Algebraic Geometry II

Exercise Sheet 4

Due Date: 19.05.2014

Exercise 1:

Let Y be a noetherian scheme and let $f : X \to Y$ be a morphism of finite type. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be line bundles on X and let $\mathscr{M}$ be a line bundle on Y.

(i) Assume that $\mathscr{L}_1$ and $\mathscr{L}$ are ample over Y. Show that $\mathscr{L}_1 \otimes_{\mathcal{O}_X} \mathscr{L}_2$ is ample over Y.

(ii) Assume that $\mathscr{L}_1$ is (very) ample over Y and that there exists a surjection $f^*\mathscr{G} \twoheadrightarrow \mathscr{L}_2$ for some coherent sheaf $\mathscr{G}$ on Y. Show that $\mathscr{L}_1 \otimes_{\mathcal{O}_X} \mathscr{L}_2$ is (very) ample over Y.

(iii) Assume that $\mathscr{L}_1$ is ample over Y. Show that that $\mathscr{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathscr{L}_2$ is ample over Y for $n \gg 0$.

(iv) Assume that $\mathscr{L}_1$ is (very) ample over Y. Show that $\mathscr{L}_1 \otimes_{\mathcal{O}_X} f^*\mathscr{M}$ is (very) ample over Y.

(v) Assume that $\mathscr{L}_1$ is ample over Y. Show that there exists $n > 0$ such that $\mathscr{L}_1^{\otimes m}$ is very ample for all $m \geq n$.

Exercise 2:

(i) For $m, n \in \mathbb{Z}$ consider the line bundle $\mathscr{L}_{(m,n)} = \mathrm{pr}_1^*\mathcal{O}(m) \otimes_{\mathcal{O}_X} \mathrm{pr}_2^*\mathcal{O}(n)$ on $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$. Show that $\mathscr{L}_{(m,n)}$ is very ample if and only if $\mathscr{L}_{(m,n)}$ is ample if and only if $m, n > 0$.

(ii) Let $Y = V_+(T_2^2T_3 - (T_1^3 - T_1T_3^2)) \subset \mathbb{P}^2_k$. Let $P = (0:1:0) \in \mathbb{P}^2_k$ and let $\mathscr{I}_P \subset \mathcal{O}_X$ be the sheaf of ideals corresponding to the closed subscheme $\{P\}$ with the reduced scheme structure. Show that $\mathscr{I}_P$ is a line bundle and let $\mathscr{L} = \mathscr{I}_P^\vee$. Show that $\mathscr{L}^{\otimes 3} \cong \mathcal{O}_{\mathbb{P}^2_k}(1)|_X$ but that $\mathscr{L}$ is not generated by global sections.
(*This shows that* $\mathscr{L}$ *is ample but not very ample*)

Exercise 3:

Let k be an algebraically closed field and let X be a proper k-scheme. Let $\mathscr{L}$ be a line bundle on X and let $\varphi : \mathcal{O}_X^{n+1} \to \mathscr{L}$ be a surjection corresponding to a morphism $g : X \to \mathbb{P}^n_k$. Let $V \subset \Gamma(X, \mathscr{L})$ denote the sub-k-vectorspace generated by the images of the standard basis of $\mathcal{O}_X^{n+1}$. Assume that

(a) for any two closed points $x \neq y \in X$ there exists $s \in V$ such that $0 = s(x) \in \mathscr{L} \otimes \kappa(x)$ and $0 \neq s(y) \in \mathscr{L} \otimes \kappa(y)$ (or vice versa).

(b) for any closed point $x \in X$ the set $\{s_x \bmod \mathfrak{m}_x^2\mathscr{L}_x \mid s \in V,\ s_x \in \mathfrak{m}_x\mathscr{L}_x\}$ spans the $\kappa(x)$-vector space $\mathfrak{m}_x\mathscr{L}_x/\mathfrak{m}_x^2\mathscr{L}_x$, where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal.

Show that g is a closed immersion (especially $\mathscr{L}$ is very ample).
(*Hint: In order to show that* $\mathcal{O}_{\mathbb{P}^n_k, g(x)} \to \mathcal{O}_{X,x}$ *is surjective for all closed points* $x \in X$, *show that a local homomorphism of local rings* $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ *is surjective, if it induces an isomorphism on residue fields, it is finite (i.e. B is finitely generated as an A-module) and the canonical morphism* $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ *is surjective.*)

Exercise 4:

(i) Let $m \geq 2$ and $L = V_+(T_1, \dots, T_m) = \{(1 : 0 : \cdots : 0)\} \subset \mathbb{P}_k^m$. Show that there is a linear projection $\pi_L : X = \mathbb{P}_k^m \backslash L \to \mathbb{P}_k^{m-1}$ given by the morphism $\bigoplus_{i=1}^m \mathcal{O}_X e_i \to \mathcal{O}(1)|_X$ mapping the basis vector e_i to $T_i \in \Gamma(X, \mathcal{O}(1)) \subset \Gamma(\mathbb{P}_k^m, \mathcal{O}(1))$. Show further that $\mathbb{P}_k^{m-1} \cong V_+(T_0)$ is a section of the morphism π_L.

(ii) Let $m > n \geq 1$. Generalize (i) to describe an arbitrary linear projection $\mathbb{P}_k^m \backslash L \to \mathbb{P}_k^n$ where $L \subset \mathbb{P}_k^m$ is a linear subspace of dimension $m - n - 1$.

(iii) Let X be a k-scheme of finite type and let $\mathscr{L}$ be a line bundle on X. Let $V \subset \Gamma(X, \mathscr{L})$ be a finite dimensional k-vector space such that the canonical morphism $\mathcal{O}_X \otimes_k V \to \mathscr{L}$ is surjective. Let $s_0, \dots, s_n$ resp. $t_0, \dots, t_m$ be generators of V inducing morphisms $f : X \to \mathbb{P}_k^n$ resp. $g : X \to \mathbb{P}_k^m$. Assume that $n \leq m$. Show that there is a linear subspace $L \subset \mathbb{P}_k^m$ of dimension $m-n-1$ such that (up to an automorphism of $\mathbb{P}_k^n$) the morphism f can be written as the composition of g with a linear projection $\mathbb{P}_k^m \backslash L \to \mathbb{P}_k^n$.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeomII`