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Exercises for **Topology I**
Sheet 11

You can obtain up to 0 points per exercise (plus bonus points, where applicable).

Exercise 1. Let $Y \subseteq X$ be a closed subspace such that the inclusion $i\colon Y \hookrightarrow X$ enjoys the homotopy extension property and admits a retraction (i.e. there exists a continuous map $r\colon X \to Y$ such that $ri = \mathrm{id}_Y$).

Construct an isomorphism $H_n(X, A) \cong H_n(Y, A) \oplus H_n(X/Y, A)$ for every coefficient group A and every $n > 0$. What happens for $n = 0$?

Exercise 2. The *Klein bottle* K is the space obtained from $[0, 1] \times [0, 1]$ by dividing out the equivalence relation generated by $(s, 0) \sim (s, 1)$ and $(1, t) \sim (0, 1 - t)$ for all $s, t \in [0, 1]$.

Describe a CW structure on K and compute $H_*(K, \mathbb{Z})$ and $H_*(K, \mathbb{Z}/2)$ via the corresponding cellular chain complex.

Exercise 3. Let X be a topological space and let $Y \subseteq X$ be a closed subspace. In the lecture we saw that the map

$$H_n(X, Y; A) \to H_n(X/Y, Y/Y; A) \tag{$*$}$$

induced by the collapse map is an isomorphism for every $n \geq 0$ and every abelian group A *provided that* (X, Y) *enjoys the homotopy extension property.* In this exercise we will consider an example showing that $(*)$ is not an isomorphism without this assumption in general.

1. Let $X = [0, 1]$. Show that $Y := \{0\} \cup \{n^{-1} : n \geq 1\}$ is a closed subspace of X.
2. Construct a surjective homomorphism from $\pi_1(X/Y, [0])$ to $\prod_{\mathbb{N}} \mathbb{Z}$.
3. Conclude that $H_1(X/Y, Y/Y; \mathbb{Z})$ is uncountable.
4. Show that $H_1(X, Y; \mathbb{Z})$ is free abelian with countable basis, and use this to show that $H_1(X, Y; \mathbb{Z}) \not\cong H_1(X/Y, Y/Y; \mathbb{Z})$.

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Exercise 4. Let $p\colon E \to X$ be a finite covering map, i.e. a covering map such that all fibers $p^{-1}(x)$ are finite. In this exercise we will define a 'wrong way' *transfer map* $p^!\colon H_n(X,A) \to H_n(E,A)$ on homology with coefficients in any abelian group A.

1. Let $\sigma\colon \nabla^n \to X$ be continuous. We define $p^!(\sigma) := \sum_{\bar\sigma\colon \nabla^n \to E} \bar\sigma$ where the sum runs over all lifts of σ, i.e. all $\bar\sigma\colon \nabla^n \to E$ with $p \circ \bar\sigma = \sigma$.

   Extend this assignment to a chain map $p^!\colon C(\mathcal{S}(X),A) \to C(\mathcal{S}(E),A)$ natural in the group A. We will denote the associated map $H_n(X,A) \to H_n(E,A)$ on homology groups again by $p^!$.

2. Let $q\colon X \to Y$ be another finite covering. Show that $qp\colon E \to Y$ is again a finite covering and that $(qp)^! = p^! \circ q^!$ as maps $H_n(Y,A) \to H_n(E,A)$.

3. Assume now that X is connected, so that the cardinality $n := |p^{-1}(x)|$ of the fiber of $p\colon E \to X$ is independent of the choice of $x \in X$. Show that the composite

$$H_n(X,A) \xrightarrow{p^!} H_n(E,A) \xrightarrow{p_*} H_n(X,A)$$

   of the transfer map with the map induced by the usual functoriality of homology is given by multiplication by n.

4. Assume now that $n = 2$. Show that we have a short exact sequence of chain complexes

$$0 \longrightarrow C(\mathcal{S}(X),\mathbb{Z}/2) \xrightarrow{p^!} C(\mathcal{S}(E),\mathbb{Z}/2) \xrightarrow{p_*} C(\mathcal{S}(X),\mathbb{Z}/2) \longrightarrow 0.$$

∗5. (*0 bonus points*) Let $m \geq 1$ and let $f\colon S^m \to S^m$ be an odd map (i.e. $f(-x) = -f(x)$ for all $x \in S^m$). Show that f has odd degree.

   Hint. Specialize the previous exercise to the 2-sheeted covering $S^m \to \mathbb{R}\mathrm{P}^m$ and consider the associated long exact sequence in homology.