Dr. E. Hellmann WS 2013/14

Algebraic Geometry I

Exercise Sheet 1

Due Date: 24.10.2013

Throughout the whole exercise sheet k denotes an algebraically closed field.

Exercise 1:
Let $X \subset \mathbb{A}^n$ be an affine algebraic set.

(i) Let $f \in \mathcal{O}(X) = k[T_1, \dots, T_n]/I(X)$. Show that the set $D(f) = \{x \in X \mid f(x) \neq 0\} \subset X$ is open with respect to the Zariski topology.

(ii) Show that the sets $D(f)$, $f \in \mathcal{O}(X)$ form a basis of the topology on X.

(iii) Given $f \in \mathcal{O}(X)$, show that the map $f : X \to k$ defined by f is continuous with respect to the Zariski topology on X and k.

Exercise 2:

(i) Let $f \in k[T_1, \dots, T_n]$ be a polynomial and write $f = \prod_{i=1}^m f_i^{r_i}$ with pairwise distinct irreducible polynomials f_i and $r_i \geq 1$. Show that $I(V(f)) = (\prod_{i=1}^m f_i)$ and that $V(f)$ has precisely m irreducible components given by the $V(f_i)$.

(ii) Consider the affine algebraic set $X = V(T_1^2 - T_2, T_1T_3 - T_1) \subset \mathbb{A}^3$. Determine the irreducible components of X.

(iii) Let $Y = \{(t, t^2, t^3) \mid t \in k\} \subset \mathbb{A}^3$. Show that Y is an affine algebraic sets and find generators for $I(Y) \subset k[T_1, T_2, T_3]$. Show that $\mathcal{O}(Y) \cong k[T]$.

Exercise 3:
Let n, m be positive integers.

(i) Show that the Zariski topology on $\mathbb{A}^{n+m}$ is strictly finer than the product topology on $\mathbb{A}^n \times \mathbb{A}^m$.

(ii) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be irreducible affine algebraic sets. Show that the product $X \times Y \subset \mathbb{A}^{n+m}$ is an affine algebraic set which is irreducible (with respect to the subspace topology induced by the Zariski topology on $\mathbb{A}^{n+m}$).
(Hint: If $X \times Y = Z_1 \cup Z_2$, show that $X_i = \{x \in X \mid \{x\} \times Y \subset Z_i\}$ is closed in X.)

Exercise 4:

Let X be a non-empty topological space.

(i) Show that the following are equivalent:

(a) The space X is irreducible.

(b) Every non-empty open subset $U \subset X$ is dense.

(c) Every non-empty open subset $U \subset X$ is irreducible.

(c) Every non-empty open subset $U \subset X$ is connected.

(ii) Show that a subset $Y \subset X$ is irreducible if and only if its closure $\bar{Y} \subset X$ is irreducible.

(iii) Let $U \subset X$ be an open subset. Show that $Z \mapsto Z \cap U$ induces a bijection

$$\{Z \subset X \ \text{ closed irreducible with } Z \cap U \neq \emptyset\} \longleftrightarrow \{Y \subset U \text{ closed irreducible}\}.$$

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeom`