Dr. E. Hellmann WS 2013/14

**Algebraic Geometry I**

**Exercise Sheet 7**

**Due Date: 05.12.2013**

**Exercise 1:**

Let $(X, \mathcal{O}_X)$ be a prevariety over an algebraically closed field $k$. We construct a scheme associated with $X$ as follows. Let $t(X)$ be the set of all irreducible closed subsets of $X$. A subset of $t(X)$ is said to be closed if it coincides with $t(Y)$ for some closed subset $Y \subset X$.

(i) Show that the above induces a topology on $t(X)$.

(ii) Show that $\iota_X : x \mapsto \{x\}$ induces a homeomorphism of $X$ onto the set of closed points of $t(X)$.

We define a sheaf of $k$-algebras $\mathcal{O}_{t(X)}$ on $t(X)$ by setting $\mathcal{O}_{t(X)}(U) = \mathcal{O}_X(\iota_X^{-1}(U))$ for some open subset $U \subset t(X)$.

(iii) Assume that $X$ is affine and $\mathcal{O}_X(X) = A$. Show that $(t(X), \mathcal{O}_{t(X)})$ is a locally ringed space that is isomorphic to $\operatorname{Spec} A$.

(iv) Show that $t(X)$ is a scheme and that $x \in t(X)$ is a closed point if and only if $\kappa(x) = k$.

(v) Let $f : X \to Y$ be a morphism of prevarieties. Show that $f$ induces a natural map of schemes $t(f) : t(X) \to t(Y)$ and that the induced map

$$\{\text{morphisms } X \to Y \text{ of prevarieties}\} \longrightarrow \{\text{morphisms } t(X) \to t(Y) \text{ of } k\text{-schemes}\}$$

is bijective.

(Hint: Treat the affine case first.)

*We will see later that the essential image of the functor $t$ is the category of integral schemes of finite type over $k$.*

**Exercise 2:**

(i) Let $X$ be a scheme and $A$ a commutative ring. Show that the canonical map

$$\operatorname{Hom}_{\text{schemes}}(X, \operatorname{Spec} A) \longrightarrow \operatorname{Hom}_{\text{rings}}(A, \mathcal{O}_X(X))$$

is bijective.

(ii) Let $R$ be a base ring. Let $X$ be an $R$-scheme and $A$ be an $R$-algebra. Show that there is a canonical bijection

$$\operatorname{Hom}_R(X, \operatorname{Spec} A) \longleftrightarrow \operatorname{Hom}_{R-\text{alg}}(A, \mathcal{O}_X(X)).$$

(iii) Show that the category of schemes has a final and an initial object.

**Exercise 3:**

Let $\{X_i, i \in I\}$ be a family of schemes. For $i, j \in I$ with $i \neq j$ suppose given an open subset $U_{ij} \subset X_i$ with the induced scheme structure and isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ such that $\varphi_{ij}^{-1} = \varphi_{ji}$ and such that for $i, j, k \in I$ pairwise distinct one has $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.

Show that there is a scheme $X$ together with open embeddings $\psi_i : X_i \to X$ such that $X = \bigcup_{i \in I} \psi_i(X_i)$ and $\psi_i(X_i) \cap \psi_j(X_j) = \psi_i(U_{ij})$ as well as $\psi_i|_{U_{ij}} = \psi_j \circ \varphi_{ij}$.

**Exercise 4:**

Let $k$ be a field and let $(a_i)_{i \in \mathbb{N}}$ be a family of pairwise distinct elements of $k$. Set

$$A = k[U, T_1, T_2, \dots]/((U - a_i)T_{i+1} - T_i, T_i^2).$$

Describe $\operatorname{Spec} A$ as a topological space and show that $A_{\mathfrak{p}}$ is noetherian for every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. Show that $A$ is not noetherian.

(Hint: Show that the nilradical of $A$ is not finitely generated.)

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeom`