Summer term 2024 Dr. Andreas Mihatsch
Elliptic curves and their moduli spaces

Homework problems (due June 19)

Problem 1 (Theorem of the square)

(a) Let E be an elliptic curve over a field k and let $D = \sum_{x \in E(k)} n_x[x]$ be a divisor supported at k-rational points. Let $Q = \sum_{x \in E(k)} n_x \cdot x$ be the sum of all points of D (with multiplicities) in $E(k)$. Show that

$$\mathcal{O}_E(D) \cong \mathcal{O}_E\big([Q] + (\deg D - 1)[e]\big).$$

(b) Prove the so-called theorem of the square: For every line bundle $\mathcal{L}$ on E, and for every pair $x, y \in E(k)$ of rational points,

$$t_{x+y}^*(\mathcal{L}) \otimes t_x^*(\mathcal{L})^{-1} \otimes t_y^*(\mathcal{L})^{-1} \otimes \mathcal{L} \cong \mathcal{O}_E.$$

Problem 2 (The trivial fibers of a line bundle)

Let $X \to S$ be a family of curves and let $\mathcal{L}$ be a line bundle on X. Prove that the set

$$\{s \in S \mid \mathcal{L}(s) \cong \mathcal{O}_{X(s)}\}$$

is a closed subset of S.

Hint: First show that $\mathcal{L}(s) \cong \mathcal{O}_{X(s)}$ if and only if $\mathcal{L}(s)$ has degree 0 and $H^0(X(s), \mathcal{L}(s)) \neq 0$. Using our results on cohomology and base change, show that this defines a closed condition on S.