# Exercise Session 4

Recall Riemann-Hurwitz formula: Let $k$ be a field, $f: X \to Y$ a map of smooth prop. conn. curves/$k$. Then

$$2g_X - 2 \geq \deg f \cdot (2g_Y - 2) + \sum_{x \in X} (e_x - 1) \cdot [\kappa(x) : \kappa(f(x))]$$

Equality holds if char $k$ does not divide any $e_x$.

## Complex Case

Let $X$ be a compact Riemann surface. Then the underlying real manifold is isom. to

# holes = genus $g_X$

← Euler characteristic

Then $\chi(X) = 2 - 2g_X$. Thus, RH formula reads

$$-\chi(X) = -\deg f \cdot \chi(Y) + \sum_{x \in X} (e_x - 1)$$

Intuitive proof: Choose triangulations of $X$ and $Y$ s.t. $f$ maps vertices, edges, faces to vertices, edges, faces resp. and s.t. every $x \in X$ with $e_x > 1$ is a vertex. Then $\chi(X) = \chi(\text{graph on } X)$, $\chi(Y) = \chi(\text{graph on } Y)$. Then every vertex/edge/face on $Y$ has $\deg f$ preimages on $X$ except the points where $f$ ramifies. $\square$

(1) $k$ a field, $\operatorname{char} k \neq 2$, $\lambda \in k$, $\lambda \neq 0,1$, $\quad k = \overline{k}$

$$E = V_+(y^2 z - x(x-z)(x-\lambda z)) \subseteq \mathbb{P}^2_k$$

(a) Show that $pr_y : E \to \mathbb{P}^1_k$ exists and compute $\deg pr_y$.

Existence:

- On $E^0 = E \cap D_+(z)$: $pr_y$ is projection to $y$.
- On $E^\infty = E \cap D_+(y)$: $pr_y : [x:y:z] \longmapsto [y:z]$.

Degree: On function fields,

$$k(\mathbb{P}^1) \hookrightarrow k(E)$$
$$\overset{\shortparallel}{k(t)} \longrightarrow k(x,y)/(y^2 - x(x-1)(x-\lambda))$$
$$t \longmapsto y$$

$\rightsquigarrow \deg pr_y = 3$

(b) Compute ramifications $e_\omega$ of $pr_y$ for $\omega \in E$.

- Let $\omega = (a,b) \in E^0$. $pr_y(\omega) = b \in \mathbb{A}^1_k$.

  1. Step: Find uniformizers of $\mathcal{O}_{\mathbb{A}^1, b}$, $\mathcal{O}_{E,\omega}$.

  - $\mathcal{O}_{\mathbb{A}^1, b}$ : $\pi_b = t - b$ is uniformizer $= y - b$ because $pr_y$ maps $t \mapsto y$
  - $\mathcal{O}_{E,\omega} = \left( k[x,y]/(y^2 - x(x-1)(x-\lambda)) \right)_{(x-a,\, y-b)}$

  $\mathfrak{m}_{E,\omega} = (x-a, y-b) = (\pi_\omega)$ for our uniformizer $\pi_\omega$.

Idea: Check if one of $x-a, y-b$ is multiple of the other, so that this other element generates $m_{E,\omega}$ and hence is a uniformizer.

$\rightsquigarrow$ Look at

$$\frac{x-a}{y-b} = \frac{(x-a)(y+b)}{y^2-b^2} = \frac{(x-a)(y+b)}{\underbrace{x(x-1)(x-\lambda)-a(a-1)(a-\lambda)}_{(x-a)g(x)}}$$

$$= \frac{y+b}{g(x)}$$

Case 1: $g(a) \neq 0$. Then $g(x) \in \mathcal{O}_{E,\omega}^{\times}$

$\Rightarrow x-a = (y-b)\cdot \alpha$ for some $\alpha \in \mathcal{O}_{E,\omega}$

$\Rightarrow m_{E,\omega} = (y-b) \rightsquigarrow y-b$ is a uniformizer of $\mathcal{O}_{E,\omega}$.

Hence

$$e_\omega = v_{y-b}(y-b) = 1$$

Case 2: $g(a) = 0 \iff a$ is root of $(x(x-1)(x-\lambda))'$

$$= 3x^2 - 2(\lambda+1)x + \lambda$$

general principle for DVRs

then $y-b$ is not a uniformizer, so $x-a$ is.

$$e_\omega = v_{x-a}(y-b) = v_{x-a}\left((x-a)\,g(x)/(y+b)\right)$$

$$= 1 + \underbrace{v_{x-a}(g(x))}_{= \begin{cases} 2 & (x-a)^2 \mid g(x) \\ 1 & \text{else} \end{cases}}$$

⇝ Look at $(x(x-1)(x-\lambda))''$ to distinguish cases

Either

$a$ is double root of $g$: $e_\omega = 3$, $\omega \in \{(a,b),(a,-b)\}$

$a$ is single root: $\exists$ second such $a$, say $a'$, so

$e_\omega = 2$ for $\omega \in \{(a,\pm b),(a',\pm b)\}$.

Assuming char $\neq 3$, else $g$ is linear

(c) $2g_E - 2 = (2g_{\mathbb{P}^1} - 2)\cdot \deg pr_y + \sum_{\omega \in E} (e_\omega - 1)$

$$0 = -2 \cdot 3 + \begin{cases} 2+2+e_\infty - 1 \\ 1+1+1+1+e_\infty - 1 \end{cases}$$

$\Rightarrow e_\infty = 3$. ← Bad things happen if char $= 3$.

## Level Structures

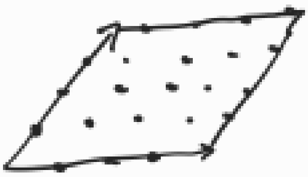
(3) (a) $E$ EC/$\mathbb{C}$. Show $E[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$

Have $E \cong \mathbb{C}/\Lambda$ for some lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, $\operatorname{im}\tau \neq 0$

Then

$$E[N] = \tfrac{1}{N}\Lambda/\Lambda \cong \tfrac{1}{N}\mathbb{Z}^2/\mathbb{Z}^2 = (\mathbb{Z}/N\mathbb{Z})^2.$$

e.g. $N=4$:



(b) Level $N$-structure on $E$ is an isom $\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N]$.

Let $\Gamma(N) = \ker\left( Gl_2(\mathbb{Z}) \twoheadrightarrow Gl_2(\mathbb{Z}/N\mathbb{Z})\right)$. Then

$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_2(\mathbb{Z}) \,\middle|\, \begin{array}{l} a \equiv d \equiv 1 \,(N) \\ c \equiv b \equiv 0 \,(N) \end{array} \right\}$

$$\Gamma(N)\backslash \mathcal{H}^{\pm} \cong \{(E,\alpha)\}/\sim$$

$$\tau \longmapsto \left( \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z},\ \alpha : \begin{array}{l} (1,0) \mapsto \frac{1}{N} \\ (0,1) \mapsto \tau/N \end{array} \right)$$

Claim:

$$\left(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}, \left(\tfrac{1}{N}, \tfrac{\tau}{N}\right)\right) \cong \left(\mathbb{C}/\mathbb{Z}+\tau'\mathbb{Z}, \left(\tfrac{1}{N}, \tfrac{\tau'}{N}\right)\right)$$

iff $\tau' = \dfrac{a\tau+b}{c\tau+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_2(\mathbb{Z})$

and $\underbrace{a\frac{\tau}{N} + b\frac{1}{N} \equiv \frac{\tau}{N} \ (\mathrm{mod}\ \Lambda),\ c\frac{\tau}{N} + d\frac{1}{N} \equiv \frac{1}{N} \ (\mathrm{mod}\ \Lambda)}$

$\Leftrightarrow a\tau + b \equiv \tau \ (\mathrm{mod}\ N\Lambda)$

$\quad c\tau + d \equiv 1 \ (\mathrm{mod}\ N\Lambda)$

$\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} \tau \\ 1 \end{pmatrix}$ in $\Lambda/N\Lambda = (\mathbb{Z}/N\mathbb{Z})^2$

$\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$ in $Gl_2(\mathbb{Z}/N\mathbb{Z})$.

(c) For $N \geq 3$ the action of $\Gamma(N)$ on $\mathcal{H}^{\pm}$ is free.

Stabilizers of $Gl_2(\mathbb{Z})$ action on $\mathcal{H}^{\pm}$ are

$$\{\pm 1\},\ \left\{\pm 1, \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}\right\},\ \left\{ \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & \mp 1 \end{pmatrix}, \begin{pmatrix} \mp 1 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}, \pm 1 \right\}$$

Stabilizers of $\Gamma(N)$ = stabilizers of $Gl_2(\mathbb{Z}) \cap \Gamma(N)$

$= \{1\}$ if $N \geq 3$.

② Use Riemann-Hurwitz formula + $\deg f = 1 \Leftrightarrow f$ isom.

E.g. (6): $X, Y$ EC.

$$\underbrace{2g_X - 2}_{=0} = \underbrace{(2g_Y - 2)}_{=0} \cdot \deg f + \underbrace{\sum_{x \in X} (e_x - 1)\, [k(x) : k(f(x))]}_{\Rightarrow \; = 0}$$

$\Rightarrow$ all $e_x = 1$.

(a): Use $\deg f = \sum_{x \in f^{-1}(y)} e_x \cdot f_x \Rightarrow e_x \leq \deg f$.