ARithmetische Geometrie OberSeminar

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Modular representations of reductive groups

For a reductive group G over a field of characteristic 0, the representation theory of G is well-understood: The category of representations is semisimple, the irreducible representations are enumerated by highest weights, and the highest weight representations V_λ admit an explicit description as $\Gamma(\mathrm{Fl}, \mathcal{O}(\lambda))$, where $\mathrm{Fl} = G/B$ is the flag variety and $\mathcal{O}(\lambda)$ the line bundle on G/B corresponding to the character λ of T (thus of B); moreover, one can describe the character of V_λ explicitly by the Weyl character formula.

For reductive groups G over a field k of positive characteristic p, the situation is much more involved; in particular, the category of representations is no longer semisimple, and lacks injective and projective objects. The irreducible objects L_λ are still enumerated by highest weights λ, given now by the irreducible socles of $\nabla_\lambda = \Gamma(\mathrm{Fl}, \mathcal{O}(\lambda))$. An immediate question is how to describe the irreducible objects explicitly, in particular to describe their character, or equivalently to write

$$[L_\lambda] = \sum_\mu a_{\mu,\lambda}[\nabla_\mu]$$

in the Grothendieck group of representations for explicit $a_{\mu,\lambda} \in \mathbb{Z}$. The central conjecture in the area is Lusztig's conjecture [9], predicting that $a_{\mu,\lambda}$ is given by the values at 1 of Kazhdan–Lusztig polynomials. While for given G, this conjecture is true for $p \gg 0$, it fails for exponentially large p.

Different proofs of Lusztig's conjecture (and disproofs for small p, and proofs of corrected variants) have appeared over the years. Note that the values at 1 of the Kazhdan–Lusztig polynomials can be understood as the dimensions of stalks of intersection complexes on affine Schubert varieties. This makes it natural to approach Lusztig's conjecture via the geometric Satake equivalence, which describes the category of representations of G as the category of equivariant perverse sheaves on the affine Grassmannian. We will study a recent proof of Lusztig's conjecture due to Riche–Williamson [12] that proceeds along these lines, using a variant of the geometric Satake equivalence employing Iwahori–Whittaker equivariance.

Talks

Talk 1: Modular representations

Following [15, Section 1], [4], recall basics of the representation theory of reductive groups in characteristic p. In particular, recall the construction of induced and Weyl representations, the classification of irreducible representation [15, Theorem 1.1], the Weyl character formula [4, Theorem 6.2], the Steinberg decomposition theorem [4, Theorem 7.6], and tilting modules [15, Section 1.5] (if time permits, explain the proof of the required vanishing [15, Eq. (1.8)] as in [5, Section 3], which relies on Kempf vanishing, as quickly proved in [4, Section 6]; and the construction of tilting modules [11, Section 7]). Moreover, explain the linkage principle and translation functors [15, Section 1.7, 1.9 – 1.11] [4, (part of) Section 9, Section 10], thus largely reducing the representation theory of G to the principal block.

Talk 2: Lusztig's conjecture

Still following [15], [4], state Lusztig's conjecture on the character of simple representations. In particular, recall the definition of Kazhdan–Lusztig polynomials [15, Section 1.8], and then state Lusztig's conjecture [15, Section 1.12 – 1.13, especially Conjecture 1.20]. Then discuss the approach to Lusztig's conjecture via geometric Satake and the Finkelberg–Mirkovic conjecture [4, Section 12] [15, Sections 2.1, 2.2, 2.4, 2.5]; in particular, recall the relation of Kazhdan–Lusztig polynomials to intersection complexes on the affine flag variety [15, Theorem 2.2]. If time permits, explain the analogous Kazhdan–Lusztig conjecture (for Lie algebra representations in characteristic 0) and sketch its proof using Beilinson–Bernstein localization [4, Section 8].

Talk 3: Geometric Satake

The goal of this talk is to recall the geometric Satake equivalence of Mirkovic–Vilonen [10] and some of the surrounding geometry. A detailed survey, including the case of modular coefficients relevant to us, is [1]. It will not be possible to cover the whole proof, but do define the Satake category and the fibre functor, and using Mirkovic–Vilonen cycles and Braden's theorem on hyperbolic localization [3] (cf. also [13]) define an $X_*(T)$-grading on the fibre functor [10, Section 2, 3]. Define the convolution product and show that it preserves perverse sheaves [10, Section 4]. Omit (or only sketch briefly) the construction of the commutativity constraint [10, Section 5, 6]. Identify the dual group in characteristic 0 [10, Section 7] (cf. also [14]). Define the standard sheaves and study the weight functors [10, Section 8,9,10]. Conclude that the Satake category is a highest weight category in the sense of [11, Section 7], cf. [2, Lemma 3.1]. Omit (or only sketch very briefly) the proof of the geometric Satake equivalence with modular coefficients [10, Section 11,12].

Talk 4: Iwahori–Whittaker geometric Satake

The goal of this talk is to prove the Iwahori–Whittaker version of the geometric Satake equivalence following [2]. First define the Iwahori–Whittaker version of the Satake category [2, Section 3.2]. As preparation, cover [2, Section 2] and prove [2, Lemma 2.3] by recalling the construction of convolution via nearby cycles as in [6]. Formulate the main theorem [2, Theorem 3.9] and give its proof following [2, Section 3.4], first recalling the geometric Casselman–Shalika formula. If possible, explain how the geometric Casselman–Shalika formula follows from [2, Corollary 3.6, Remark 3.7 (1)].

Talk 5: Iwahori–Whittaker geometric Satake and parity sheaves

Recall the notion of parity sheaves [7], [8]. Following [2, Section 4] show that in the Iwahori–Whittaker category, parity sheaves and tilting sheaves agree [12, Eq. (5.2)]. Also prove that under geometric Satake parity sheaves give rise to tilting modules by passing to perverse cohomology sheaves [2, Theorem 4.10]. As applications, deduce that tilting modules are stable under tensor product and stable under restriction to Levi subgroups [2, Theorem 4.16].

Talk 6: Equivariant sheaves and Smith–Treumann theory

In this talk, we will start with the main paper [12]. The goal of this talk is to define the Smith categories [12, Section 3], giving background on the definition of equivariant sheaves [12, Section 2] (only) as required; in particular, discuss the ciritical [12, Proposition 2.6, Lemma 3.5]. Identify the Smith category of a point [12, Section 3.5].

Talk 7: Fixed points of roots of unity on the affine Grassmannian

In order to apply Smith–Treumann theory, we need to identify the fixed points of μ_ℓ acting by rotation on the affine Grassmannian. Follow [12, Section 4]; the central result is [12, Proposition 4.6]. As preparation, recall the partial affine flag varieties, associated to parahoric group schemes [12, Sections 4.2, 4.3].

Talk 8: Smith–Treumann theory for Iwahori–Whittaker sheaves

The goal of this talk is to cover [12, Section 5,6,7]. In Section 5, the essential new result is that all objects in the Iwahori–Whittaker version of the geometric Satake are $\mathbb{G}_m$-invariant for the rotation $\mathbb{G}_m$-action [12, Lemma 5.2] (see [12, Remark 8.4] for an alternative proof). In Section 6, the Smith–Treumann formalism is applied to the Iwahori–Whittaker Satake category. A central result is [12, Lemma 6.3]. In Section 7, the notion of parity objects is extended to the Smith–Treumann setting, eventually leading to the central [12, Theorem 7.4], which is the main result of this talk. Moreover, following [12, Section 7.4], pass from the Smith–Treumann setting back to usual sheaves on the fixed points.

Talk 9: Applications to representation theory

Following [12, Section 8], prove the applications to modular representation theory. First, give a proof of the linkage principle [12, Theorem 8.5]. Then deduce a character formula for tilting modules [12, Theorem 8.8]. For this, first define the p-Kazhdan–Lusztig polynomials, using parity sheaves on the affine flag variety. Deduce Lusztig's conjecture for large p following [12, Section 1.10].

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