Dictatorship theorems
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In this note I record proofs of the two most famous dictatorship theorems: by Arrow and by Gibbard and Satterthwaite. Their proofs can be of course found in many places, but I was dissatisfied with the length of those expositions that I could find.

1 Arrow's theorem

Theorem 1 (Arrow [Arr51]). *Let L be the set of all strict total orderings on a set $\mathscr{A} = \{a, b, c, \dots\}$ with $|\mathscr{A}| > 2$. Consider a map $L^N \to L, \vec{>} = (>_1, \dots, >_N) \mapsto >$ with the following properties:*

1. *independence of irrelevant alternatives*

$$\forall a, b, \vec{>}, \vec{>}' (\forall i (a >_i b \iff a >_i' b) \implies (a > b \iff a >' b)) \tag{IIA}$$

2. *and unanimity*

$$\forall a, b, \vec{>}, \vec{>}' (\forall i (a >_i b) \implies (a > b)) \tag{UA}$$

Then there exists i (the dictator) with $> = >_i$ for every argument order $(>_1, \dots, >_N)$.

Proof. For a set $A \subset \{1, \dots, N\}$ we write $a >_A b$ if $\forall i \in A (a >_i b)$. Fix $a, b \in \mathscr{A}$. By (UA) there exists a partition $\{1, \dots, N\} = A \cup \{i\} \cup B$ in which i decides between alternatives a and b in the sense

$$a >_{A \cup \{i\}} b, b >_B a \implies a > b, \tag{2}$$

$$a >_A b, b >_{B \cup \{i\}} a \implies b > a. \tag{3}$$

We claim that i is the dictator. To this end it suffices to show that for any pair of alternatives $\alpha \in \{a, b\}$, $c \in \mathscr{A} \setminus \{a, b\}$ we have

$$\alpha \lessgtr_i c \implies \alpha \lessgtr c;$$

the remaining cases follow by transitivity and (IIA).

Consider first an argument order with

$$a >_A c >_A b \wedge b >_{B \cup \{i\}} a >_{B \cup \{i\}} c.$$

Then by (UA) we have $a > c$ and by (3) and (IIA) we have $b > a$. Hence $b > c$ by transitivity, and (IIA) gives

$$c >_A b \wedge b >_{B \cup \{i\}} c \implies b > c \tag{4}$$

Consider now an argument order with

$$a, c >_A b \wedge a >_i b >_i c \wedge b >_B a, c.$$

Then by (4) and (IIA) we have $b > c$, while by (2) and (IIA) $a > b$. By transitivity it follows that $a > c$. Hence by (IIA) we obtain

$$a >_i c \implies a > c.$$

The above reasoning is symmetric in a, b and $<, >$, so we are done. □

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2 Gibbard–Satterthwaite theorem

A function $f : L^N \to \mathscr{A}$ is called *tactical voting proof* if

$$\forall \vec{>}, i, >'_i (f(\vec{>}/_i >'_i) \not>_i f(\vec{>})), \tag{TVP}$$

where $\vec{>}/_i >'_i$ denotes the element of L^N that coincides with $\vec{>}$ in coordinates $\neq i$ and equals $>'_i$ in coordinate i.

Lemma 5 (Monotonicity, [MS77]). *Suppose that f satisfies* (TVP). *Then for any $a \neq b \in \mathscr{A}$*

$$f(\vec{>}) = a \wedge \forall i(a >_i b \implies a >'_i b) \implies f(\vec{>}') \neq b. \tag{M}$$

The converse also holds, see [MS77].

Proof. Suppose for contradiction $f(\vec{>}') = b$. The assumption can be written

$$\forall i(b >_i a \vee a >'_i b).$$

We will show that this is absurd by induction on the size of the set $I = \{i : >'_i \neq >_i\}$. If $|I| = 0$, then we have $a = f(\vec{>}) = f(\vec{>}') = b$, a contradiction. Otherwise pick $i \in I$. Suppose first $a >'_i b$. Then $f(\vec{>}/_i >'_i) = a$, since otherwise the tuple $(\vec{>}, i, >'_i)$ witnesses failure of (TVP), and we have reduced to the case $(\vec{>}/_i >'_i, \vec{>}')$, which is absurd by inductive hypothesis. Similarly $b >_i a \implies f(\vec{>}'/ >_i) = b$. □

In particular any function with property (TVP) is Pareto efficient on its range, that is,

$$a \in \operatorname{ran} f \wedge \forall i(a >_i b) \implies f(\vec{>}) \neq b. \tag{PE}$$

Theorem 6 (Gibbard [Gib73], Satterthwaite [Sat75]). *Suppose that a surjective function f satisfies* (TVP) *and $2 < |\mathscr{A}| < \infty$. Then there exists i (the dictator) with $f(\vec{>}) = \max(\mathscr{A}, >_i)$ for every argument order $\vec{>}$.*

Gibbard [Gib73] actually proves a stronger statement which I will not discuss here.

Proof. We use f to construct a map $L^N \to L$ that satisfies (IIA) and (UA). Let an argument order $\vec{>}$ be given. We have to define a total ordering $\mathscr{A} = \{a_0 > a_1 > \cdots > a_{|\mathscr{A}|-1}\}$. Let $\vec{>}^{(0)} := \vec{>}$. Inductively set $a_n := f(\vec{>}^{(n)})$ and obtain $\vec{>}^{(n+1)}$ from $\vec{>}^{(n)}$ by moving a_n to the bottom of every individual choice, in symbols

$$\forall i \forall a, b \in \mathscr{A} \setminus \{a_n\}(a >_i^{(n+1)} b \iff a >_i^{(n)} b), \quad \forall i \forall a \in \mathscr{A} \setminus \{a_n\}(a >_i^{(n+1)} a_n).$$

Then the elements a_n are distinct by (PE). Property (UA) of the resulting map $\vec{>} \mapsto >$ follows from (PE) and property (IIA) from (M).

By Theorem 1 the map constructed above admits a dictator i, and it is clear that i satisfies the conclusion of the present theorem. □

References

[Arr51] K. J. Arrow. “Social Choice and Individual Values”. Cowles Commission Monograph No. 12. John Wiley & Sons, 1951, pp. xi+99.

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