About Space Curves of Constant Torsion

See also: About Space Curves of Constant Curvature

Definition via Differential Equations

Most Space Curves that 3DXM can exhibit are given in terms of explicit formulas or explicit geometric constructions. In "About Space Curves of Constant Curvature" we explain how *curvature* and *torsion* of a space curve are defined. The definition immediately translates into a construction of the curve from curvature and torsion via the following *differential equations*, the famous

Frenet-Serret Equations:

$$\begin{aligned}
\dot{e}_1(t) &:= \qquad\qquad \kappa(t)\cdot e_2(t),\\
\dot{e}_2(t) &:= -\kappa(t)\cdot e_1(t) - \tau(t)\cdot e_3(t),\\
\dot{e}_3(t) &:= \qquad\qquad \tau(t)\cdot e_2(t).
\end{aligned}$$

For given continuous functions κ, τ these differential equations have — for given orthonormal initial values — unique orthonormal solutions $\{e_1(t), e_2(t), e_3(t)\}$. The curve $c(t) := \int^t e_1(s)ds$ is then parametrized by arc length and has the given curvature functions κ, τ.

The simplest curves in the plane are straight lines and circles, curves of constant curvature. It is therefore natural to discuss also space curves of constant curvature. In 3DXM we illustrate these by finding closed examples in the following family:

$$\kappa(t) := aa,$$
$$\tau(t) := bb + cc \cdot \sin(t) + dd \cdot \sin(2t) + ee \cdot \sin(3t).$$

To understand the Frenet-Serret equations better one can also study other special cases. Experimentation shows that the following curves of constant torsion

$$\kappa(t) := bb + cc \cdot \cos(ff \cdot t) + dd \cdot \cos(2ff \cdot t) + ee \cdot \cos(3ff \cdot t)$$
$$\tau(t) = aa$$

have an amusingly strong change of shape as one changes the parameters. Again we look for closed examples with the help of symmetries. Note that $180°$ rotations around the principal normals $e_2(t)$ at $t/ff = k\pi, k \in \mathbb{Z}$ are isometries of the curves. At $t/ff = \pi/2 + k\pi, k \in \mathbb{Z}$ the $180°$ rotations around the other normal vector of the frame, $e_3(t)$, are also isometries of the space curve. This allows us to formulate the *closing condition*:
If the normals $e_2(0)$ at $c(0)$, $e_3(\pi \cdot ff/2)$ at $c(\pi \cdot ff/2)$

intersect and if their angle is a rational multiple of π then the space curve closes up. Numerically one can use the parameter cc to keep the angle constant, e.g. at $\pi/3, \pi/4$, and then use aa to let the normals intersect. There are many closed solutions. Typically they look like a collection of bed springs which are joint by fairly straight pieces. If one allows these bed springs to have many turns then the closing values of aa and cc are almost equidistant. The default morph of 3DXM shows this, it contains two closed and three approximately closed curves which are made of *three* bed springs with an increasing number of turns. It is easy to extend this family to springs with more turns, but one can also find all the small values, down to just one half turn for each spring. — We found no closed curves made of only *two springs*.
Here is a list of numerically closed curves:

Curves with 3-fold symmetry, $ff = 0.208$,

aa,	0.178632213,	0.284031845,	0.417033334,
cc,	0.2874008,	0.90658882,	2.19234962,
aa,	0.513441035,	0.59263462,	0.628044,
cc,	3.489480574,	4.7901189,	5.4411264,
aa,	0.661324546,	0.69281176,	0.7227614
cc,	6.09244336,	6.7440016,	7.39575343

Curves with 4-fold symmetry, $ff = 0.23$,

aa,	0.2137654757,	0.3704887,	0.479019355,
cc,	0.234123448,	0.89640923,	1.59595534,
aa,	0.56642393,	0.6414483533,	0.7081321561,
cc,	2.30473675,	3.01756515691,	3.732639742,
aa,	0.76871766,	0.8246012,	0.87671763
cc,	4.449136,	5.1666082,	5.8847911

Curve with 5-fold symmetry, $ff = 0.2324$,
$aa = 0.73855871446286, \quad cc = 2.96466$

If ones does not begin with the differential equation but starts from the curve, then one cannot define the torsion at points where the curvature vanishes. This problem is caused by the use of the Frenet frame. Another frame is suggested by a mechanical consideration: If a massive sphere would move along the space curve (imagine the space curve as a wire and the sphere with a hole through which the wire slides without friction) then inertia would make the sphere avoid unnecessary rotations around the wire. In other words: A frame which is attached to the sphere so that it is normal to the wire remains normal and the derivatives of the normal vectors have *no normal* *components*. Such frames are called "parallel as normal vectors", or simply "parallel frames". In 3DXM one can choose `Parallel Frame` in the Action Menu . Now `Show Curve as Tube` illustrates the behaviour of the chosen frame. In particular the torus knots show how the parallel frames avoid "unnecessary" rotations which the Frenet frames must make.

An advantage of such parallel frames is that they neither require to assume more than *two* continuous derivatives of the curve nor that κ never vanishes—even straight lines are not exceptional curves if one works with these frames. Let $\phi(t)$ be an antiderivative of the torsion function, i.e., $\dot{\phi}(t) = \tau(t)$. Then the differential equation that determines this frame has the following simple form:

Frenet-Serret Equations for Parallel Frames:

$$\begin{aligned}
\dot{e}_1(t) &:= \kappa(t)\cos(\phi(t)) \cdot e_2(t) + \kappa(t)\sin(\phi(t)) \cdot e_3(t) \\
\dot{e}_2(t) &:= -\kappa(t)\cos(\phi(t)) \cdot e_1(t) \\
\dot{e}_3(t) &:= -\kappa(t)\sin(\phi(t)) \cdot e_1(t).
\end{aligned}$$