty M(n, \mathbb{H}), w_{n,\mathbb{H}})$ *represents the functor*

$$\ker\left(\mathrm{res}^{Sp(n)}_{Sp(n-1)} : \pi_0^{Sp(n)} \longrightarrow \pi_0^{Sp(n-1)}\right) : \mathcal{GH} \longrightarrow \mathcal{Ab}.$$

Proof. We give the argument in the orthogonal case; the unitary and symplectic cases are analogous. We apply the functor $\Sigma^\infty \mathbf{L}(\nu_{n,\mathbb{R}}, -)_+ \wedge_{O(n)} -$ from the category of based $O(n)$-spaces to the category of orthogonal spectra to the mapping cone sequence

$$O(n)/O(n-1)_+ \longrightarrow S^0 \longrightarrow S^{\nu_{n,\mathbb{R}}} \longrightarrow O(n)/O(n-1)_+ \wedge S^1.$$

The result is a mapping cone sequence of orthogonal spectra

$$\Sigma_+^\infty B_{gl}O(n-1) \xrightarrow{\Sigma_+^\infty B_{gl}i_n} \Sigma_+^\infty B_{gl}O(n) \xrightarrow{\Sigma^\infty j} \Sigma^\infty M(n,\mathbb{R}) \longrightarrow \Sigma_+^\infty B_{gl}O(n-1) \wedge S^1 .$$

Mapping cone sequences of orthogonal spectra define distinguished triangles in the global stable homotopy category; taking morphism groups $[\![-,E]\!]$ in $\mathcal{GH}$ to an orthogonal spectrum E turns the distinguished triangle into a long exact sequence of abelian groups. The orthogonal spectra $\Sigma_+^\infty B_{gl}O(n-1)$ and $\Sigma_+^\infty B_{gl}O(n)$ represent the functors $\pi_0^{O(n-1)}$ and $\pi_0^{O(n)}$, respectively, by [18, Theorem 4.4.3], and the morphism $\Sigma_+^\infty B_{gl}i_n$ represents the restriction homomorphism $\mathrm{res}_{O(n-1)}^{O(n)} : \pi_0^{O(n)}(E) \longrightarrow \pi_0^{O(n-1)}(E)$. Since the equivariant homotopy groups of E are part of a global functor, the restriction homomorphism is surjective by Theorem 1.4. So the long exact sequence decomposes into short exact sequences

$$0 \longrightarrow [\![\Sigma^\infty M(n,\mathbb{R}),E]\!] \xrightarrow{f \mapsto f_*(w_{n,\mathbb{R}})} \pi_0^{O(n)}(E) \xrightarrow{\mathrm{res}_{O(n-1)}^{O(n)}} \pi_0^{O(n-1)}(E) \longrightarrow 0 .$$

This proves the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For the homotopy group global functor $\underline{\pi}_0(\Sigma_+^\infty B_{gl}O(n))$, Theorem 1.4 specializes to a splitting

$$\sum \psi_{k,n} : \bigoplus_{k=0}^n \ker\left(\mathrm{res}_{O(k-1)}^{O(k)} : \pi_0^{O(k)}(\Sigma_+^\infty B_{gl}O(n)) \longrightarrow \pi_0^{O(k-1)}(\Sigma_+^\infty B_{gl}O(n))\right)$$

$$\xrightarrow{\cong} \pi_0^{O(n)}(\Sigma_+^\infty B_{gl}O(n)) .$$

So there is a unique collection of classes $s_k \in \pi_0^{O(k)}(\Sigma_+^\infty B_{gl}O(n))$ such that $\mathrm{res}_{O(k-1)}^{O(k)}(s_k) = 0$ for all $k = 0, \dots, n$ and

$$\sum_{k=0}^n \psi_{k,n}(s_k) = e_{O(n)} .$$

Corollary 2.4 provides a unique morphism $\Psi_{k,n} : \Sigma^\infty M(k,\mathbb{R}) \longrightarrow \Sigma_+^\infty B_{gl}O(n)$ in the global stable homotopy category such that

$$\pi_0^{O(k)}(\Psi_{k,n})(w_{k,\mathbb{R}}) = s_k .$$

The analogous unitary and symplectic arguments provide morphisms $\Psi_{k,n} : \Sigma^\infty M(k,\mathbb{C}) \longrightarrow \Sigma_+^\infty B_{gl}U(n)$ and $\Psi_{k,n} : \Sigma^\infty M(k,\mathbb{H}) \longrightarrow \Sigma_+^\infty B_{gl}Sp(n)$ in $\mathcal{GH}$.

Corollary 2.5. *For every $n \geqslant 0$, the morphisms*

$$\bigvee \Psi_{k,n} : \bigvee_{k=0,\dots,n} \Sigma^\infty M(k,\mathbb{R}) \longrightarrow \Sigma_+^\infty B_{gl}O(n) ,$$

$$\bigvee \Psi_{k,n} : \bigvee_{k=0,\dots,n} \Sigma^\infty M(k,\mathbb{C}) \longrightarrow \Sigma_+^\infty B_{gl}U(n) \qquad and$$

$$\bigvee \Psi_{k,n} : \bigvee_{k=0,\dots,n} \Sigma^\infty M(k,\mathbb{H}) \longrightarrow \Sigma_+^\infty B_{gl}Sp(n)$$

are isomorphisms in the global stable homotopy category.

Proof. We give the argument in the orthogonal case; the unitary and symplectic cases are analogous. The composite homomorphism

$$\llbracket \Sigma_+^\infty B_{\mathrm{gl}} O(n), E \rrbracket \xrightarrow{\llbracket \bigvee \Psi_{k,n}, E \rrbracket} \llbracket \bigvee_{k=0,\ldots,n} \Sigma^\infty M(k, \mathbb{R}), E \rrbracket$$

$$\xrightarrow{\text{eval at } w_{k,\mathbb{R}}} \bigoplus_{k=0,\ldots,n} \ker\left(\mathrm{res}_{O(k-1)}^{O(k)} : \pi_0^{O(k)}(E) \longrightarrow \pi_0^{O(k-1)}(E) \right) \xrightarrow{\Sigma \psi_{k,n}} \pi_0^{O(n)}(E)$$

is evaluation at the stable tautological class $e_{O(n)}$, and hence an isomorphism by [18, Theorem 4.4.3]. In this composite, the second map is an isomorphism by Corollary 2.4, and the third map is an isomorphism by Theorem 1.4. So the map $\llbracket \bigvee \Psi_{k,n}, E \rrbracket$ is an isomorphism. Because E is an arbitrary object of the global stable homotopy category, this proves the claim. $\square$

If we apply the forgetful functor

$$U \; : \; \mathcal{GH} \longrightarrow \mathcal{SH}$$

from the global stable homotopy category to the non-equivariant stable homotopy category to Corollary 2.5, we obtain the stable splittings due to Snaith [19, Theorem 4.2], [20, Theorem 2.2] and Mitchell-Priddy [16, Theorem 4.1]. If G is a compact Lie group, we can apply the forgetful functor [18, Theorem 4.5.23]

$$U_G \; : \; \mathcal{GH} \longrightarrow G\text{-}\mathcal{SH}$$

from the global stable homotopy category to the genuine G-equivariant stable homotopy category. This forgetful functor turns the splittings of $B_{\mathrm{gl}} O(n)$, $B_{\mathrm{gl}} U(n)$ and $B_{\mathrm{gl}} Sp(n)$ of Corollary 2.5 into G-equivariant stable splittings of the classifying G-spaces for G-equivariant real, complex and quaternionic vector bundles. To the best of my knowledge, these G-equivariant stable splittings have not been observed before.

3 │ REGULARITY OF EULER CLASSES

In this section we apply our splitting results to derive the regularity of certain equivariant Euler classes of the global Thom spectra **MU** and **MO**, see Corollaries 3.2–3.5.

As we already mentioned, the equivariant homotopy groups of a global spectrum (that is, an object of the global stable homotopy category, represented by an orthogonal spectrum) form a graded global functor. If the global spectrum E is a global homotopy ring spectrum (that is, a monoid in the global stable homotopy category under the globally derived smash product), the equivariant homotopy groups form graded rings, and for all compact Lie groups G and K, the groups $\pi_*^{G \times K}(E)$ are naturally a graded module over the graded ring $\pi_*^G(E)$, via inflation along the projection $G \times K \longrightarrow G$.

In the next corollary, we continue to write $\nu_{n,\mathbb{R}}$, $\nu_{n,\mathbb{C}}$ and $\nu_{n,\mathbb{H}}$ for the tautological representation of $O(n)$, $U(n)$ and $Sp(n)$ on $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$, respectively. We write $a_{n,\mathbb{R}}$ for the Euler class of $\nu_{n,\mathbb{R}}$, that is, the element of $\pi_0^{O(n)}(\Sigma^\infty S^{\nu_{n,\mathbb{R}}})$ represented by the fixed point inclusion $S^0 \longrightarrow S^{\nu_{n,\mathbb{R}}}$, and similarly for $a_{n,\mathbb{C}}$ and $a_{n,\mathbb{H}}$.

Corollary 3.1. *For every orthogonal spectrum E, every compact Lie group G and every $n \geqslant 1$, the sequences of graded abelian groups*

$$0 \longrightarrow \pi_{*+\nu_{n,\mathbb{R}}}^{G \times O(n)}(E) \xrightarrow{-\cdot a_{n,\mathbb{R}}} \pi_*^{G \times O(n)}(E) \xrightarrow{\operatorname{res}_{G \times O(n-1)}^{G \times O(n)}} \pi_*^{G \times O(n-1)}(E) \longrightarrow 0$$

$$0 \longrightarrow \pi_{*+\nu_{n,\mathbb{C}}}^{G \times U(n)}(E) \xrightarrow{-\cdot a_{n,\mathbb{C}}} \pi_*^{G \times U(n)}(E) \xrightarrow{\operatorname{res}_{G \times U(n-1)}^{G \times U(n)}} \pi_*^{G \times U(n-1)}(E) \longrightarrow 0$$

and

$$0 \longrightarrow \pi_{*+\nu_{n,\mathbb{H}}}^{G \times Sp(n)}(E) \xrightarrow{-\cdot a_{n,\mathbb{H}}} \pi_*^{G \times Sp(n)}(E) \xrightarrow{\operatorname{res}_{G \times Sp(n-1)}^{G \times Sp(n)}} \pi_*^{G \times Sp(n-1)}(E) \longrightarrow 0$$

are split exact. If E is a global homotopy ring spectrum, then the splittings can be chosen as homomorphisms of graded $\pi_*^G(E)$-modules.

Proof. As usual, we prove the orthogonal case, and the unitary and symplectic cases are analogous. The cofiber sequence of based $O(n)$-spaces

$$O(n)/O(n-1)_+ \longrightarrow S^0 \xrightarrow{\operatorname{incl}} S^{\nu_{n,\mathbb{R}}} \longrightarrow S^1 \wedge O(n)/O(n-1)_+$$

becomes a cofiber sequence of $(G \times O(n))$-spaces by letting G act trivially. It induces a long exact sequence in $(G \times O(n))$-equivariant E-cohomology that we can interpret as a long exact sequence of RO-graded equivariant homotopy groups:

$$\dots \longrightarrow \pi_{*+1}^{G \times O(n-1)}(E) \xrightarrow{\partial} \pi_{*+\nu_{n,\mathbb{R}}}^{G \times O(n)}(E) \xrightarrow{-\cdot a_{n,\mathbb{R}}} \pi_*^{G \times O(n)}(E) \xrightarrow{\operatorname{res}_{G \times O(n-1)}^{G \times O(n)}} \pi_*^{G \times O(n-1)}(E) \longrightarrow \dots$$

The restriction homomorphism $\operatorname{res}_{G \times O(n-1)}^{G \times O(n)}$ is split surjective by Theorem 1.4 and Proposition 2.2, so the long exact sequence decomposes into short exact sequences. $\square$

We let **MU** denote the global Thom ring spectrum defined in [18, Example 6.1.53]. For every compact Lie group G, the underlying G-homotopy type of **MU** is that of tom Dieck's homotopical equivariant bordism [22]. For abelian compact Lie groups, the equivariant cohomology theory represented by **MU** is the universal complex-oriented equivariant cohomology theory [4, Theorem 1.2]. On the family of all abelian compact Lie groups, the equivariant homotopy groups of **MU** carry the universal global group law [10, Theorem C].

Since the global theory **MU** is complex-oriented, every unitary representation W of a compact Lie group G has a Euler class $e_{G,W} \in \mathbf{MU}_G^{2n}$, where $n = \dim_\mathbb{C}(W)$; by definition, $e_{G,W}$ is the restriction of the Thom class

$$\sigma_{G,W} \in \widetilde{\mathbf{MU}}_G^{2n}(S^W)$$

along the inclusion $S^0 \longrightarrow S^W$.

Corollary 3.2. *For every compact Lie group G, every character $\chi : G \longrightarrow U(1)$ and every $n \geq 1$, the Euler class of the $(G \times U(n))$-representation $\chi \otimes \nu_{n,\mathbb{C}}$ is a non zero-divisor in the graded-commutative ring* $\mathbf{MU}^*_{G \times U(n)}$.

Proof. We start with the special case where χ is the trivial character. Then the representation in question is $p^*(\nu_{n,\mathbb{C}})$, the restriction of the tautological $U(n)$-representation along the projection $G \times U(n) \longrightarrow U(n)$. The equivariant Thom isomorphism identifies the group $\pi^{G \times U(n)}_{k+p^*(\nu_{n,\mathbb{C}})}(\mathbf{MU}) = \widetilde{\mathbf{MU}}^0_{G \times U(n)}(S^{k+p^*(\nu_{n,\mathbb{C}})})$ with the group $\mathbf{MU}^{-k-2n}_{G \times U(n)}$, in a way that takes multiplication by the class $a_{n,\mathbb{C}}$ to multiplication by the Euler class of the representation $p^*(\nu_{n,\mathbb{C}})$. Corollary 3.1 thus shows that the Euler class of $p^*(\nu_{n,\mathbb{C}})$ is a non zero-divisor.

In the general case, the map

$$\psi : G \times U(n) \longrightarrow G \times U(n), \quad \psi(g, A) = (g, \chi(g) \cdot A)$$

is an isomorphism of Lie groups, and $\chi \otimes \nu_{n,\mathbb{C}}$ is the restriction of the representation $p^*(\nu_{n,\mathbb{C}})$ along ψ. Restriction along ψ is an isomorphism of graded rings

$$\psi^* : \mathbf{MU}^*_{G \times U(n)} \longrightarrow \mathbf{MU}^*_{G \times U(n)}$$

that sends the Euler class of $p^*(\nu_{n,\mathbb{C}})$ to the Euler class of $\psi^*(p^*(\nu_{n,\mathbb{C}})) = \chi \otimes \nu_{n,\mathbb{C}}$. Since the Euler class of $p^*(\nu_{n,\mathbb{C}})$ is a non zero-divisor by the first part, the Euler class of $\chi \otimes \nu_{n,\mathbb{C}}$ is a non zero-divisor, too. $\qquad \square$

The special case $G = e$ of Corollary 3.2 shows that the Euler class of the tautological complex $U(n)$-representation $\nu_{n,\mathbb{C}}$ is a non zero-divisor in the ring $\mathbf{MU}^*_{U(n)}$. The following corollary generalizes this:

Corollary 3.3. *For all $k_1, \dots, k_m \geq 1$ with $k_1 + \cdots + k_m = n$, the Euler class of the tautological representation of the group $U(k_1) \times \cdots \times U(k_m)$ on $\mathbb{C}^n$ is a non zero-divisor in the graded ring* $\mathbf{MU}^*_{U(k_1) \times \cdots \times U(k_m)}$.

Proof. The tautological representation of $U(k_1) \times \cdots \times U(k_m)$ splits as a direct sum

$$p_1^*(\nu_{k_1,\mathbb{C}}) \oplus \cdots \oplus p_m^*(\nu_{k_m,\mathbb{C}}),$$

where $p_i : U(k_1) \times \cdots \times U(k_m) \longrightarrow U(k_i)$ is the projection to the i-th factor. The Euler class of a direct sum is the product of the Euler classes, so

$$e_{U(k_1) \times \cdots \times U(k_m), \nu_{n,\mathbb{C}}} = p_1^*(\nu_{k_1,\mathbb{C}}) \cdot \ldots \cdot p_m^*(\nu_{k_m,\mathbb{C}}).$$

Each factor is a non zero-divisor by Corollary 3.2, hence so is the product. $\qquad \square$

Corollaries 3.2 and 3.3 work more generally for all globally complex-oriented homotopy-commutative global homotopy-ring spectra, that is, commutative monoids, under derived smash

product, in the global stable homotopy category that come equipped with coherent and natural Thom isomorphisms for equivariant complex vector bundles.

We let **MO** denote the global Thom ring spectrum defined in [18, Example 6.1.7]. For every compact Lie group G, the underlying G-homotopy type of **MO** is the real analog of tom Dieck's homotopical equivariant bordism [22]. By a theorem of Bröcker and Hook [2, Theorem 4.1], the G-equivariant homology theory represented by **MO** is stable equivariant bordism. Restricted to elementary abelian 2-groups, the equivariant homotopy groups of **MO** carry the universal global 2-torsion group law [10, Theorem D].

Since the global theory **MO** is real-oriented, every orthogonal representation V of a compact Lie group G has a Euler class $e_{G,V} \in \mathbf{MO}_G^n$, where $n = \dim_{\mathbb{R}}(V)$. The analogous arguments as in the complex case in Corollaries 3.2 and 3.3 prove the following real counterparts:

Corollary 3.4. *For every compact Lie group G, every continuous homomorphism $\chi : G \longrightarrow O(1)$ and every $n \geqslant 1$, the Euler class of the $(G \times O(n))$-representation $\chi \otimes \nu_{n,\mathbb{R}}$ is a non zero-divisor in the graded-commutative ring $\mathbf{MO}_{G \times O(n)}^*$.*

Corollary 3.5. *For all $k_1, \ldots, k_m \geqslant 1$ with $k_1 + \cdots + k_m = n$, the Euler class of the tautological representation of the group $O(k_1) \times \cdots \times O(k_m)$ on $\mathbb{R}^n$ is a non zero-divisor in the graded ring $\mathbf{MO}_{O(k_1) \times \cdots \times O(k_m)}^*$.*

Corollaries 3.4 and 3.5 work more generally for all globally real-oriented homotopy-commutative global homotopy-ring spectra, that is, commutative monoids, under derived smash product, in the global stable homotopy category that come equipped with coherent and natural Thom isomorphisms for equivariant real vector bundles.

4 | ALTERNATING, SPECIAL ORTHOGONAL AND SPECIAL UNITARY GROUPS

The families of alternating groups, special orthogonal groups and special unitary groups have the same kind of structure as the symmetric, orthogonal, unitary and symplectic groups; so one might wonder about the existence of splittings for the values of global functors at A_n, $SO(n)$ and $SU(n)$. In this section we complete the picture by showing that the restriction homomorphisms between adjacent groups in these families do *not* split naturally, except in some low-dimensional cases and for half of the special orthogonal groups.

In [11], Dold's method is adapted to obtain non-equivariant stable splittings of the classifying spaces of alternating, special orthogonal and special unitary groups after localizing at specific primes. Since these are *not* integral splittings and they do not involve adjacent groups from the respective family, the results are coarser than those of Dold, Snaith and Mitchell-Priddy for Σ_n, $O(n)$, $U(n)$ and $Sp(n)$.

Example 4.1 (Alternating groups). The standard embeddings $i_3 : e = A_2 \longrightarrow A_3$ and $i_4 : A_3 \longrightarrow A_4$ admit unique retractions by group homomorphisms; so for every global functor F, the restriction homomorphisms $F(i_3^*) : F(A_3) \longrightarrow F(A_2)$ and $F(i_4^*) : F(A_4) \longrightarrow F(A_3)$ are naturally split by the corresponding inflation homomorphisms.

For $n \geqslant 5$, the restriction homomorphism $F(i_n^*)$ is not in general surjective. For $n \geqslant 5$ and $n \neq 6, 8$, the complex representation ring global functor [18, Example 4.2.8 (iv)] is a witness. Indeed, for such n, there are non-conjugate elements of A_{n-1} that become conjugate in A_n. There is thus a complex representation of A_{n-1} whose character takes different values on these elements, and the class of this representation cannot be in the image of the restriction homomorphism $\mathrm{res}_{A_{n-1}}^{A_n}$: $RU(A_n) \longrightarrow RU(A_{n-1})$.

The two remaining cases can be settled by group cohomology with mod-2 and mod-3 coefficients: the map $i_6^* : H^3(A_6, \mathbb{F}_2) \longrightarrow H^3(A_5, \mathbb{F}_2)$ is not surjective, and the map $i_8^* : H^2(A_8, \mathbb{F}_3) \longrightarrow H^2(A_7, \mathbb{F}_3)$ is not surjective.

Example 4.2 (Special orthogonal groups). For the special orthogonal groups, Theorem 1.4 implies a natural splitting in half of the cases, ultimately because the group $O(2m - 1)$ is isomorphic to $\{\pm 1\} \times SO(2m - 1)$. Indeed, the continuous homomorphism $r : O(2m - 1) \longrightarrow SO(2m - 1)$ defined by $r(A) = \det(A) \cdot A$ is a retraction to the inclusion. So for every global functor F, inflation along r is a natural section to restriction from $F(O(2m - 1))$ to $F(SO(2m - 1))$. In combination with Theorem 1.4, this shows that restriction from $F(O(2m))$ to $F(SO(2m - 1))$ is naturally split; hence also the restriction homomorphism

$$F(i_{2m}^*) : F(SO(2m)) \longrightarrow F(SO(2m - 1))$$

is a naturally split epimorphism. Because $F(i_{2m}^*)$ is surjective, the restriction homomorphism

$$F(i_{2m}^* \circ i_{2m+1}^*) : F(SO(2m + 1)) \longrightarrow F(SO(2m - 1))$$

is also surjective, and hence a naturally split epimorphism by Proposition 2.2. The group $F(SO(2m + 1))$ then naturally splits as

$$F(SO(2m + 1)) \cong \bigoplus_{k=0,\ldots,m} \ker \left(\mathrm{res}_{SO(2k-1)}^{SO(2k+1)} : F(SO(2k + 1)) \longrightarrow F(SO(2k - 1)) \right),$$

with the interpretation that the summand for $k = 0$ is the value at the trivial group $SO(1)$. The underlying non-equivariant stable splitting of $BSO(2m + 1)$ goes back to Snaith [19, Theorem 4.3].

In the case of opposite parities, the map $Bi_{2m+1} : BSO(2m) \longrightarrow BSO(2m + 1)$ induced by the standard embedding is not even stably split in the non-equivariant sense. So its global analog cannot split in the global stable homotopy category, either, and the corresponding restriction homomorphism for global functors are not generally surjective, compare Proposition 2.2. To see this, we recall that the mod-2 cohomology of $BSO(k)$ is a polynomial algebra on the Stiefel–Whitney classes $w_2, \ldots, w_k$, and for $k = 2m + 1$, the relation $Sq^1(w_{2m}) = w_{2m+1}$ holds. Since w_{2m+1} is in the kernel of the restriction homomorphism

$$\mathrm{res}_{SO(2m)}^{SO(2m+1)} : H^*(BSO(2m + 1), \mathbb{F}_2) \longrightarrow H^*(BSO(2m), \mathbb{F}_2),$$

but w_{2m} restricts non-trivially, this restriction does not admit a section that is linear over the mod-2 Steenrod algebra.

Example 4.3 (Special unitary groups). As the group $SU(1)$ is trivial, the standard embedding i_2 : $SU(1) \longrightarrow SU(2)$ has a retraction by a continuous homomorphism, and for every global functor F, the corresponding restriction homomorphism $F(i_3^*) : F(SU(2)) \longrightarrow F(SU(1))$ is a naturally split epimorphism.

For $n \geqslant 3$, restriction homomorphism $F(i_n^*) : F(SU(n)) \longrightarrow F(SU(n-1))$ is not in general surjective. For $n = 3$, the relation $Sq^2(\bar{c}_2) = \bar{c}_3$ in $H^*(BSU(3), \mathbb{F}_2)$ shows that the map $Bi_3 :$ $BSU(2) \longrightarrow BSU(3)$ is not even stably split in the non-equivariant sense; here $\bar{c}_k$ is the mod-2 reduction of the k-th Chern class. For $n \geqslant 4$, the Burnside ring global functor $\mathbb{A} = \mathbf{A}(e, -)$ [18, Example 4.2.8 (i)] is a witness that there is no natural splitting, that is, the map

$$\mathrm{res}^{SU(n)}_{SU(n-1)} \; : \; \mathbb{A}(SU(n)) \longrightarrow \mathbb{A}(SU(n-1))$$

is not surjective. A specific element that is not in the image can be obtained as follows. The *reduced natural representation* of the alternating group A_n is the $(n-1)$-dimensional complex vector space

$$\{(x_1, \ldots, x_n) \in \mathbb{C}^n \; : \; x_1 + \cdots + x_n = 0\}$$

with A_n-action by permutation of coordinates. This action is faithful and by isometries of determinant 1. A choice of orthonormal basis identifies A_n with a subgroup of $SU(n-1)$, well-defined up to conjugacy. For $n \geqslant 4$, the Weyl group of A_n in $SU(n-1)$ is finite, and the transfer $\mathrm{tr}^{SU(n-1)}_{A_n}(1)$ is an element of infinite order in $\mathbb{A}(SU(n-1))$ that is not the restriction of any class in $\mathbb{A}(SU(n))$.

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