It is a matter of individual taste which of the various characterizations of T_pM one chooses to take as the definition. The definition we have chosen, however abstract it may seem at first, has several advantages: it is relatively concrete (tangent vectors are actual derivations of $C^\infty(M)$, with no equivalence classes involved); it makes the vector space structure on T_pM obvious; and it leads to straightforward coordinate-independent definitions of differentials, velocities, and many of the other geometric objects we will be studying.

Categories and Functors

Another useful perspective on tangent spaces and differentials is provided by the theory of categories. In this section we summarize the basic definitions of category theory. We do not do much with the theory in this book, but we mention it because it provides a convenient and powerful language for talking about many of the mathematical structures we will meet.

A ***category*** C consists of the following things:

- a class $\mathrm{Ob}(\mathsf{C})$, whose elements are called ***objects of*** C,
- a class $\mathrm{Hom}(\mathsf{C})$, whose elements are called ***morphisms of*** C,
- for each morphism $f \in \mathrm{Hom}(\mathsf{C})$, two objects $X, Y \in \mathrm{Ob}(\mathsf{C})$ called the ***source*** and ***target of*** f, respectively,
- for each triple $X, Y, Z \in \mathrm{Ob}(\mathsf{C})$, a mapping called ***composition***:

$$\mathrm{Hom}_{\mathsf{C}}(X,Y) \times \mathrm{Hom}_{\mathsf{C}}(Y,Z) \to \mathrm{Hom}_{\mathsf{C}}(X,Z),$$

  written $(f,g) \mapsto g \circ f$, where $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ denotes the class of all morphisms with source X and target Y.

The morphisms are required to satisfy the following axioms:

(i) ASSOCIATIVITY: $(f \circ g) \circ h = f \circ (g \circ h)$.
(ii) EXISTENCE OF IDENTITIES: For each object $X \in \mathrm{Ob}(\mathsf{C})$, there exists an ***identity morphism*** $\mathrm{Id}_X \in \mathrm{Hom}_{\mathsf{C}}(X,X)$, such that $\mathrm{Id}_Y \circ f = f = f \circ \mathrm{Id}_X$ for all $f \in \mathrm{Hom}_{\mathsf{C}}(X,Y)$.

A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X,Y)$ is called an ***isomorphism in*** C if there exists a morphism $g \in \mathrm{Hom}_{\mathsf{C}}(Y,X)$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.

Example 3.26 (Categories). In most of the categories that one meets "in nature," the objects are sets with some extra structure, the morphisms are maps that preserve that structure, and the composition laws and identity morphisms are the obvious ones. Some of the categories of this type that appear in this book (implicitly or explicitly) are listed below. In each case, we describe the category by giving its objects and its morphisms.

- Set: sets and maps
- Top: topological spaces and continuous maps
- Man: topological manifolds and continuous maps

- $\mathsf{Man_b}$: topological manifolds with boundary and continuous maps
- Diff: smooth manifolds and smooth maps
- $\mathsf{Diff_b}$: smooth manifolds with boundary and smooth maps
- $\mathsf{Vec}_{\mathbb{R}}$: real vector spaces and real-linear maps
- $\mathsf{Vec}_{\mathbb{C}}$: complex vector spaces and complex-linear maps
- Grp: groups and group homomorphisms
- Ab: abelian groups and group homomorphisms
- Rng: rings and ring homomorphisms
- CRng: commutative rings and ring homomorphisms

There are also important categories whose objects are sets with distinguished base points, in addition to (possibly) other structures. A ***pointed set*** is an ordered pair (X, p), where X is a set and p is an element of X. Other pointed objects such as ***pointed topological spaces*** or ***pointed smooth manifolds*** are defined similarly. If (X, p) and (X', p') are pointed sets (or topological spaces, etc.), a map $F: X \to X'$ is said to be a ***pointed map*** if $F(p) = p'$; in this case, we write $F: (X, p) \to (X', p')$. Here are some important examples of categories of pointed objects.

- Set_*: pointed sets and pointed maps
- Top_*: pointed topological spaces and pointed continuous maps
- Man_*: pointed topological manifolds and pointed continuous maps
- Diff_*: pointed smooth manifolds and pointed smooth maps //

We use the word *class* instead of *set* for the collections of objects and morphisms in a category because in some categories they are "too large" to be considered sets. For example, in the category Set, $\mathrm{Ob}(\mathsf{Set})$ is the class of all sets; any attempt to treat it as a set in its own right leads to the well-known Russell paradox of set theory. (See [LeeTM, Appendix A] or almost any book on set theory for more.) Even though the classes of objects and morphisms might not constitute sets, we still use notations such as $X \in \mathrm{Ob}(\mathsf{C})$ and $f \in \mathrm{Hom}(\mathsf{C})$ to indicate that X is an object and f is a morphism in C. A category in which both $\mathrm{Ob}(\mathsf{C})$ and $\mathrm{Hom}(\mathsf{C})$ are sets is called a ***small category***, and one in which each class of morphisms $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ is a set is called ***locally small***. All the categories listed above are locally small but not small.

If C and D are categories, a ***covariant functor from*** C ***to*** D is a rule $\mathcal{F}$ that assigns to each object $X \in \mathrm{Ob}(\mathsf{C})$ an object $\mathcal{F}(X) \in \mathrm{Ob}(\mathsf{D})$, and to each morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ a morphism $\mathcal{F}(f) \in \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X), \mathcal{F}(Y))$, so that identities and composition are preserved:

$$\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}; \qquad \mathcal{F}(g \circ h) = \mathcal{F}(g) \circ \mathcal{F}(h).$$

We also need to consider functors that reverse morphisms: a ***contravariant functor from*** C ***to*** D is a rule $\mathcal{F}$ that assigns to each object $X \in \mathrm{Ob}(\mathsf{C})$ an object $\mathcal{F}(X) \in \mathrm{Ob}(\mathsf{D})$, and to each morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ a morphism $\mathcal{F}(f) \in \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(Y), \mathcal{F}(X))$, such that

$$\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}; \qquad \mathcal{F}(g \circ h) = \mathcal{F}(h) \circ \mathcal{F}(g).$$

► **Exercise 3.27.** Show that any (covariant or contravariant) functor from C to D takes isomorphisms in C to isomorphisms in D.