Violating the Singular Cardinals Hypothesis Without Large Cardinals

Talk at the University of Bristol

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Cantor's Continuum Hypothesis

Georg Cantor proved:

Theorem 1. *The power set* $\{x \mid x \subseteq \mathbb{N}\}$ *of* $\mathbb{N}$ *is* not *denumerable.*

In the language of cardinal arithmetic this reads:

Theorem 2. $2^{\aleph_0} \geqslant \aleph_1$.

Cantor conjectured

Conjecture 3. (Cantor's Continuum Hypothesis, CH) $2^{\aleph_0} = \aleph_1$.

Kurt Gödel proved the consistency of CH, assuming the consistency of the Zermelo-Fraenkel axioms ZFC, by constructing the model L of constructible sets

Theorem 4. $L \vDash \mathrm{CH}$.

Paul Cohen proved the opposite relative consistency

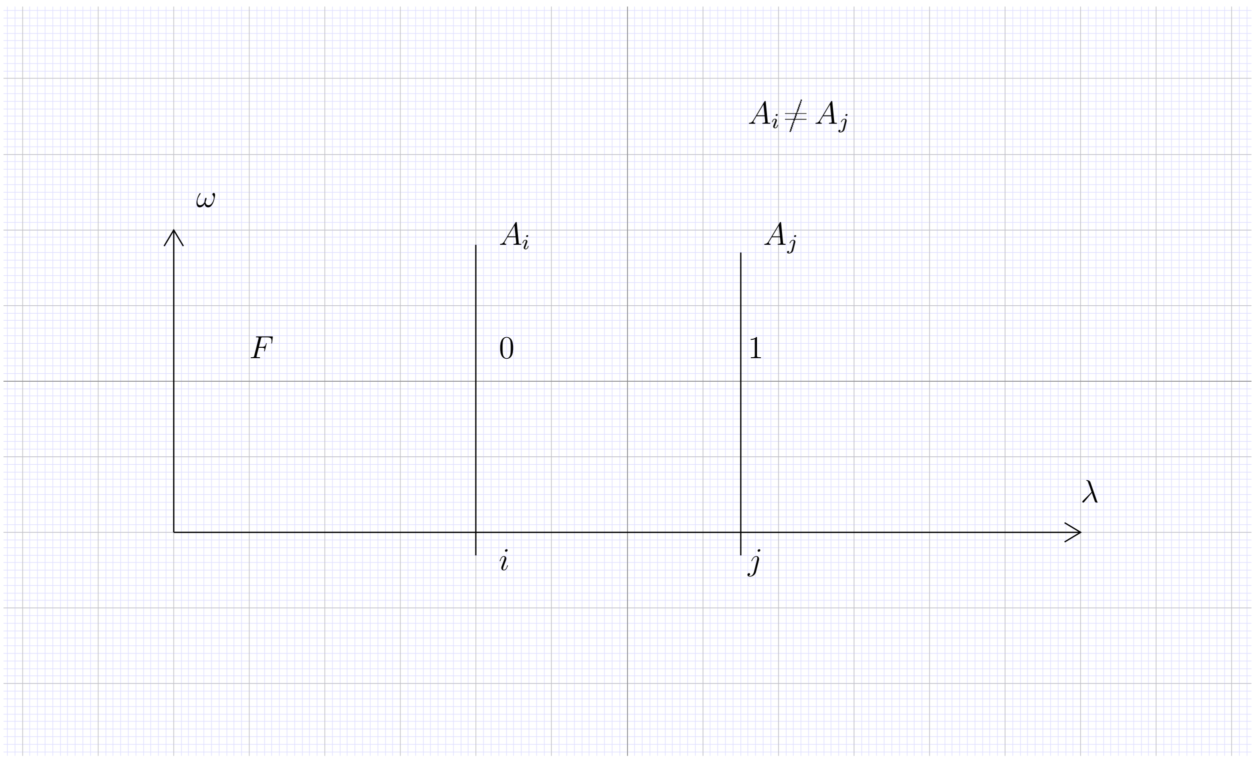
Theorem 5. *Any (countable) model* M *of* ZFC *can be extended to a model* $M[G]$ *of* $\mathrm{ZFC} + 2^{\aleph_0} > \aleph_1$.

For this, Cohen invented the method of *forcing* to adjoin further objects G to M.

G is a (generic) limit of approximations (conditions) in some partial order.

For the $\neg$CH construction we want to adjoin a characteristic function F satisfying

1. $F\colon \lambda \times \omega \to 2$ for some $\lambda \geqslant \aleph_2^M$
2. $\forall i < j < \lambda (\lambda n.F(i,n) \neq \lambda n.F(j,n))$



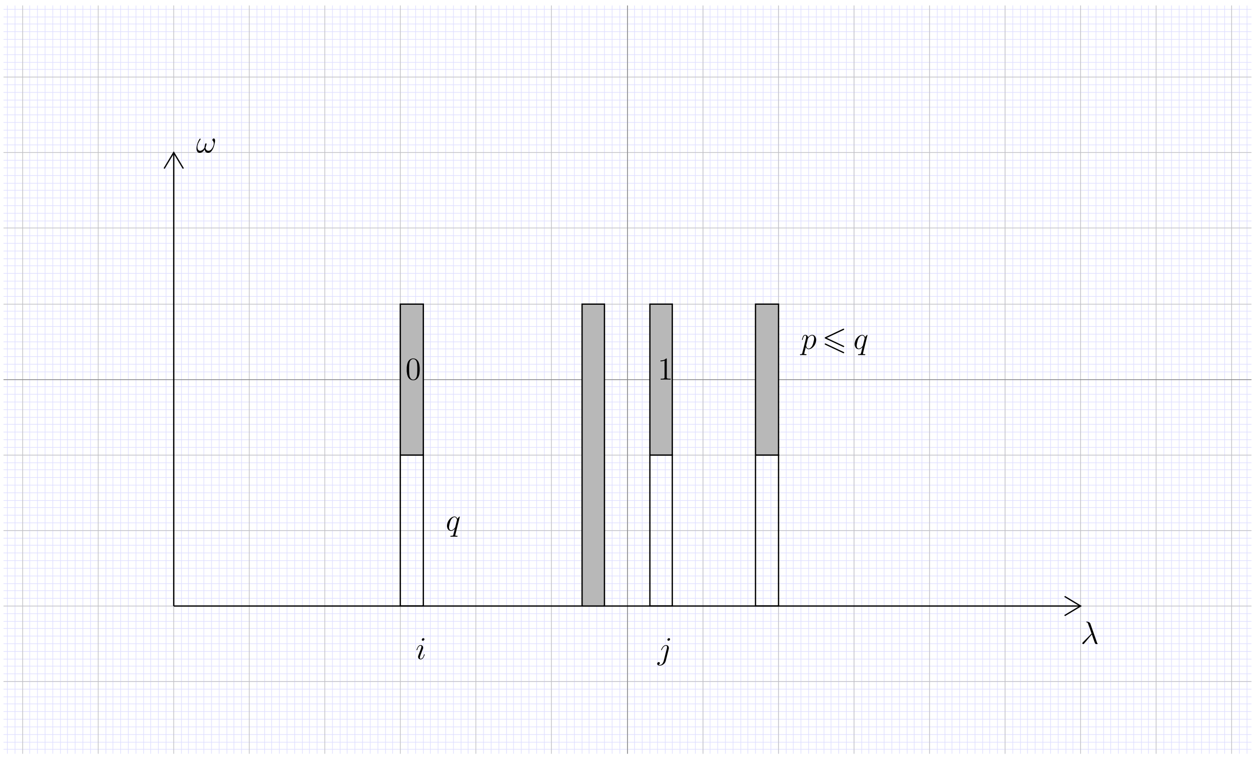
Cohen's partial order for this is essentially

$$P = \{p \mid \exists n < \omega \, \exists D \in [\lambda]^{<\omega} \, p\colon D \times n \to 2\}$$

partially ordered by *reverse inclusion*:

$$p \leqslant q \ (p \text{ is stronger than } q) \ \text{ iff } \ p \supseteq q$$

This may be pictured as



If $G \subseteq P$ is a "generic path" through P then $F = \bigcup \{p | p \in G\}$ is as required.

Hausdorff's Generalized Continuum Hypothesis

Felix Hausdorff conjectured an extension of CH

Conjecture 6. (Hausdorff's Generalized Continuum Hypothesis, GCH)

$$\forall \alpha . 2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

Since GCH holds in GÖDEL's model L,

Theorem 7. GCH *is independent of* ZFC*.*

EASTON proved

Theorem 8. *Let* $E\colon \mathrm{Ord} \to \mathrm{Ord}$ *be a sufficiently absolute function such that*

- $E(\alpha) > \alpha$
- $\alpha < \beta \to E(\alpha) \leqslant E(\beta)$
- $\mathrm{Lim}(E(\alpha)) \to \mathrm{cof}(E(\alpha)) > \aleph_\alpha$

Then one can construct a forcing extension $M[G]$ *such that*

$$\forall \alpha\, (\aleph_\alpha \textit{ is } \textbf{\textit{regular}} \to 2^{\aleph_\alpha} = \aleph_{E(\alpha)})$$

The Singular Cardinal Hypothesis

is the statement

(SCH) if κ is a **singular** strong limit cardinal then $2^\kappa = \kappa^+$

MOTI GITIK and BILL MITCHELL showed

Theorem 9. *The following two theories are equiconsistent:*

- $\mathrm{ZFC} + \neg\mathrm{SCH}$
- $\mathrm{ZFC} +$ *there are "many" measurable cardinals*

SCH Without the Axiom of Choice

Theorem 10. *The following theories are equiconsistent:*

- ZF

– ZF + "GCH *holds below* $\aleph_\omega$" + *"there is a surjection from* $\mathcal{P}(\aleph_\omega)$ *onto* $\aleph_\alpha$*", for some fixed big ordinal* α

This is a strong *surjective* failure of SCH, without requiring large cardinals. *Injective* failures possess much higher consistency strengths.

The forcing

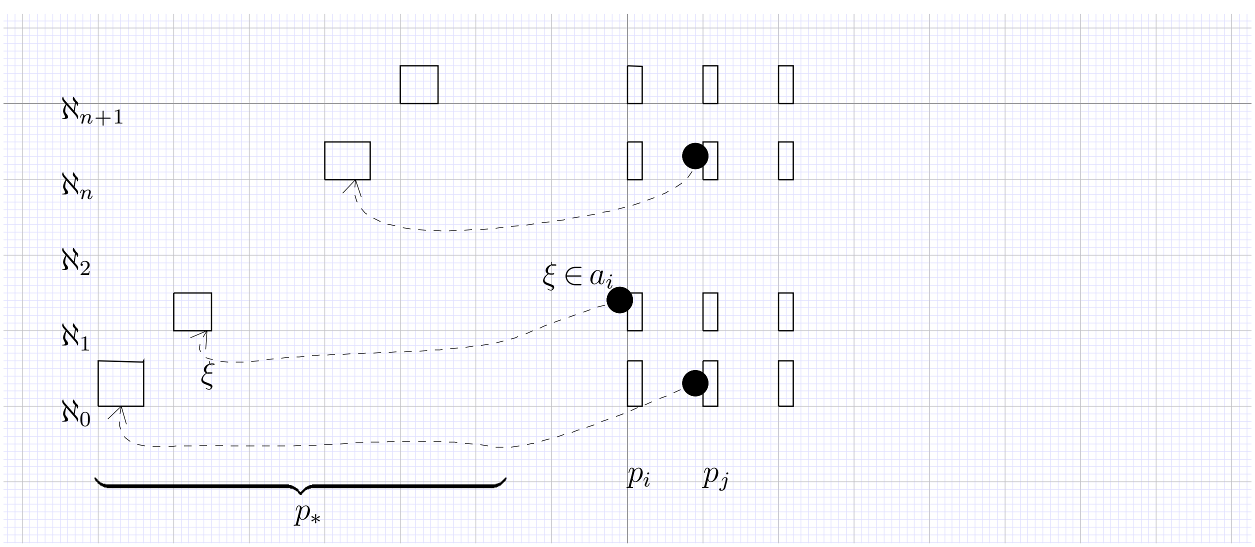
Fix a ground model V of ZFC + GCH and let $\lambda = \aleph_\alpha$ be some regular cardinal in V.

The forcing $P_0 = (P_0, \supseteq, \emptyset)$ adjoins one Cohen subset of $\aleph_{n+1}$ for every $n < \omega$.

$$P_0 = \{ p \mid \exists (\delta_n)_{n<\omega} \, (\forall n < \omega \colon \delta_n \in [\aleph_n, \aleph_{n+1}) \wedge p \colon \bigcup_{n<\omega} [\aleph_n, \delta_n) \to 2)\}.$$

The forcing $(P, \leqslant_P, \emptyset)$ is defined by

$$\begin{aligned} P = \{(p_*, (a_i, p_i)_{i<\lambda}) \mid \; & \exists (\delta_n)_{n<\omega} \, \exists D \in [\lambda]^{<\omega} \, (\forall n < \omega \colon \delta_n \in [\aleph_n, \aleph_{n+1}), \\ & p_* \colon \bigcup_{n<\omega} [\aleph_n, \delta_n)^2 \to 2, \\ & \forall i \in D \colon p_i \colon \bigcup_{n<\omega} [\aleph_n, \delta_n) \to 2 \wedge p_i \neq \emptyset, \\ & \forall i \in D \colon a_i \in [\aleph_\omega \setminus \aleph_0]^{<\omega} \wedge \forall n < \omega \colon \mathrm{card}(a_i \cap [\aleph_n, \aleph_{n+1})) \leqslant 1, \\ & \forall i \notin D \, (a_i = p_i = \emptyset))\} \end{aligned}$$



P is partially ordered by

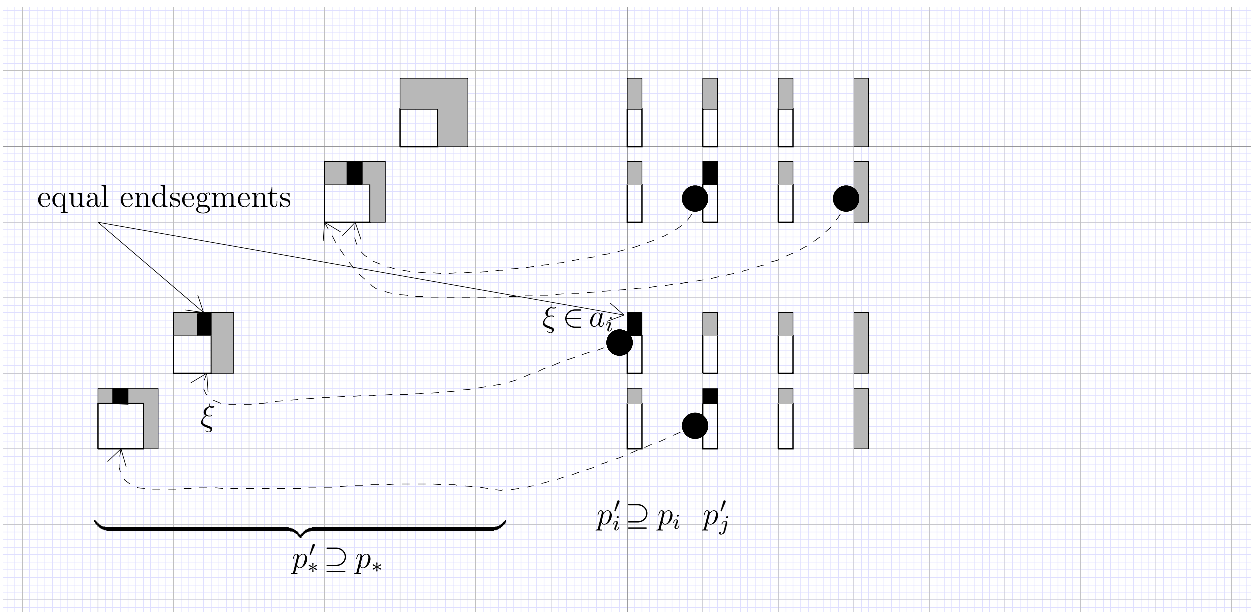
$$p' = (p'_*, (a'_i, p'_i)_{i<\lambda}) \leqslant_P (p_*, (a_i, p_i)_{i<\lambda}) = p$$

iff

a) $p'_* \supseteq p_*, \forall i < \lambda (a'_i \supseteq a_i \wedge p'_i \supseteq p_i)$,

b) $\forall i < \lambda \forall n < \omega \forall \xi \in a_i \cap [\aleph_n, \aleph_{n+1}) \forall \zeta \in \mathrm{dom}(p'_i \setminus p_i) \cap [\aleph_n, \aleph_{n+1})\colon p'_i(\zeta) = p'_*(\xi)(\zeta)$, and

c) $\forall j \in \mathrm{supp}(p)\colon (a'_j \setminus a_j) \cap \bigcup_{i \in \mathrm{supp}(p), i \neq j} a'_i = \emptyset$.

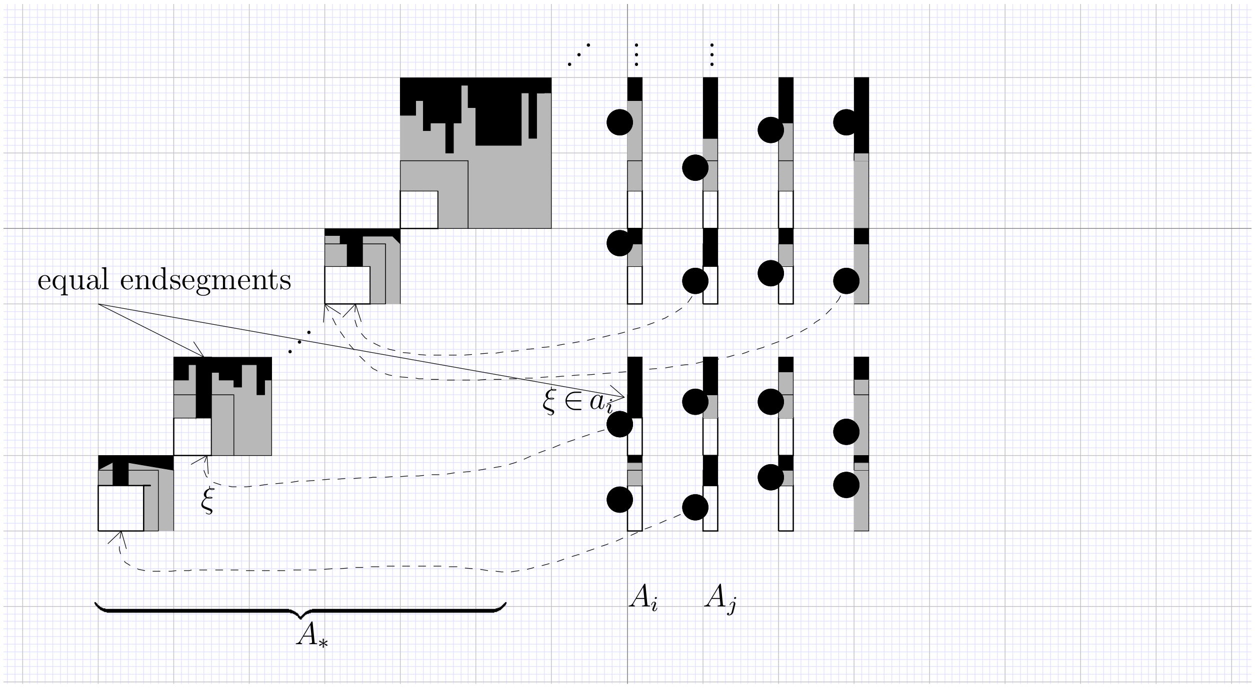


Lemma 11. *P satisfies the $\aleph_{\omega+2}$-chain condition.*

Let G be V-generic for P.

Define

$$\begin{aligned} G_* &= \{p_* \in P_* \,|\, (p_*, (a_i, p_i)_{i<\lambda}) \in G\} \\ A_* &= \bigcup G_*\colon \bigcup_{n<\omega} [\aleph_n, \aleph_{n+1})^2 \to 2 \\ A_*(\xi) &= \{(\zeta, A_*(\xi, \zeta)) | \zeta \in [\aleph_n, \aleph_{n+1})\}\colon [\aleph_n, \aleph_{n+1}) \to 2 \\ A_i &= \bigcup \{p_i \,|\, (p_*, (a_j, p_j)_{j<\lambda}) \in G\}\colon [\aleph_0, \aleph_\omega) \to 2 \end{aligned}$$



Fuzzifying the A_i

Define the *exclusive or* function $\oplus: 2 \times 2 \to 2$ by

$$a \oplus b = 0 \text{ iff } a = b.$$

For functions $A, A': \mathrm{dom}(A) = \mathrm{dom}(A') \to 2$

define the pointwise exclusive or $A \oplus A': \mathrm{dom}(A) \to 2$ by

$$(A \oplus A')(\xi) = A(\xi) \oplus A'(\xi).$$

For functions $A, A': (\aleph_\omega \setminus \aleph_0) \to 2$ define an equivalence relation $\sim$ by

$$A \sim A' \text{ iff } \exists n < \omega\, ((A \oplus A') \upharpoonright \aleph_{n+1} \in V[G_*] \wedge (A \oplus A') \upharpoonright [\aleph_{n+1}, \aleph_\omega) \in V).$$

Let $\tilde{A} = \{A' | A' \sim A\}$ be the $\sim$-equivalence class of A.

The "symmetric" submodel

Set

- $T_* = \mathcal{P}(<\kappa)^{V[A_*]}$, setting $\kappa = \aleph_\omega^V$;

- $\vec{A} = (\tilde{A}_i \,|\, i < \lambda)$.

The final model is

$$N = \mathrm{HOD}^{V[G]}(V \cup \{T_*, \vec{A}\} \cup T_* \cup \bigcup_{i<\lambda} \tilde{A}_i)$$

consisting of all sets which, in $V[G]$ are hereditarily definable from parameters in the transitive closure of $V \cup \{T_*, \vec{A}\}$.

Lemma 12. *Every set $X \in N$ is definable in $V[G]$ in the following form: there are an $\in$-formula φ, $x \in V$, $n < \omega$, and $i_0, \ldots, i_{l-1} < \lambda$ such that*

$$X = \{u \in V[G] \,|\, V[G] \vDash \varphi(u, x, T_*, \vec{A}, A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}})\}.$$

Lemma 13. *N is a model of* ZF*, and there is a surjection $f: \mathcal{P}(\kappa) \to \lambda$ in N.*

Proof. Note that for every $i < \lambda$: $A_i \in N$.
(1) Let $i < j < \lambda$. Then $A_i \nsim A_j$.
Proof. Assume instead that $A_i \sim A_j$. Then take $n < \omega$ such that $v = (A_i \oplus A_j) \upharpoonright [\aleph_{n+1}, \aleph_\omega) \in V$. The set

$$D = \{(p_*, (a_k, p_k)_{k<\lambda}) \,|\, \exists \xi \in [\aleph_{n+1}, \aleph_\omega)\, (\xi \in \mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) \wedge v(\xi) \neq p_i(\xi) \oplus p_j(\xi))\} \in V$$

is readily seen to be dense in P. Take $(p_*, (a_k, p_k)_{k<\lambda}) \in D \cap G$. Take $\xi \in [\aleph_{n+1}, \aleph_\omega)$ such that

$$\xi \in \mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) \wedge v(\xi) \neq p_i(\xi) \oplus p_j(\xi)).$$

Since $p_i \subseteq A_i$ and $p_j \subseteq A_j$ we have $v(\xi) \neq A_i(\xi) \oplus A_j(\xi)$ and $v \neq (A_i \oplus A_j) \upharpoonright [\aleph_{n+1}, \aleph_\omega)$. Contradiction. *qed*(1)

Thus

$$f(z) = \begin{cases} i, \text{ if } z \in \tilde{A}_i; \\ 0, \text{ else}; \end{cases}$$

is a well-defined surjection $f: \mathcal{P}(\kappa) \to \lambda$, and f is definable in N from the parameters κ and $\vec{A}$. □

Approximating N

Lemma 14. *Let $X \in N$ and $X \subseteq \mathrm{Ord}$. Then there are $n < \omega$ and $i_0, \ldots, i_{l-1} < \lambda$ such that*

$$X \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}].$$

Proof. Let

$$X = \{u \in \mathrm{Ord} \,|\, V[G] \vDash \varphi(u, x, T_*, \vec{A}, A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}})\}.$$

By taking n sufficiently large, we may assume that

$$\forall j < k < l \forall m \in [n, \omega) \forall \delta \in [\aleph_m, \aleph_{m+1}) \colon A_{i_j} \upharpoonright [\delta, \aleph_{m+1}) \neq A_{i_k} \upharpoonright [\delta, \aleph_{m+1}).$$

For $j < l$ set

$$a_{i_j}^* = \{\xi \,|\, \exists m \leqslant n \exists \delta \in [\aleph_m, \aleph_{m+1}) \colon A_{i_j} \upharpoonright [\delta, \aleph_{m+1}) = A_*(\xi) \upharpoonright [\delta, \aleph_{m+1})\}$$

where $A_*(\xi) = \{(\zeta, A_*(\xi, \zeta)) | (\xi, \zeta) \in \mathrm{dom}(A_*)\}$.

Define

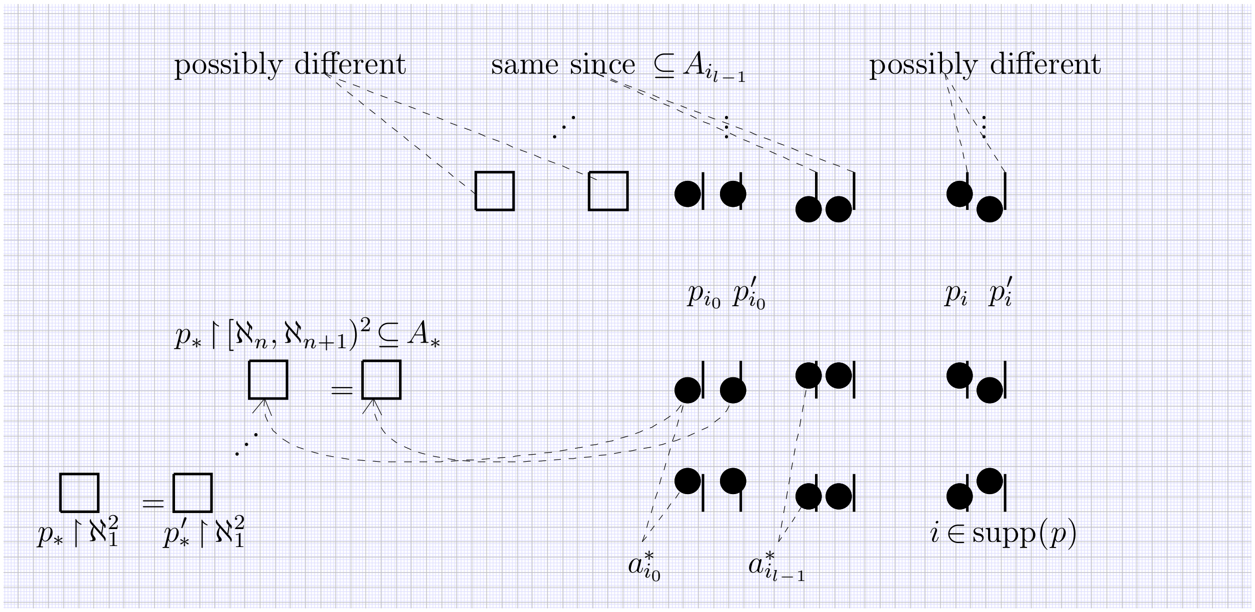
$$\begin{aligned} X' \;=\; \{ u \in \mathrm{Ord} \;|\; & \text{there is } p = (p_*, (a_i, p_i)_{i<\lambda}) \in P \text{ such that} \\ & p_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2, \\ & a_{i_0} \supseteq a_{i_0}^*, \ldots, a_{i_{l-1}} \supseteq a_{i_{l-1}}^*, \\ & p_{i_0} \subseteq A_{i_0}, \ldots, p_{i_{l-1}} \subseteq A_{i_{l-1}}, \text{ and} \\ & p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}) \}, \end{aligned}$$

where $\sigma, \tau, \dot{A}, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}$ are canonical names for $T_*, \vec{A}, A_*, A_{i_0}, \ldots, A_{i_{l-1}}$ resp.

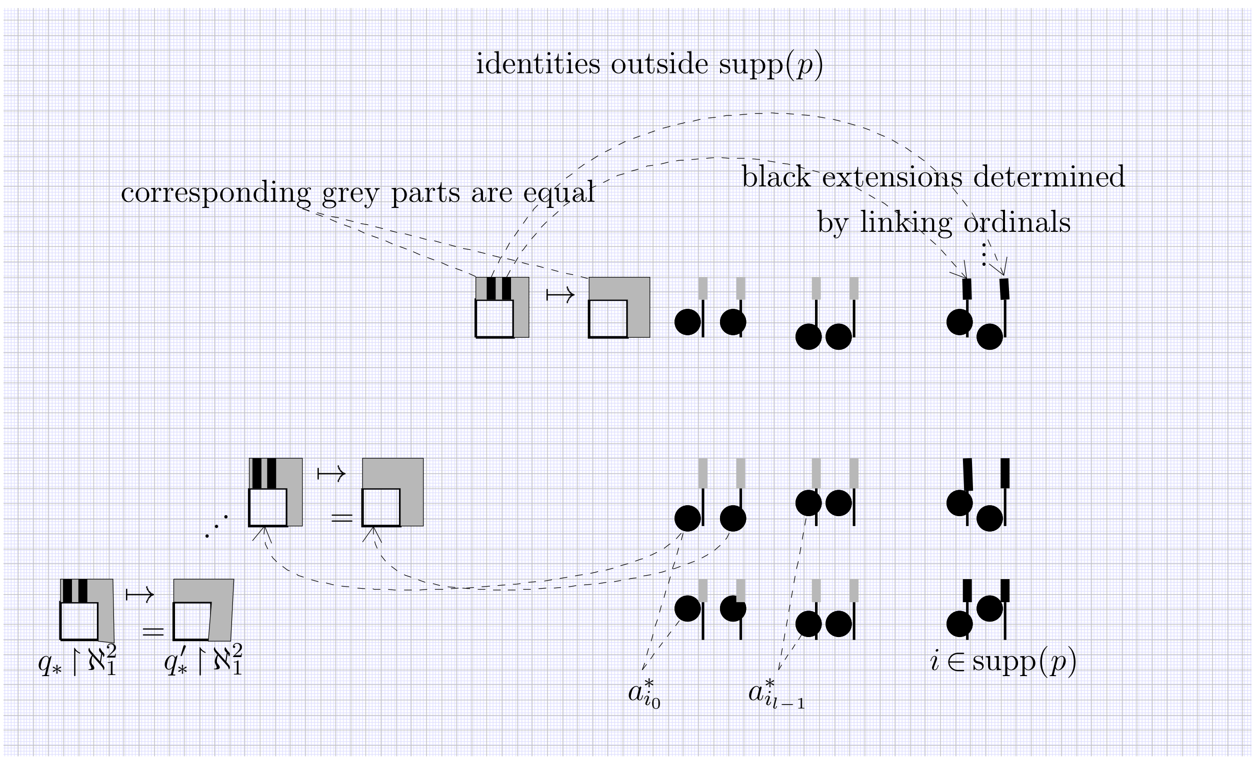
Then $X' \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}]$.

(1) $X \subseteq X'$.
Proof. Straightforward. *qed*(1)

The converse direction, $X' \subseteq X$, uses an automorphism argument.



One defines an isomorphism π of P below p and below p', respectively.



□

Wrapping up

Lemma 15. *Let $n < \omega$ and $i_0, \ldots, i_{l-1} < \lambda$. Then cardinals are absolute between V and $V[A^* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}]$.*

Lemma 16. *Cardinals are absolute between N and V, and in particular $\kappa = \aleph_\omega^V = \aleph_\omega^N$.*

Proof. If not, then there is a function $f \in N$ which collapses a cardinal in V. By Lemma 14, f is an element of some model $V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}]$ as above. But this contradicts Lemma 15. □

Lemma 17. GCH *holds in N below $\aleph_\omega$.*

Proof. If $X \subseteq \aleph_n$ and $X \in N$ then X is an element of some model $V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}]$ as above. Since $A_{i_0}, \ldots, A_{i_{l-1}}$ do not adjoin new subsets of $\aleph_n$ we have that

$$X \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2].$$

Hence $\mathcal{P}(\aleph_n^V) \cap N \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$. GCH holds in $V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$. Hence there is a bijection $\mathcal{P}(\aleph_n^V) \cap N \leftrightarrow \aleph_{n+1}^V$ in $V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$ and hence in N. □

Discussion and Remarks

To work with singular cardinals κ of *uncountable* cofinality, various finiteness properties in the construction have to be replaced by the property of being of cardinality $< \mathrm{cof}(\kappa)$. This yields choiceless violations of SILVER's theorem.

Theorem 18. *Let V be any ground model of* ZFC + GCH *and let λ be some cardinal in V. Then there is a model $N \supseteq V$ of the theory* ZF + "GCH *holds below $\aleph_{\omega_1}$" + "there is a surjection from $\mathcal{P}(\aleph_{\omega_1})$ onto λ". Moreover, the axiom of dependent choices* DC *holds in N.*