ARITHMETISCHE GEOMETRIE OBERSEMINAR

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TOPIC: TORSION IN THE COHOMOLOGY OF LOCALLY SYMMETRIC VARIETIES

This semester, we want to go through the manuscript [8] constructing Galois representations associated to Hecke eigenvalue systems appearing in the mod p cohomology of locally symmetric varieties.

Recall the global Langlands conjecture over number fields.

Conjecture 1. *Let F be a number field, let p be some prime number, and fix an isomorphism $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_p$. Then there is a bijection between the set of algebraic cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, and the set of isomorphism classes of irreducible continuous representations of the absolute Galois group of F on n-dimensional $\overline{\mathbb{Q}}_p$-vector spaces which are almost everywhere unramified, and de Rham at places above p. Under this bijection, eigenvalues of Hecke operators agree with traces of Frobenius elements.*

In the seminar, we will look at (one direction of) a 'mod p' version of this correspondence. Recall the locally symmetric varieties

$$X_K = \mathrm{GL}_n(F)\backslash(\mathrm{GL}_n(F\otimes\mathbb{R})/K_\infty\mathbb{R}_{>0} \times \mathrm{GL}_n(\mathbb{A}_{F,f})/K)\ ,$$

where $K_\infty \subset \mathrm{GL}_n(F\otimes\mathbb{R})$ is a maximal compact subgroup, and $K \subset \mathrm{GL}_n(\mathbb{A}_{F,f})$ is a (sufficiently small) compact open subgroup. If $F = \mathbb{Q}$ and $n = 2$, these varieties carry a natural complex structure, and in fact are the $\mathbb{C}$-valued points of an algebraic variety over $\mathbb{Q}$, called the modular curve. In general, they are just real manifolds.

By a theorem of Franke, [5], one can express the singular cohomology groups

$$H^i(X_K, \mathbb{C})\ ,$$

with their action of the Hecke operators, in terms of automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$; in fact, only (a subset of the) algebraic representations appear. Thus, one expects to have Galois representations associated to Hecke eigenvalue systems appearing in $H^i(X_K,\mathbb{C})$, or more canonically to $H^i(X_K,\overline{\mathbb{Q}}_p)$.

Now, one can also look at the integral singular cohomology groups $H^i(X_K,\overline{\mathbb{Z}}_p)$. If $F=\mathbb{Q}$, $n=2$, these groups are torsion-free. However, computations show that in general, these groups have a large amount of p-torsion, so the dimension of $H^i(X_K,\overline{\mathbb{F}}_p)$ may be much larger than the dimension of $H^i(X_K,\overline{\mathbb{Q}}_p)$, and not every system of Hecke eigenvalues mod p lifts to characteristic 0. Nonetheless, the associated Galois representations are conjectured to exist.

Conjecture 2. *For any system of Hecke eigenvalues appearing in $H^i(X_K,\overline{\mathbb{F}}_p)$, there is a continuous semisimple representation of the absolute Galois group of F on an n-dimensional $\overline{\mathbb{F}}_p$-vector space, such that the eigenvalues of the Hecke operators agree with the traces of Frobenius elements.*[1]

Conjectures in this direction were first made by Grunewald, and then more precisely by Ash, [1], (for $\mathrm{GL}_n/\mathbb{Q}$, $n\geq 3$) and Figueiredo, [4], (for GL_2 over an imaginary-quadratic number field). Very recently, it was formulated by Calegari and Geragthy, [2], where they prove modularity lifting theorems over general fields conditional on (a strengthening of) this conjecture. Our aim in this seminar is to prove this conjecture when F is totally real or CM.

We note that recently, Harris-Lan-Taylor-Thorne, [6], have proved that Galois representations exist for all Hecke eigenvalues appearing in $H^i(X_K,\overline{\mathbb{Q}}_p)$, if F is totally real or

[1] There is also a conjectural converse to Conjecture 2, saying that all (irreducible, odd) Galois representations mod p arise in this way. This is the immediate generalization of Serre's conjecture.

CM. Our method follows theirs in that we realize the cohomology of X_K in the boundary of a $GU(n,n)$-Shimura variety (in case F is CM), and then make a p-adic deformation to a cusp form on $GU(n,n)$. For cusp forms on $GU(n,n)$, it is known that the Galois representations exist. (Roughly, one can move them to $GU(1,2n-1)$, where one finds them in the corresponding Shimura variety.) However, the cohomology theories we use, as well as the way the p-adic deformation argument works, are very different from their case. One important novelty of our approach is that the ordinary locus does not play a special role in the deformation argument anymore.

Technically, our approach relies on realizing (the minimal compactifications of) Shimura varieties with infinite level at p as perfectoid spaces, and the Hodge-Tate period map defined there. In fact, for most of the semester, we will be occupied with proving these results on Shimura varieties. Unless otherwise specified, all talks rely on [8].

0. Talk: Introduction

Explanation of main results.

1. Talk: A Hebbarkeitssatz for perfectoid spaces

The goal of this talk is to prove a version of Riemann's Hebbarkeitssatz for perfectoid spaces. However, most of this talk should be devoted to reminders on perfectoid spaces ([9], [10]).

2. Talk: p-adic Hodge theory and the Hodge-Tate filtration

Give some reminders on p-adic Hodge theory. The key results needed are those contained in [10], Sections 3.2, 3.3, 4.2, i.e.: The comparison theorem for constructible coefficients, [10, Theorem 3.13] (in the absolute case, in fact Lemma 3.16 is enough), the existence of the Hodge-Tate filtration, [10, Theorem 3.20], and the comparison with Fargues's Hodge-Tate filtration for p-divisible groups, [10, Proposition 4.15]. [This talk may well take 2 sessions, and might even be split among two students.]

3. Talk: The canonical subgroup

Prove the existence of the canonical subgroup over an arbitrary formal scheme whenever the Hasse invariant is sufficiently small. This relies on Illusie's deformation theory for finite locally free group schemes, cf. [7, Section 3].

4. Talk: A strict neighborhood of the (étale-)ordinary locus in the $\Gamma_0(p^\infty)$-level Siegel moduli space is perfectoid

Use the canonical subgroup to define canonical Frobenius lifts on formal models for a strict neighborhood of the ordinary locus. Show that the tower of these Frobenius lifts can be considered as (a formal model of) an open subset of the $\Gamma_0(p^\infty)$-level tower, and deduce the result stated in the title of the talk. Moreover, describe the tilt of the strict neighborhood of the étale-ordinary locus at $\Gamma_0(p^\infty)$-level as its obvious characteristic p-analogue.

5. Talk: Tate's normalized traces

Show that Tate's normalized traces exist on the $\Gamma_0(p^\infty)$-tower constructed in the previous talk.

6. Talk: Extension to the minimal compactification

Extend the results of the fourth talk to the minimal compactification of the Siegel moduli space. In particular, give the necessary reminders on this compactification, as constructed by Faltings-Chai, [3], and prove a rigid-analytic version of Hartogs's extension theorem. Moreover, prove that Riemann's Hebbarkeitssatz holds with respect to the boundary.

7. Talk: The $\Gamma(p^\infty)$-level Siegel moduli space above a strict neighborhood of the étale-ordinary locus

Deduce that one can go to $\Gamma(p^\infty)$-level at the locus considered in the previous talks. Do this in two steps: First, go to $\Gamma_1(p^\infty)$-level by arguing first on the tilt in characteristic p, and tilting back to characteristic 0. Second, go to $\Gamma(p^\infty)$-level by a direct application of the almost purity theorem. Show that Riemann's Hebbarkeitssatz stays true at $\Gamma(p^\infty)$-level, and deduce the existence of the Hodge-Tate period map.

8. Talk: Conclusion in Siegel case, and extension to Shimura varieties of Hodge type

Prove (using the Hodge-Tate periods) that any abelian variety is isogenous to one in a given strict neighborhood of the ordinary locus. Deduce that the whole Siegel moduli space is perfectoid at $\Gamma(p^\infty)$-level, and that the Hodge-Tate period map exists on it.

Recall the notion of a Shimura variety of Hodge type, and deduce all results in that generality.

9. Talk: The deformation argument

Use the results about Shimura varieties to get applications on the torsion in the cohomology of locally symmetric varieties. In particular, explain how one gets analogues of the Hasse invariant (which work outside the ordinary locus) from the Hodge-Tate period map.

References

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