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List of talks for the seminar

Local Langlands Correspondence for GL(2)

— summer term 2008, Di 14–16, SR B —

All §x.y without further reference refer to [BH06].

1. **Smooth representations.** Locally profinite groups, examples §1.1-4; characters, level, additive duality §1.6+7, characters of the multiplicative group §1.6-8.

   Smooth representations, admisible representations §2.1; example $\mathrm{GL}_2(F)$: a smooth irreducible representation has dimension 1 or ∞, characters factor over det §9.2.

   Semisimple representations, smooth representations of profinite groups are semisimple §2.2; properties of the decomposition according to isotypic components of an open compact subgroup §2.3.

2. **Induction and Frobenius reciprocity.** Smooth vectors, the functor 'smooth vectors' is right adjoint to the inclusion of smooth representations in all representations §2.3 exercise; induction, Frobenius reciprocity, compact induction, compact Frobenius reciprocity, exactness of (compact) induction §2.4-5.

   Going up and down of semisimplicity with respect to open subgroups §2.7.

   Hypothesis: countable modulo compact open subgroup, satisfied by $\mathrm{GL}_2(F)$ §2.6, §7.2; Schur's Lemma, central character, smooth representations of the discrete group $\mathbb{Z}$ §2.6.

   Contragredient representation §2.8-10.

3. **The Weil group of a local field and its representations.** Absolute Galois groups Gal_K of fields §28.1; geometric/arithmetic Frobenius and Galois group of finite fields, Galois group of a local field, inertia subgroup §28.2, [Se79] I§8, IV§1+2; tame character, wild inertia subgroup §28.3.

   The Weil group W_F of a local field F, unramified characters $\| \|^s$ §28.4; Galois theory in terms of the Weil group §28.5; irreducible smooth representations of W_F are of finite dimension, comparison of the categories of smooth representations of Weil group and absolute Galois group §28.6.

4. **Weil group representations of dimension 2.** Review of local class field theory[1], local class field theory as Langlands correspondence for characters §29.1+2.

[1] The Artin reciprocity map is normalized such that a prime element maps to a geometric Frobenius — the negative of the classical reciprocity map.

Statement[2] of the Langlands Correspondence[3], in particular the correspondence is compatible with twists by characters which correspond under local class field theory §33.1; implication of the Converse Theorem[4] §33.1+2+4.

Admissible pairs §18.2 Definition; classification of 2-dimensional Weil representations: reducible, induced, tetrahedral or octahedral §34.1 Theorem, §42.3 Comment[5]; for $p \neq 2$ all irreducible representations of dimenson 2 are induced representations §34.1 Theorem; hint at automorphic induction.

5. **Representation theory of** $\mathrm{GL}_2(\mathbb{F}_q)$**.** Standard subgroups of GL_2: Borel B, split torus T, unipotente radical N of the Borel, center Z, §5.1; Bruhat decomposition §5.2; conjugacy classes of elements §5.3.

   Restriction-Induction Formula of Mackey [Se77] II.7.4; irreducible representations of the principal series[6], Steinberg representation §6.2-3; non-split tori in GL_2, cuspidal representations associated to a non-split torus with character[7] §6.4; principal series and cuspidal representations exhaust the list of all irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ §6.1+4, [Se77] I.1-3.

6. **Chain orders in** $\mathrm{M}_2(F)$ **and characters.** Lattices §1.5; lattice chaines in $F \oplus F$ §12.1; chain orders in $\mathrm{M}_2(F)$ §12.2; congruence subgroups in $\mathrm{GL}_2(F)$ associated to a lattice chain §12.3; fractional ideals of a quadratic field extension as a lattice chain §12.4.

   Trace form and characters of $\mathrm{M}_2(F)$, logarithm and characters on congruence subgroups of $\mathrm{GL}_2(F)$ §12.5.

7. **Automorphic induction I: case of level** 0**.** Intertwining §11.1 without Prop 2; groups which are compact modulo center and their representations that admit a central character §2.7 Proposition; intertwining criterion for irreducibility of compactly induced representations[8] §11.4.

   Construction of automorphic induction[9]π_χ for admissible pairs $(E/F, \chi)$ of level $\ell(\chi) = 0$ §19.1, §11.5 Lemma; normalized level §12.6; tame parametrization for level $\ell(\pi) = 0$ §19.1 Proposition[10], §11.5 Theorem.

8. **Fundamental strata.** Strata, intertwining of strata §12.7; fundamental strata §12.8; fundamental strata exist only for level $\ell(\pi) > 0$ and coincide with strata of smallest level §12.9.

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[2] We ignore definitions of L-functions and local ε-factors, and give no details on Weil-Deligne representations.

[3] In the seminar we focus on the following: for a local field F of residue characteristic $p \neq 2$ irreducible Weil representations of dimension 2 are naturally in bijection with irreducible cuspidal representations of $\mathrm{GL}_2(F)$.

[4] The Langlands Correspondence is unique and compatible with the dichotomies *reducible versus irreducible* on the Galois side and *principal series versus cuspidal*. Again, we ignore definitions of *principal series* and *cuspidal*. We use the ad hoc definition that cuspidal representations are those we construct lateron.

[5] The proof can be improved to add some insight for the case $p = 2$, an irreducible representation is induced or the corresponding projective representation has finite image in $\mathrm{PGL}_2(\mathbb{C})$, hence is cyclic (reducible), dihedral (induced) or A_4 (tetrahedral), S_4 (octahedral) or S_5 (never).

[6] Replace §6.3 Lemma (2) for the following: π as representation of N admits the trivial character as *quotient*.

[7] It would be nice to see a short exact sequence that defines π_θ.

[8] Here we avoid duality and the Hecke algebra. Replace §11.4 Theorem by (1) $X = \text{c-Ind}_K^G(\rho)$ is irreducible if and only if $\dim_{\mathbb{C}} \mathrm{End}_G X = 1$ (2) Mackey formula $\text{c-Ind}_K^G(\rho)|_K = \bigoplus_{g \in K\backslash G/K} \mathrm{Ind}_{K \cap K^g}^G(\rho^g)$ (3) $End_G(X) = \mathrm{Hom}_K(\rho, X|_K) = \bigoplus_{g \in K\backslash G/K} \mathrm{Hom}_{K \cap K^g}(\rho, \rho^g)$.

[9] Only true up to a twist.

[10] For most of the seminar we make no effort to discuss the dichotomy of principal series representations versus cuspidal representations. Hence we believe §14.3 Proposition, i.e., we raise the result of Theorem 11.5 to a definition of cuspidal (1) and principal series (2) for representations of level 0; and for §15.5 Corollary refer to talk 12.

Classify irreducible representations according to level: $\ell(\pi) = 0$ or $\ell(\pi) > 0$ and in the latter case the kind of fundamental strata it contains: (a) ramified simple, (b) unramified simple, (c) split, (d) essentially scalar §13.1+2.

Criterion for minimal level among all twists §13.3.

9. **Simple strata: minimal elements and intertwining properties.** Minimal elements in $\mathrm{GL}_2(F)$ respectively in quadratic extensions E/F §13.4; bijection between simple strata and minimal elements of $\mathrm{GL}_2(F)$ and up to conjugation[11] also minimal admissible pairs §13.4+5, §18.2 Proposition.

   Intertwining of simple strata: Intertwining and Conjugacy Theorem §15.1+2, §16.1-3.

10. **Cuspidal types.** Definition of cuspidal types §15.5 Definition; irreducibility of the compactly induced representation associated to a cuspidal type[12] §15.3.

    Group theory of representations occuring in cuspidal types[13] §15.6+7, §16.4; cuspidal inducing data §15.8+9.

11. **Automorphic induction II: case of level** > 0**.** Automorphic induction[9] for minimal admissible pairs: odd level[14] §19.2+3, §15.6 Proposition 1; even level[14] §19.2+4, §15.6 Proposition 2, §16.4 Lemma, §22.1+2+4. General case of automorphic induction[9] §19.5+6.

12. **Tame parametrization.** Classification Theorem for cuspidal representations[15] §15.4+5.

    Unramified irreducible representations of $\mathrm{GL}_2(F)$ §20.1; The Tame Parametrization Theorem in the case $p \neq 2$, compatibility of automorphic induction[9] with twists and contragredient, description of level and central character §20.2, §29.2; proof of the Tame Parametrization Theorem §20.3, §21.1-4.

Literatur

[BH06] Bushnell, C. J., Henniart, G., *The local Langlands Conjecture for* GL(2), Grundlehren der Math. Wissenschaften **335**, Springer, 2006.

[Ha02] Harris, M., On the Local Langlands Correspondence, in *ICM 2002, vol III*, arXiv:math.NT/0304324v1, April 2003.

[Se77] Serre, J.-P., *Linear representations of finite groups*, translated from the French edition *Représentations linéaires des groupes finis*, GTM **42**, Springer, 1977.

[Se79] Serre, J.-P., *Local Fields*, translated from the French edition *Corps locaux*, GTM **67**, Springer, 1979.

[We00] Wedhorn, T., The local Langlands correpondence for GL(n) over p-adic fields, arXiv:math.AG/0011210v2, 25 Nov 2000.

[11] In particular this requires a proof, that a quadratic extension E/F has up to conjugation a unique F-linear embedding into $\mathrm{M}_2(F)$.

[12] That Λ is of finite dimension follows from §2.7 Proposition and should be mentioned.

[13] Some effort is required in order to present the material more conceptually.

[14] §15.6 has been provided by talk 10.

[15] A cuspidal representation for us is still a representation of level 0 and type (1) in Theorem §11.5 or of level > 0 and obtained as a compactly induced representation from a cuspidal type.