Endoscopic character identities for depth-zero supercuspidal L-packets

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Abstract

One of the conjectural properties of a Langlands correspondence is its compatibility with endoscopic induction. DeBacker and Reeder have recently constructed a partial local Langlands correspondence for p-adic groups, focusing on L-packets consisting of depth-zero supercuspidal representations. In this paper we prove the conjectural endoscopic transfer for these L-packets.

The local Langlands correspondence, which is known in the real case and partially constructed in the p-adic case, assigns to each Langlands parameter for a reductive group G over a local field F a finite set of admissible irreducible representations of $G(F)$, called an L-packet. When such a parameter factors through an endoscopic group H, the broad principle of Langlands functoriality asserts that the packet on H should "transfer" to the packet on G. The endoscopic character identities are an instance of this principle – they state that the "stable" character of the packet on H is identified via endoscopic induction with an "unstable" character of the packet on G.

To be more precise, let F be a p-adic field with Weil-group W_F and let G be a connected reductive group over F. For the purposes of this introduction, we assume that G is unramified, although in the body of this paper the more general case of a pure inner form of an unramified group is handled. Let ${}^LG$ be an L-group for G, that is ${}^LG = \widehat{G} \rtimes W_F$, where $\widehat{G}$ is the complex Langlands dual of G and W_F acts on $\widehat{G}$ via its action on the based root datum of $\widehat{G}$ which is dual to that of G. The Langlands parameters considered in this paper are continuous sections

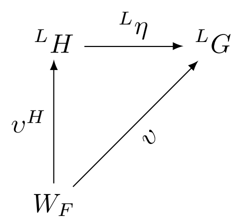
$$\upsilon : W_F \to {}^LG$$

of the natural projection ${}^LG \to W_F$ and subject to certain conditions, called TRSELP in [DR09], which will be reviewed in detail later on. To such a parameter DeBacker and Reeder construct in loc.cit. an L-packet $\Pi_G(\upsilon)$ of representations of $G(F)$ and a bijection

$$\mathrm{Irr}(C_\upsilon, 1) \to \Pi_G(\upsilon), \qquad \rho \mapsto \pi_\rho$$

where C_υ is the component group of the centralizer in $\widehat{G}$ of υ and $\mathrm{Irr}(C_\upsilon, 1)$ are those irreducible representations of the finite group C_υ which are trivial on elements of C_υ coming from the center of $\widehat{G}$. This bijection maps the trivial representation of C_υ to a generic representation of $G(F)$.

Let $(H, s, \widehat{\eta})$ be an unramified endoscopic triple for G. Recall that H is an unramified reductive group over F, s is a Galois-fixed element of the center of $\widehat{H}$, and $\widehat{\eta}$ is an inclusion $\widehat{H} \to \widehat{G}$ which identifies $\widehat{H}$ with $(\widehat{G}_{\widehat{\eta}(s)})^\circ$. It was shown by Hales in [Hal93] that $\widehat{\eta}$ extends to an embedding ${}^L\eta : {}^LH \to {}^LG$. Thus for any parameter υ^H for H we may consider the parameter $\upsilon = {}^L\eta \circ \upsilon^H$, i.e. we have



If υ^H is a TRSELP, then so is υ, and we have the L-packets $\Pi_G(\upsilon)$ and $\Pi_H(\upsilon^H)$. Associated to these, we have the stable character

$$\mathcal{S}\Theta_{\upsilon^H} := \sum_{\rho \in \mathrm{Irr}(C_{\upsilon^H},1)} [\dim \rho] \chi_{\pi_\rho}$$

of $\Pi_H(\upsilon^H)$, which is a stable function on $H(F)$ (this is one of the main results of [DR09]), as well as the s-unstable character

$$\Theta^s_{\upsilon,1} := \sum_{\rho \in \mathrm{Irr}(C_\upsilon,1)} [\mathrm{tr}\, \rho(s)] \chi_{\pi_\rho}$$

of $\Pi_G(\upsilon)$, which is an invariant function on $G(F)$.

Recall that the representation π_1 of $G(F)$ is generic. Thus there is a Borel subgroup $B = TU$ of G defined over F and a generic character $\psi : U(F) \to \mathbb{C}^\times$ which occurs as a quotient of the restriction of π_1 to $U(F)$. Associated to the character ψ there is a unique normalization Δ_ψ of the transfer factor for G and H, called the Whittaker normalization. The endoscopic lift of the stable function $\mathcal{S}\Theta_{\upsilon^H}$ is given by

$$\mathrm{Lift}^G_H \mathcal{S}\Theta_{\upsilon^H}(\gamma) := \sum_{\gamma^H} \Delta_\psi(\gamma^H, \gamma) \frac{D^H(\gamma^H)^2}{D^G(\gamma)^2} \mathcal{S}\Theta_{\upsilon^H}(\gamma^H)$$

where D is the usual Weyl-discriminant, $\gamma \in G(F)$ is any strongly regular semi-simple element and γ^H runs through the set of stable classes of G-strongly regular semi-simple elements in $H(F)$.

The main result of this paper asserts that

$$\Theta^s_{\upsilon,1} = \mathrm{Lift}^G_H \mathcal{S}\Theta_{\upsilon^H}$$

As a corollary of the main result in the case where G is a pure inner form of an unramified group G^* and $H = G^*$ we obtain a proof (for the L-packets considered) of the conjecture of Kottwitz [Kot83] about sign changes in stable characters on inner forms.

We now describe the contents of the paper. After fixing some basic notation in Section 1, we discuss pure inner twists and the associated notions of conjugacy and stable conjugacy. We have allowed trivial inner twists in the discussion so as to accommodate the natural construction of the L-packets in [DR09] and not just their normalized form. With these notions in place we implement an observation of Kottwitz which allows one to define compatible normalizations of the absolute transfer factors for all pure inner twists. To do this in the necessary generality, we need to study homotopically trivial twists of complexes of tori. In Section 3 we briefly review the construction of the local Langlands correspondence in [DR09], and after gathering the necessary notation we state the main result of this paper. The remaining sections are devoted to its proof, which is similar in spirit to the proof of the stability result in loc. cit.. In Section 4 we study three signs which are defined for a pair (G, H) of a group G and an endoscopic group H and play an important role in the theory of endoscopy – one of them is defined in terms of the split ranks of these groups and goes back to [Kot83], the other one occurs in Waldspurger's work [Wal95] on the endoscopic transfer for p-adic Lie algebras, and the third is a certain local ϵ-factor used in the Whittaker normalization of the transfer factors [KS99]. We show that when both G and H are unramified, these three signs coincide. This supplements the results of [DR09, §12] to assert in particular that the Waldspurger-sign and the relative-ranks-sign coincide whenever G is a pure inner form of an unramified group and H is an unramified endoscopic group. Because this section may be of independent interest we have minimized the notation that it borrows from previous sections. In Section 5 we provide a proof of the character identities at topologically semi-simple elements. The proof of this special case is considerably simpler than that of the general case. Although the proof of the general case does not depend on establishing this special case first, we hope that it may help elucidate some aspects of the general proof, and thank the referee for suggesting that it be included in the paper. Section 6 establishes a reduction formula for the unstable character of an L-packet with respect to the topological Jordan decomposition. For that we first need explicit formulas for some basic constructions in endoscopy, which are established in a preparatory subsection. Among other things we show that the isomorphism $H^1(F, G) \rightarrow \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma))$ constructed in [DR09] via Bruhat-Tits theory coincides with the one constructed in [Kot86] using Tate-Nakayama duality. With these preliminaries in place we derive the reduction formula for the unstable character using the results of [DR09, §9,§10]. The ingredients from the previous sections are combined in Section 7 to establish the proof of the main result. After reducing to the case of compact elements, the reduction formula from Section 6 is combined with the work of Langlands and Shelstad [LS90] and Hales [Hal93] on endoscopic descent. The topologically unipotent part of the resulting expression is then transferred to the Lie algebra, where we invoke the deep results of Waldspurger on endoscopic transfer for p-adic Lie-algebras together with the fundamental lemma, which has been recently proved by the combined effort of many people, the decisive last step being carried out by Ngô Bao Châu [Ngo08].

We would like to bring to the attention of the reader some related work. In [KV1], Kazhdan and Varshavsky construct an endoscopic decomposition for the L-packets considered here. In particular, they consider the s-unstable characters of these packets and show that they belong to a space of functions which contains the image of endoscopic induction. The existence of such a decomposition is a necessary condition for the validity of the character identities considered here and also gave us yet more reason to hope that indeed these identities should be true. In [KV2] the aforementioned authors prove a formula for the geometric endoscopic transfer of Deligne-Lusztig functions, in particular answering a conjecture of Kottwitz. After the current paper was written, the author was informed in a private conversation with Kazhdan that the results in [KV2], when combined with the material from sections 2, 3 and 6, could be used to derive character identities similar to the ones proved here, on the set of elliptic elements.

The author would like to thank Professor Robert Kottwitz for his generous support and countless enlightening and inspiring discussions. This work would not have been possible without his dedication and kindness. We would also like to thank Professor Stephen DeBacker for suggesting this problem and discussing at length the character formulas in [DR09], as well as for his continual support and encouragement. Finally, we thank the referee for suggesting multiple improvements to the exposition, which we believe have made this paper more understandable and pleasant to read. In particular, Sections 3.5 and 5 were added following the referee's suggestion.

Contents

1 Notation

Let F be a p-adic field (i.e. a finite extension of $\mathbb{Q}_p$) with ring of integers O_F, uniformizer π_F, and residue field $k_F = O_F/\pi_F O_F$ with cardinality q_F. We use analogous notation for any other discretely valued field, in particular for the maximal unramified extension F^u of F in a fixed algebraic closure $\overline{F}$. Since we will consider only extensions of F which lie in F^u, π_F will be a uniformizer in each of them and so we will drop the index F and simply call it π. For any such finite extension E, $v_E : E^\times \to \mathbb{Z}$ will be the discrete valuation normalized so that $v_E(\pi) = 1$, and $|x|_E$ will be the norm given by $q_E^{-v_E(x)}$. Thus v_E extends v_F and so we may again drop the index F. On the other hand, for $x \in F^\times$ we have $|x|_E = |x|_F^{[E:F]}$; if dx is any additive Haar measure on E then $d(ax) = |a|_E dx$. The absolute Galois group of F will be denoted by Γ, its Weil group by W_F and inertia group by I_F. We choose an element $\mathrm{Fi} \in \Gamma$ whose inverse induces on $\overline{k_F}$ the map $x \mapsto x^{q_F}$.

For a reductive group G defined over F, we will denote its Lie algebra by the Fraktur letter $\mathfrak{g}$. Our convention will be that $a \in G$ resp. $a \in \mathfrak{g}$ will mean that a is an $\overline{F}$-point of the corresponding space, while a maximal torus $T \subset G$ will be tacitly assumed to be defined over F. The action of Fi on both $G(F^u)$ and $\mathfrak{g}(F^u)$ will be denoted by Fi_G. For a semi-simple $a \in G$, we will write $\mathrm{Cent}(a, G) = G^a$ for the centralizer of a in G and G_a for its connected component. If $T \subset G$ is a maximal torus then the roots resp. coroots of T in G will be denoted by $R(T, G)$ resp. $R^\vee(T, G)$, and the Weyl-group of T will be denoted by $\Omega(T, G)$. The center of G will be Z_G, or simply Z if G is understood, and the maximal split torus in Z_G will be A_G. The sets of strongly-regular semi-simple elements of G resp. $\mathfrak{g}$ will be denoted by G_{sr} resp. $\mathfrak{g}_{\mathrm{sr}}$. For any $g \in G$ the map $G \to G, x \mapsto gxg^{-1}$ as well as its tangent map $\mathfrak{g} \to \mathfrak{g}$ will be called $\mathrm{Ad}(g)$. Abusing words, will will refer to the orbits of $\mathrm{Ad}(G)$ in $\mathfrak{g}$ as conjugacy classes, and then notions such as stable classes and rational classes will have their obvious meaning. Elements of G will be oftentimes denoted by lowercase Latin or Greek letters, like g or γ, while for elements of $\mathfrak{g}$ we will often use uppercase Latin letters, like P or Q.

The set of compact elements in $G(F)$ will be denoted by $G(F)_0$. Given a semi-simple $\gamma \in G(F)_0$, we will write $\gamma = \gamma_s\gamma_u$ for its topological Jordan decomposition, where γ_s is the topologically semi-simple part and γ_u is the topologically unipotent part of γ. For a quick overview of the topological Jordan decomposition and its properties, we refer the reader to [DR09, §7], as well as to [Hal93, §§2,3], but warn the reader that the language of the latter reference is slightly different. For a detailed discussion of the topological Jordan decomposition see [Sp08].

To maintain notational similarity with [DR09], we will sometimes use the following conventions. If $\psi : G \to G'$ is an inner twist, then we may identify $G(\overline{F})$ and $G'(\overline{F})$ via ψ and suppress ψ from the notation, thereby treating $\gamma \in G(\overline{F})$ and $\psi(\gamma) \in G'(\overline{F})$ as the same element. If $u \in Z^1(\Gamma, G)$ is a cocycle, then we will use the same letter u also for the value of that cocycle at Fi.

If $(H, s, \widehat{\eta})$ is an endoscopic triple for a reductive group G/F, we will often attach a superscript H to objects related to H, such as maximal tori, Borel subgroups, or elements of $H(F)$. If ${}^L\eta : {}^LH \to {}^LG$ is an L-embedding extending $\widehat{\eta}$, then we will call $(H, s, {}^L\eta)$ an extended triple for G. The set of G-strongly regular semi-simple elements of H resp. $\mathfrak{h}$ will be denoted by $H_{G-\mathrm{sr}}$ resp. $\mathfrak{h}_{G-\mathrm{sr}}$.

Let $t^H \in H(F)$ and $t \in G(F)$ be semi-simple elements. We will call t an image of t^H if there exist maximal tori $T^H \subset H$ and $T \subset G$ and an admissible isomorphism $T^H \to T$ defined over F and mapping t^H to t (the notion of an admissible isomorphism is recalled in the introduction of Section 3). This definition is the same as in [LS90], but our wording is opposite – in [LS90] the element t^H is called an image of t. If t is an image of t^H we will also call (t^H, t) a pair of related elements. For such a pair, we consider the set of $\varphi : T^H \to T$, where T^H is a maximal torus in H containing t^H, T is a maximal torus of G containing t, and φ is an admissible isomorphism defined over F and mapping t^H to t. On this set we define an equivalence relation, by saying that two such isomorphisms φ and φ' are (G_t, H_{t^H})-equivalent if there exist $g \in G_t(\overline{F})$ and $h \in H_{t^H}(\overline{F})$ s.t. $\varphi' = \mathrm{Ad}(g)\varphi\mathrm{Ad}(h)$. If φ is an element of this set, and H_{t^H} is quasi-split, then H_{t^H} can be identified with an endoscopic group of G_t in such a way that φ becomes an admissible isomorphism with respect to (G_t, H_{t^H}). Then we can talk about images, admissible isomorphisms, etc. with respect to the group G_t and its endoscopic group H_{t^H}. When we do so, we will use the prefix (G_t, H_{t^H}, φ).

If γ, γ' are two strongly G-regular semi-simple elements, each of which belongs to either $G(F)$ or $H(F)$, and T, T' are their centralizers, then there exists at most one admissible isomorphism $T \to T'$ which maps γ to γ'. We will call this isomorphism $\varphi_{\gamma,\gamma'}$. If it exists, then so does $\varphi_{\gamma',\gamma}$ and $\varphi_{\gamma',\gamma} = \varphi_{\gamma,\gamma'}^{-1}$. Moreover, if $\gamma, \gamma', \gamma''$ are three elements as above and $\varphi_{\gamma,\gamma'}$ and $\varphi_{\gamma',\gamma''}$ exist, then so does $\varphi_{\gamma,\gamma''}$ and

$$\varphi_{\gamma,\gamma''} = \varphi_{\gamma',\gamma''} \circ \varphi_{\gamma,\gamma'}$$

The same can also be done with regular semi-simple elements of the Lie algebras of G and H and we will use the same notation for that case. Moreover, for any isomorphism $\varphi : T \to T'$ between tori, we will denote its dual isomorphism by $\widehat{\varphi} : \widehat{T'} \to \widehat{T}$.

2 Pure inner twists

Let A, B be reductive groups over F. A pure inner twist

$$(\psi, z) : A \to B$$

consists of an isomorphism of $\overline{F}$-groups $\psi : A \times \overline{F} \to B \times \overline{F}$ and an element $z \in Z^1(\Gamma, A)$ s.t.

$$\forall \sigma \in \Gamma : \psi^{-1}\sigma(\psi) = \mathrm{Ad}(z_\sigma)$$

We will from now on abbreviate "pure inner twist" to simply "twist", since these will be the only twists of reductive groups that will concern us here.

Starting from $(\psi, z) : A \to B$ we can form the inverse twist $(\psi, z)^{-1} : B \to A$, which is given by $(\psi^{-1}, \psi(z_\sigma^{-1}))$.

If $(\psi, z) : A \to B$ and $(\varphi, u) : B \to C$ are twists, then we can form their composition

$$(\varphi, u) \circ (\psi, z) : A \to C$$

which is given by $(\varphi \circ \psi, \psi^{-1}(u)z)$. One immediately checks

$$\begin{aligned}
(\psi, z) \circ (\psi, z)^{-1} &= (\mathrm{id}_B, 1) \\
(\psi, z)^{-1} \circ (\psi, z) &= (\mathrm{id}_A, 1) \\
[(\varphi, u) \circ (\psi, z)]^{-1} &= (\psi, z)^{-1} \circ (\varphi, u)^{-1} \\
(\chi, v) \circ [(\varphi, u) \circ (\psi, z)] &= [(\chi, v) \circ (\varphi, u)] \circ (\psi, z)
\end{aligned}$$

In particular, reductive groups and pure inner twists form a groupoid.

Let $(\psi, z), (\psi', z') : A \to B$ be two twists. They are called equivalent if $(\psi', z') \circ (\psi, z)^{-1}$ equals $(\mathrm{Ad}(g), g^{-1}\sigma(g))$ for some $g \in B(\overline{F})$. One immediately checks the equality

$$(\psi, z)^{-1} \circ (\mathrm{Ad}(g), g^{-1}\sigma(g)) \circ (\psi, z) = (\mathrm{Ad}(h), h^{-1}\sigma(h)), \quad h = \psi^{-1}(g)$$

from which it follows that this defines an equivalence relation on all inner twists which is invariant under composition and taking inverses.

We call the twist (ψ, z) strongly trivial if $z = 1$. In that case of course ψ is defined over F. We call the twist (ψ, z) trivial if it is equivalent to a strongly trivial twist. An example of a trivial twist is given by $(\mathrm{Ad}(g), g^{-1}\sigma(g)) : A \to A$ for any $g \in A(\overline{F})$. This twist is strongly trivial if and only if $g \in A(F)$.

2.1 Conjugacy along pure inner twists

Now consider a twist $(\psi, z) : A \to B$ and two elements $a \in A(F), b \in B(F)$. We call a, b conjugate (with respect to (ψ, z)) if there exists a twist (ψ', z') equivalent to (ψ, z) which maps a to b and is strongly trivial. We call a, b stably conjugate (with respect to (ψ, z)) if there exists a twist (ψ', z') equivalent to (ψ, z) which maps a to b and descends to a twist $A_a \to B_b$. The latter condition simply means that z' takes values in A_a (a-priori it only takes values in $\mathrm{Cent}(a, A)$).

Fact 2.1.1.

1. *Applied to the twist* $(\mathrm{id}, 1) : A \to A$ *the notions defined above coincide with the usual ones for the group* A
2. *If* $a \in A(F), b \in B(F)$ *are conjugate with respect to* $(\psi, z) : A \to B$*, then they are also stably conjugate and moreover* (ψ, z) *is a trivial twist*
3. *If* $a \in A(F), b \in B(F)$ *are conjugate (resp. stably conjugate) with respect to* $(\psi, z) : A \to B$*, then so are they with respect to any twist equivalent to* (ψ, z)*.*
4. *If* $a \in A(F)$ *and* $b \in B(F)$ *are conjugate (resp. stably-conjugate) with respect to* $(\psi, z) : A \to B$*, then so are they with respect to* $(\psi, z)^{-1} : B \to A$
5. *If* $(\psi, z) : A \to B$ *and* $(\varphi, u) : B \to C$ *are two twists and* $a \in A(F), b \in B(F), c \in C(F)$ *are s.t.* a, b *and* b, c *are conjugate (resp. stably-conjugate), then so are* a, c*.*

Let $a \in A(F)$ and $b \in B(F)$ be stably conjugate and assume that $\mathrm{Cent}(a, A)$ is connected. Choose a twist $(\varphi, u) : A \to B$ which is equivalent to (ψ, z) and sends a to b, and write $\mathrm{inv}(a, b)$ for the image of u in $H^1(\Gamma, A_a)$.

Fact 2.1.2.

1. *The element* $\mathrm{inv}(a, b)$ *is independent of the choice of the twist* (φ, u)*.*
2. *Applied to the twist* $(\mathrm{id}, 1) : A \to A$*,* inv *coincides with the usual definition for the group* A*.*
3. *The image of* $\mathrm{inv}(a, b)$ *under* $H^1(\Gamma, A_a) \to H^1(\Gamma, A)$ *equals the class of* z*.*

The proofs of both facts are straightforward and left to the reader.

Fact 2.1.3. *Let $a \in A(F)$, $b \in B(F)$ and $c \in C(F)$ be s.t. the inner twists*

$$A \overset{(\varphi,u)}{\longrightarrow} B \overset{(\psi,z)}{\longrightarrow} C$$

send a to b to c. Assume that $\mathrm{Cent}(a, A)$ is connected. Then

$$\mathrm{inv}(a, c) = \varphi^{-1}(\mathrm{inv}(b, c))\mathrm{inv}(a, b)$$

Proof: This follows at once from the composition formula for twists. □

Now let A be quasi-split. We consider a set I of triples (A^z, ψ_z, z) s.t. $(\psi_z, z) : A \to A^z$ is a twist. Put

$$A^I = \bigsqcup_{(A^z,\psi_z,z)} A^z$$

This is a variety over F (it will not be of finite type if I is infinite). For $a \in A^z$ and $b \in A^{z'}$ we obtain notions of conjugacy and stable conjugacy, namely those relative to the twist $(\psi_{z'}, z') \circ (\psi_z, z)^{-1}$. Thus we can talk about conjugacy classes and stable conjugacy classes of elements of $A^I(F)$. For the sake of abbreviation, we will call a twist $(\varphi, u) : A^z \to A^{z'}$ allowable if it is equivalent to $(\psi_{z'}, z') \circ (\psi_z, z)^{-1}$. Note that the set of allowable twists is invariant under composing and taking inverses.

Fact 2.1.4. *Every stable semi-simple conjugacy class of $A^I(F)$ meets $A(F)$.*

Proof: This is a consequence of a well known theorem of Steinberg, which implies that any maximal torus of a reductive group transfers to its quasi-split inner form. □

The rule $(A^z, \psi_z, z) \mapsto [z]$ defines a map $I \to H^1(\Gamma, A)$. Let $\bar{I}$ be the image of this map.

Lemma 2.1.5. *For each $a \in A(F)$ whose centralizer is connected, the map $b \mapsto \mathrm{inv}(a, b)$ is a bijection from the set of conjugacy classes inside the stable class of a in $A^I(F)$ to the preimage of $\bar{I}$ under $H^1(\Gamma, A_a) \to H^1(\Gamma, A)$.*

Remark: One can prove a similar lemma for $a \in A^{z'}(F)$ and any z', but the statement is more awkward and we will not need it.

Proof: Let $b \in A^z(F)$ and $b' \in A^{z'}(F)$ be conjugate elements belonging to the stable class of a. Thus there exists an allowable strongly trivial twist $(\chi, 1) : A^z \to A^{z'}$ mapping b to b'. Let $(\varphi, u) : A \to A^z$ be an allowable twist mapping a to b, thus $\mathrm{inv}(a, b) = [u]$. Then $(\chi, 1) \circ (\varphi, u)$ is an allowable twist $A \to A^{z'}$, mapping a to b', so $\mathrm{inv}(a, b')$ equals the class of the cocycle of $[(\chi, 1) \circ (\varphi, u)]$, which is also $[u]$. This shows that $\mathrm{inv}(a, b) = \mathrm{inv}(a, b')$ and we see that $b \mapsto \mathrm{inv}(a, b)$ gives a well-defined map on the set of conjugacy classes inside the stable class of a. By above facts it lands in the preimage of $\bar{I}$. We will show that it is injective. To that end, let $b \in A^z(F)$ and $b' \in A^{z'}(F)$ be s.t. $\mathrm{inv}(a, b) = \mathrm{inv}(a, b')$. Let $(\varphi, u) : A \to A^z$ and $(\varphi', u') : A \to A^{z'}$ be allowable twists sending a to b resp. b'. By assumption there exists $i \in A_a$ s.t. $u = i^{-1}u'\sigma(i)$. Then $(\varphi', u') \circ (\mathrm{Ad}(i), i^{-1}\sigma(i))$ is again an allowable twist $A \to A^{z'}$ sending a to b', and so replacing (φ', u') by it we achieve $u = u'$. Now it is clear that $(\varphi', u') \circ (\varphi, u)^{-1}$ is an allowable strongly trivial twist $A^z \to A^{z'}$ sending b to b', thus showing that b and b' are conjugate. Finally we show that the map $b \mapsto \mathrm{inv}(a, b)$ is surjective. Thus let $[u] \in H^1(\Gamma, A_a)$ be an element mapping to $[z] \in H^1(\Gamma, A)$, where $(A^z, \psi_z, z) \in I$. Then there exists $g \in A$ s.t. $u = g^{-1}z\sigma(g)$. Put $b = \psi_z(\mathrm{Ad}(g)a)$. One computes immediately that $b \in A^z(F)$. By construction $(\psi_z, z) \circ (\mathrm{Ad}(g), g^{-1}\sigma(g))$ maps a to b, which shows that a and b are stably conjugate and that $\mathrm{inv}(a,b) = g^{-1}z\sigma(g) = u$. □

2.2 Transfer factors for pure inner twists

Let G' be a reductive F-group, and $(H, s, {}^L\eta)$ an extended endoscopic triple for G'. Recall that from this data Langlands and Shelstad construct in [LS87] a function

$$\Delta : H_{G'-\mathrm{sr}} \times G'_{\mathrm{sr}} \times H_{G'-\mathrm{sr}} \times G'_{\mathrm{sr}} \to \mathbb{C}$$

called the canonical relative transfer factor. It is given as a product of five terms

$$\frac{\Delta_I(\gamma^H,\gamma)}{\Delta_I(\bar\gamma^H,\bar\gamma)}\frac{\Delta_{II}(\gamma^H,\gamma)}{\Delta_{II}(\bar\gamma^H,\bar\gamma)}\Delta_{III_1}(\gamma^H,\gamma,\bar\gamma^H,\bar\gamma)\frac{\Delta_{III_2}(\gamma^H,\gamma)}{\Delta_{III_2}(\bar\gamma^H,\bar\gamma)}\frac{\Delta_{IV}(\gamma^H,\gamma)}{\Delta_{IV}(\bar\gamma^H,\bar\gamma)}$$

each of which is constructed using auxiliary choices and encodes information from Galois-cohomology or harmonic analysis. Their product is, as its name suggests, independent of all auxiliary choices. The construction of the individual terms is technically involved, and we refer the reader to [LS87] for the details.

For the purposes of endoscopy one needs an absolute transfer factor, which is a function

$$\Delta : H_{G'-\mathrm{sr}} \times G'_{\mathrm{sr}} \to \mathbb{C}$$

with the property that $\Delta(\gamma^H,\gamma) = 0$ if (γ^H,γ) is not a pair of related elements and for any two pairs (γ^H,γ),$(\bar\gamma^H,\bar\gamma)$ of related elements

$$\frac{\Delta(\gamma^H,\gamma)}{\Delta(\bar\gamma^H,\bar\gamma)} = \Delta(\gamma^H,\gamma,\bar\gamma^H,\bar\gamma)$$

Such a function inherits from the relative transfer factor the following important properties.

- $\Delta(\gamma_1^H,\gamma) = \Delta(\gamma_2^H,\gamma)$ if γ_1^H and γ_2^H are stably conjugate.
- $\Delta(\gamma^H,\gamma_2) = \Delta_H^G(\gamma^H,\gamma_1) \cdot \langle \mathrm{inv}(\gamma_1,\gamma_2), \widehat{\varphi}_{\gamma_1,\gamma^H}(s)\rangle^{-1}$

where $\langle\rangle : H^1(\Gamma, T) \times \pi_0(\widehat{T}^\Gamma) \to \mathbb{C}^\times$ is the Tate-Nakayama pairing, and $T = \mathrm{Cent}(\gamma_1, G)$.

An absolute transfer factor is clearly determined by the relative transfer factor up to a non-zero complex scalar, but fixing a specific normalization requires further choices. If G' is quasi-split, one can obtain a normalization of the absolute transfer factor by fixing Whittaker data. This is the so called Whittaker normalization constructed in [KS99, 5.3] and is the correct normalization for the study of spectral questions via endoscopy. However, when G' is not quasi-split, there is no known natural normalization. It was observed by Kottwitz that if we consider instead of just G' a twist $(\psi, z) : G \to G'$ with G quasi-split, the additional data provided by the twist can be used to carry the Whittaker normalization (or in fact any fixed normalization) of the absolute transfer factor for G over to G'. In other words, given a fixed absolute transfer factor

$\Delta : H_{G-\mathrm{sr}} \times G_{\mathrm{sr}} \to \mathbb{C}$ we obtain an absolute transfer factor $\Delta : H_{G'-\mathrm{sr}} \times G'_{\mathrm{sr}} \to \mathbb{C}$ and the two transfer factors are compatible in a sense which we describe below.

Let G be a quasi-split F-group, $(\psi, z) : G \to G'$ a twist, and $(H, s, {}^L\eta)$ an extended triple for G. Then $(H, s, {}^L\eta)$ is also an extended triple for G'. This data gives canonical relative transfer factors $\Delta_H^G(\gamma^H, \gamma, \bar{\gamma}^H, \bar{\gamma})$ for (G, H) and $\Delta_H^{G'}(\gamma^H, \gamma', \bar{\gamma}^H, \bar{\gamma}')$ for (G', H). Let $\Delta_H^G(\gamma^H, \gamma)$ be an arbitrary normalization for the absolute transfer factor for (G, H). For any pair $\gamma^H \in H(F)$ and $\gamma' \in G'(F)$ of strongly G'-regular related elements we choose an element $\gamma \in G(F)$ stably conjugate to γ' (which exists by Fact 2.1.4) and define

$$\Delta_H^{G'}(\gamma^H, \gamma') = \Delta_H^G(\gamma^H, \gamma) \cdot \langle \mathrm{inv}(\gamma, \gamma'), \widehat{\varphi}_{\gamma, \gamma^H}(s) \rangle^{-1}$$

where $\langle\rangle : H^1(\Gamma, T) \times \pi_0(\widehat{T}^\Gamma) \to \mathbb{C}^\times$ is the Tate-Nakayama pairing, and $T = \mathrm{Cent}(\gamma, G)$.

Lemma 2.2.1. *$\Delta_H^{G'}(\cdot, \cdot)$ is well defined.*

Proof: To show that $\Delta_H^{G'}(\gamma^H, \gamma')$ is independent of the choice of γ, let $\tilde{\gamma} \in G(F)$ be another element in the stable class of γ'. We know from [LS87]

$$\Delta_H^G(\gamma^H, \tilde{\gamma}) = \Delta_H^G(\gamma^H, \gamma) \langle \mathrm{inv}(\gamma, \tilde{\gamma}), \widehat{\varphi}_{\gamma, \gamma^H}(s) \rangle^{-1}$$

On the other hand if $(\varphi, u) : A \to A$ is an admissible twist mapping $\tilde{\gamma}$ to γ, then $\varphi_{\tilde{\gamma}, \gamma^H} = \varphi_{\gamma, \gamma^H} \circ \varphi$ and by functoriality of the Tate-Nakayama pairing we get

$$\langle \mathrm{inv}(\tilde{\gamma}, \gamma'), \widehat{\varphi}_{\tilde{\gamma}, \gamma^H}(s) \rangle^{-1} = \langle \varphi(\mathrm{inv}(\tilde{\gamma}, \gamma')), \widehat{\varphi}_{\gamma, \gamma^H}(s) \rangle^{-1}$$

Thus

$$\begin{aligned}
&\Delta_H^G(\gamma^H, \tilde{\gamma}) \langle \mathrm{inv}(\tilde{\gamma}, \gamma'), \widehat{\varphi}_{\tilde{\gamma}, \gamma^H}(s) \rangle^{-1} = \\
&\Delta_H^G(\gamma^H, \gamma) \langle \mathrm{inv}(\gamma, \tilde{\gamma}), \widehat{\varphi}_{\gamma, \gamma^H}(s) \rangle^{-1} \; \langle \varphi(\mathrm{inv}(\tilde{\gamma}, \gamma')), \widehat{\varphi}_{\gamma, \gamma^H}(s) \rangle^{-1} = \\
&\Delta_H^G(\gamma^H, \gamma) \langle \mathrm{inv}(\gamma, \gamma'), \widehat{\varphi}_{\gamma, \gamma^H}(s) \rangle^{-1}
\end{aligned}$$

by Fact 2.1.3. □

Proposition 2.2.2. *$\Delta_H^{G'}(\cdot, \cdot)$ is an absolute transfer factor for (G', H).*

The proof of this proposition will be given in section 2.5 after the necessary cohomological facts have been gathered.

Now let I be a set of pure inner twists for G and construct G^I as above. Taking the disjoint union over I of all functions $\Delta_H^{G^z}$ we obtain a function

$$\Delta_H^{G^I} : H_{G-\mathrm{sr}}(F) \times G^I_{\mathrm{sr}}(F) \longrightarrow \mathbb{C}^\times$$

Fact 2.2.3. *For all stably conjugate $\gamma, \gamma' \in G^I_{\mathrm{sr}}(F)$ and $\gamma^H \in H_{G-\mathrm{sr}}(F)$ we have*

$$\Delta_H^{G^I}(\gamma^H, \gamma') = \Delta_H^{G^I}(\gamma^H, \gamma) \cdot \langle \mathrm{inv}(\gamma, \gamma'), \widehat{\varphi}_{\gamma, \gamma^H}(s) \rangle^{-1}$$

Proof: Let $\gamma_0 \in G(F)$ be an element stably conjugate to γ (it exists by Fact 2.1.4). Then by construction of $\Delta_H^{G^I}$ we have

$$\Delta_H^{G^I}(\gamma^H, \gamma') \Delta_H^{G^I}(\gamma^H, \gamma)^{-1} = \langle \mathrm{inv}(\gamma_0, \gamma') \mathrm{inv}(\gamma_0, \gamma)^{-1}, \widehat{\varphi}_{\gamma_0, \gamma_H}(s) \rangle^{-1}$$

By Fact 2.1.3 the right hand side equals $\langle \varphi_{\gamma, \gamma_0}(\mathrm{inv}(\gamma, \gamma')), \widehat{\varphi}_{\gamma_0, \gamma_H}(s) \rangle^{-1}$ and the claim now follows from the functoriality of the Tate-Nakayama pairing. □

Remark: We see in particular the the function $\gamma \mapsto \Delta_H^{G^I}(\gamma^H, \gamma)$ is constant on the conjugacy classes of $G^I(F)$.

2.3 Cohomological lemmas I

We need to recall some well-known basic facts about Tate-Nakayama duality as used in endoscopy. For this, we will deviate from the notation established so far in order to make the statements in their natural generality. Let E/F be a finite Galois extension of local fields of characteristic 0, $\Gamma = \mathrm{Gal}(E/F)$, $u_{E/F} \in H^2(\Gamma, E^\times)$ the canonical class of the extension E/F, T a torus over F which splits over E, and $\widehat{T}$ its dual complex torus. In the following, we will use the notation H_{Tate} to denote Tate-cohomology groups.

Lemma 2.3.1. *We have the exact sequences*

$$1 \longrightarrow (\widehat{T}^\Gamma)^\circ \longrightarrow \widehat{T}^\Gamma \longrightarrow H^1(\Gamma, X_*(\widehat{T})) \longrightarrow 0$$

$$0 \longrightarrow X^*(\widehat{T}/\widehat{T}^\Gamma) \longrightarrow X^*(\widehat{T}/(\widehat{T}^\Gamma)^\circ) \longrightarrow H^{-1}_{\mathrm{Tate}}(\Gamma, X^*(\widehat{T})) \longrightarrow 0$$

Proof: For the first one, tensor the exponential sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{e^{2\pi i z}} \mathbb{C}^\times \to 1$$

with $X_*(T)$ and take Γ-invariants, noting that the image of $[X_*(\widehat{T}) \otimes \mathbb{C}]^\Gamma = \mathrm{Lie}(\widehat{T})^\Gamma$ in $\widehat{T}^\Gamma$ under the exponential map is $(\widehat{T}^\Gamma)^\circ$.

For the second one, observe that an element of $X^*(\widehat{T})$ is in the kernel of the norm map precisely when it is trivial on $(\widehat{T}^\Gamma)^\circ$ and in the augmentation submodule precisely when it is trivial on $\widehat{T}^\Gamma$. $\square$

Lemma 2.3.2. *The following three pairings*

$$H^1(\Gamma, X_*(\widehat{T})) \otimes H^{-1}_{\mathrm{Tate}}(\Gamma, X^*(\widehat{T})) \to \mathbb{C}^\times$$

are equal:

1. *The pairing induced by the standard pairing $\widehat{T} \times X^*(\widehat{T}) \to \mathbb{C}^\times$ via the above sequences.*

2. $$\begin{array}{c} H^1(\Gamma, X_*(\widehat{T})) \\ \otimes \\ H^{-1}_{\mathrm{Tate}}(\Gamma, X^*(\widehat{T})) \end{array} \xrightarrow{\cup} H^0_{\mathrm{Tate}}(\Gamma, \mathbb{Z}) = \mathbb{Z}/|\Gamma|\mathbb{Z} \xrightarrow{\cdot|\Gamma|^{-1}} \mathbb{Q}/\mathbb{Z} \xrightarrow{e^{2\pi i z}} \mathbb{C}^\times$$

3. $$\begin{array}{ccc} H^1(\Gamma, X_*(\widehat{T})) & = & H^1(\Gamma, X^*(T)) \\ & & \otimes \\ H^{-1}_{\mathrm{Tate}}(\Gamma, X_*(T)) & \xrightarrow{\cup u_{E/F}} & H^1(\Gamma, T) \end{array} \xrightarrow{\cup} H^2(\Gamma, E^\times) \xrightarrow{\mathrm{inv}} \mathbb{Q}/\mathbb{Z} \xrightarrow{e^{2\pi i z}} \mathbb{C}^\times$$

Proof: The equality of the pairings in 2. and 3. is an immediate consequence of local class field theory, more precisely of the following commutative square.

$$\begin{array}{ccc} H^2(\Gamma, E^\times) & \xleftarrow{\cup u_{E/F}} & H^0_{\mathrm{Tate}}(\Gamma, \mathbb{Z}) \\ \mathrm{inv} \downarrow & & \| \\ |\Gamma|^{-1}\mathbb{Z}/\mathbb{Z} & \xleftarrow{\cdot|\Gamma|^{-1}} & \mathbb{Z}/|\Gamma|\mathbb{Z} \end{array}$$

In order to relate pairings 1. and 2. take $t \in \widehat{T}^\Gamma$ and $\varphi \in X^*(\widehat{T}/(\widehat{T}^\Gamma)^\circ)$. Choose $z \in \mathrm{Lie}(\widehat{T}) = X_*(\widehat{T}) \otimes \mathbb{C}$ mapping to t under the exponential map. Then the image of t in $H^1(\Gamma, X_*(\widehat{T}))$ is represented by the cocycle $z_\tau : \tau \mapsto \tau z - z$. Now using the appropriate cup product formula and denoting the canonical pairing $X^*(\widehat{T}) \otimes X_*(\widehat{T}) \to \mathbb{Z}$ by $\langle\rangle$ we compute

$$z_\tau \cup \varphi = \sum_{\tau \in \Gamma} \langle \tau\varphi, \tau z - z \rangle = |\Gamma| \langle \varphi, z \rangle$$

It follows that

$$\exp(2\pi i |\Gamma|^{-1}(\tau z - z) \cup \varphi) = \exp(2\pi i \langle \varphi, z \rangle) = \langle \varphi, t \rangle$$

□

Lemma 2.3.3. *Assume that E/F is either an unramified extension of p-adic fields, or $\mathbb{C}/\mathbb{R}$. In the p-adic case, let $\pi \in E^\times$ be a uniformizer and $\sigma \in \Gamma$ be the Frobenius element. In the real case, let $\pi = -1$ and $\sigma \in \Gamma$ be complex conjugation. Then the map*

$$[\lambda] \mapsto \lambda(\pi)$$

induces the same isomorphism

$$[X_*(T)_\Gamma]_{\mathrm{tor}} = H^{-1}_{\mathrm{Tate}}(\Gamma, X_*(T)) \to H^1(\Gamma, T)$$

as the isomorphism given by $\cup u_{E/F}$. Here we regard $\lambda(\pi) \in T(E)$ as the class in $H^1(\Gamma, T)$ represented by the unique element $z \in Z^1(\Gamma, T)$ s.t. $z(\sigma) = \lambda(\pi)$.

Proof: By definition, $H^{-1}_{\mathrm{Tate}}(\Gamma, X_*(T)) = \mathrm{Ker}(N : X_*(T) \to X_*(T))/IX_*(T)$ where N is the norm map and $I \subset \mathbb{Z}[\Gamma]$ is the augmentation ideal. If $\lambda \in X_*(T)$ is torsion modulo $IX_*(T)$, then some multiple of it is killed by N, and since $X_*(T)$ is torsion-free this means that λ itself is killed by N. Thus

$$[X_*(T)_\Gamma]_{\mathrm{tor}} \subset H^{-1}_{\mathrm{Tate}}(\Gamma, X_*(T))$$

The converse inclusion follows from the finiteness of $H^{-1}_{\mathrm{Tate}}(\Gamma, X_*(T))$. This justifies the first equality.

It is well known from local class field theory ([MiCFT, III.2.]) that the fundamental class of E/F is represented by the 2-cocycle

$$(\sigma^a, \sigma^b) \mapsto \begin{cases} 1 & , 0 \le a + b < |\Gamma| \\ \pi & , \text{else} \end{cases}$$

If $\lambda \in X_*(T)$ is torsion modulo $IX_*(T)$, then applying the appropriate cup-product formula one sees

$$([\lambda] \cup u_{E/F})(\sigma) = \sum_{i=0}^{|\Gamma|-1} \sigma^{i+1} \lambda(\pi^{\mathrm{char}_{\{i+1 \ge |\Gamma|\}}}) = \lambda(\pi)$$

□

This isomorphism is sometimes called the Tate-Nakayama isomorphism. We will denote it by TN. In the case that E/F is an unramified extension of p-adic fields, DeBacker and Reeder construct in [DR09, Cor 2.4.3] another isomorphism

$$[X_*(T)_\Gamma]_{\mathrm{tor}} \to H^1(\Gamma, T)$$

We will call this isomorphism DR. It turns out that these two isomorphisms are almost identical, namely

Lemma 2.3.4. *The following diagram commutes*

$$\begin{array}{ccc} [X_*(T)_\Gamma]_{\mathrm{tor}} & \xrightarrow{\lambda \mapsto -\lambda} & [X_*(T)_\Gamma]_{\mathrm{tor}} \\ & {\searrow}^{\mathrm{DR}} \quad {\swarrow}^{\mathrm{TN}} & \\ & H^1(\Gamma, T) & \end{array}$$

Proof: By construction, DR sends $[\lambda] \in [X_*(T)_\Gamma]_{\mathrm{tor}}$ to the class in $H^1(\Gamma, T)$ of the unique cocycle z whose value at Fi equals $t_\lambda = \lambda(\pi)$, while TN sends $[\lambda]$ to the class in $H^1(\Gamma, T)$ of the unique cocycle z' whose value at $\sigma = \mathrm{Fi}^{-1}$ equals $\lambda(\pi)$. But

$$z(\sigma) = \sigma(z(\mathrm{Fi})^{-1}) = \sigma(\lambda(\pi)^{-1}) = \sigma(\lambda)^{-1}(\pi)$$

Since λ and $\sigma(\lambda)$ give rise to the same element of $X_*(T)_\Gamma$, the lemma follows. $\square$

2.4 Homotopically trivial twists and cup-products

Let $T \subset G$ be a maximal torus. Then $T \to T_{\mathrm{ad}}$ is a resolution of the diagonalizable group $Z(G)$ by tori. Such a resolution is useful in connection with Langlands duality, because $Z(G)$ may be disconnected. If $S \subset G$ is another maximal torus, then the quasi-isomorphisms

$$[T \to T_{\mathrm{ad}}] \leftarrow Z(G) \to [S \to S_{\mathrm{ad}}]$$

induces an isomorphism of Tate-cohomology

$$H^1(\Sigma, T \to T_{\mathrm{ad}}) \to H^1(\Sigma, S \to S_{\mathrm{ad}})$$

where Σ is the Galois-group of a finite extension splitting both S and T. It will be important for computations to have an explicit formula for this isomorphism on the level of cocycles. To obtain one, let $g \in G$ be s.t. $\mathrm{Ad}(g)T = S$. Then $\mathrm{Ad}(g)$ gives an isomorphism from the complex $T \to T_{\mathrm{ad}}$ to a Galois-twist of the complex $S \to S_{\mathrm{ad}}$. It will turn out that the twisted Galois-action is in a suitable sense homotopic to the original one, and using this homotopy one can write down the sought explicit expression for the isomorphism on cohomology. In fact, we will show that there is a diagram

$$\begin{array}{ccc} H^1(\Sigma, T \to T_{\mathrm{ad}}) & \longrightarrow & H^1(\Sigma, S \to S_{\mathrm{ad}}) \\ \uparrow & & \uparrow \\ H^{-1}_{\mathrm{Tate}}(\Sigma, X_*(T) \to X_*(T_{\mathrm{ad}})) & \longrightarrow & H^{-1}_{\mathrm{Tate}}(\Sigma, X_*(S) \to X_*(S_{\mathrm{ad}})) \end{array}$$

where the horizontal maps are isomorphisms given explicitly on the level of cocycles, and the vertical maps are the Tate-Nakayama-isomorphisms. Problems of this nature arise also in connection with endoscopy, in particular in the proof of Proposition 2.2.2. For example, we may be interested in complexes of the form $T_{\mathrm{sc}} \to T$, or $T^H_{\mathrm{sc}} \to T_{\mathrm{sc}}$, or $T^H_{\mathrm{sc}} \to T$, that can be associated to an admissible isomorphism $T^H \to T$ from a torus in an endoscopic group H to a torus in G. To handle these situations, we will now study homotopically trivial twists in a general setting.

Let Σ, Ω be finite groups, such that Σ acts on Ω by automorphisms. We will write this action as conjugation, i.e. $g : w \mapsto gwg^{-1}$. Let $A = A^{-1} \xrightarrow{\varphi} A^0$ be a length-two complex of abelian groups with $\Omega \rtimes \Sigma$-action. If w_σ is a 1-cocycle of Σ in Ω then we can twist the action of Σ on both Ω and the complex A to obtain a new action of Σ on these, which we will denote using the symbol $*$, i.e. $g * w * g^{-1}$ or $g * a$, respectively.

When Σ acts on A via the original action, we will denote the Tate-hypercochains and Tate-hypercocycles of Σ in A of degree r by $C^r_{\mathrm{Tate}}(\Sigma, A)$ and $Z^r_{\mathrm{Tate}}(\Sigma, A)$ respectively. When Σ acts on A via the $*$-action, we will write $C^r_{\mathrm{Tate}}(\Sigma_*, A)$ and $Z^r_{\mathrm{Tate}}(\Sigma_*, A)$ instead.

Recall (see for example [KS99, §A]) that for the given complex A we have $C^r_{\mathrm{Tate}}(\Sigma, A) = \mathrm{Hom}_\Sigma(P_{r+1}, A^{-1}) \oplus \mathrm{Hom}_\Sigma(P_r, A^0)$, where P is the standard complete resolution of Σ. The following notation will be useful to describe elements of $P_{-1} = \mathrm{Hom}(\mathbb{Z}[\Sigma], \mathbb{Z})$: For $\sigma \in \Sigma$ we denote by $\sigma^\vee \in \mathrm{Hom}(\mathbb{Z}[\Sigma], \mathbb{Z})$ the function which sends σ to 1 and all other elements of Σ to 0.

Proposition 2.4.1. *Assume that for each $\sigma \in \Sigma$ there is a homotopy k_σ s.t.*

$$\begin{aligned} w_\sigma - 1 &= \varphi k_\sigma + k_\sigma \varphi \\ k_{\sigma\tau} &= k_\sigma + w_\sigma \sigma(k_\tau) \end{aligned}$$

Then the maps

$$\ddagger : Z^1_T(\Sigma, A) \to Z^1_T(\Sigma_*, A), \qquad (f, g) \mapsto (F, G)$$

$$\sharp : Z^{-1}_T(\Sigma, A) \to Z^{-1}_T(\Sigma_*, A^{-1}), \qquad (p, q) \mapsto (P, Q)$$

where

$$\begin{aligned} F(x,y,z) &= w_x f(x,y,z) - (k_y - k_x) g(y,z) \\ G(x,y) &= w_x g(x,y) \\ P(x) = P(1) &= p(1) - \sum\nolimits_{\sigma\in\Sigma} k_\sigma q(\sigma^\vee) \\ Q(x^\vee) &= w_x q(x^\vee) \end{aligned}$$

are well-defined isomorphisms of abelian groups which respect coboundaries and therefore induce isomorphisms on the level of cohomology.

Proof: This is a direct computation and is left to the reader. □

Remark: It will be useful for later to have the formula for $\ddagger$ also in terms of inhomogenous cochains. If $\bar{f}(\sigma, \tau) = f(1, \sigma, \sigma\tau)$ and $\bar{g}(\sigma) = g(1, \sigma)$ then we have

$$\bar{F}(\sigma, \tau) = \bar{f}(\sigma, \tau) - k_\sigma(\bar{g}(\tau)), \qquad \bar{G}(\sigma) = \bar{g}(\sigma)$$

Let B be another abelian group with Σ-action. We extend this to a $\Omega \rtimes \Sigma$-action by letting Ω act trivially, and give the complex $A \otimes B$ the diagonal $\Omega \rtimes \Sigma$-action. We define a cup-product pairing

$$C^r_{\mathrm{Tate}}(\Sigma, A) \otimes C^s_{\mathrm{Tate}}(\Sigma, B) \to C^{r+s}_{\mathrm{Tate}}(\Sigma, A \otimes B)$$

by

$$[f \oplus g] \cup h = [f \cup h] \oplus [g \cup h]$$

One checks at once that for $\alpha \in C^r_{\mathrm{Tate}}(\Sigma, A)$ and $\beta \in C^s_{\mathrm{Tate}}(\Sigma, B)$ the equality

$$\partial(\alpha \cup \beta) = \partial(\alpha) \cup \beta + (-1)^r \alpha \cup \partial(\beta)$$

holds, and so we obtain a well defined pairing on the level of cohomology. Moreover it is evident that for $h \in C^s_{\mathrm{Tate}}(\Sigma, B)$ the following diagram commutes

$$\begin{array}{ccccc}
C^{r+s}_{\mathrm{Tate}}(\Sigma, A^0 \otimes B) & \longrightarrow & C^{r+s}_{\mathrm{Tate}}(\Sigma, A \otimes B) & \longrightarrow & C^{r+s+1}_{\mathrm{Tate}}(\Sigma, A^{-1} \otimes B) \\
\uparrow{\scriptstyle \cup h} & & \uparrow{\scriptstyle \cup h} & & \uparrow{\scriptstyle \cup h} \\
C^r_{\mathrm{Tate}}(\Sigma, A^0) & \longrightarrow & C^r_{\mathrm{Tate}}(\Sigma, A) & \longrightarrow & C^{r+1}_{\mathrm{Tate}}(\Sigma, A^{-1})
\end{array}$$

Clearly $k_\sigma \otimes \mathrm{id}_B : A \otimes B \to A \otimes B$ is a homotopy with the same properties as k_σ, and so we obtain maps ‡ and ♯ for the Tate-cohomology of the complex $A \otimes B$ as well.

Proposition 2.4.2. *For any $\beta \in Z^2_T(\Sigma, B)$ the diagram*

$$\begin{array}{ccc}
Z^1(\Sigma, A \otimes B) & \xrightarrow{\ddagger} & Z^1(\Sigma_*, A \otimes B) \\
{\scriptstyle \cup\beta}\uparrow & & \uparrow{\scriptstyle \cup\beta} \\
Z^{-1}_{\mathrm{Tate}}(\Sigma, A) & \xrightarrow{\sharp} & Z^{-1}_{\mathrm{Tate}}(\Sigma_*, A)
\end{array}$$

commutes up to a coboundary, hence the induced diagram on cohomology commutes.

Proof: This is a lengthy and tedious but straightforward computation. □

Remark: To apply these results to the problem described in the beginning of this section, let E/F be an extension splitting T, S with Galois group Σ, and let $\mathrm{Ad}g$ identify $[T \to T_{\mathrm{ad}}]$ with the twist of $[S \to S_{\mathrm{ad}}]$ by an element of $Z^1(\Sigma, \Omega)$, where $\Omega = \Omega(S, G) \cong \Omega(S_{\mathrm{ad}}, G_{\mathrm{ad}})$. Put $A = X_*(S) \to X_*(S_{\mathrm{ad}})$, $B = E^\times$. For any $s_{\mathrm{ad}} \in S_{\mathrm{ad}}$ we choose a lift $s \in S$ and define $k_\sigma(s_{\mathrm{ad}}) = w_\sigma(s)s^{-1}$. This defines a homotopy $w_\sigma \to 1$ which is necessarily unique and thus satisfies the assumptions in Proposition 2.4.1. The corresponding homotopy on A is $X_*(k_\sigma)$. Now Propositions 2.4.1 and 2.4.2 provide the required commutative diagram. A similar procedure works in the other cases mentioned, and we will see an example in the next section.

2.5 Proof of Proposition 2.2.2

In this section we are going to prove Proposition 2.2.2 using the cohomological facts laid out in the previous two sections. In order to show that $\Delta_H^{G'}(\gamma^H, \gamma')$ is an absolute transfer factor for (G', H) we must prove for any two strongly G-regular related pairs (γ_1^H, γ_1') and (γ_2^H, γ_2') in $H(F) \times G'(F)$ the equality

$$\frac{\Delta_H^{G'}(\gamma_1^H, \gamma_1')}{\Delta_H^{G'}(\gamma_2^H, \gamma_2')} = \Delta_H^{G'}(\gamma_1^H, \gamma_1', \gamma_2^H, \gamma_2')$$

where the right hand side is the canonical relative transfer factor for (G', H). By construction of $\Delta_H^{G'}$ this is equivalent to proving

$$\frac{\langle \mathrm{inv}(\gamma_1, \gamma_1'), \widehat{\varphi}_{\gamma_1, \gamma_1^H}(s) \rangle^{-1}}{\langle \mathrm{inv}(\gamma_2, \gamma_2'), \widehat{\varphi}_{\gamma_2, \gamma_2^H}(s) \rangle^{-1}} = \frac{\Delta_H^{G'}(\gamma_1^H, \gamma_1', \gamma_2^H, \gamma_2')}{\Delta_H^{G}(\gamma_1^H, \gamma, \gamma_2^H, \gamma_2)}$$

where $\gamma_1 \in G(F)$ is any element in the stable class of γ_1', and $\gamma_2 \in G(F)$ is any element in the stable class of γ_2'. Applying [LS87, Lemma 4.2.A] we need to show

$$\frac{\langle \mathrm{inv}(\gamma_1,\gamma_1'), \widehat{\varphi}_{\gamma_1,\gamma_1^H}(s)\rangle^{-1}}{\langle \mathrm{inv}(\gamma_2,\gamma_2'), \widehat{\varphi}_{\gamma_2,\gamma_2^H}(s)\rangle^{-1}} = \left\langle \mathrm{inv}\left(\frac{\gamma_1,\gamma_1'}{\gamma_2,\gamma_2'}\right), s_U\right\rangle \tag{2.1}$$

The right hand side of this equality is constructed in [LS87, §3.4]. Let us briefly discuss it. Let T_i be the centralizer of γ_i, T_i^H be the centralizer of γ_i^H, and $p_i \in G_{\mathrm{der}}(\overline{F})$ be s.t. $\gamma_i' = \psi(\mathrm{Ad}(p_i)\gamma_i)$. Then $\mathrm{inv}(\gamma_i,\gamma_i') = p_i^{-1}z_\sigma\sigma(p_i)$. Choose an arbitrary admissible isomorphism $\varphi_S : S^H \to S$ from a maximal torus $S^H \subset H$ to a maximal torus $S \subset G$. Let U_S and $U_{1,2}$ be the cokernels of

$$Z(G_{\mathrm{sc}}) \to S_{\mathrm{sc}} \times S_{\mathrm{sc}}, \qquad Z(G_{\mathrm{sc}}) \to [T_1]_{\mathrm{sc}} \times [T_2]_{\mathrm{sc}}$$

respectively, where in both cases the map is given by $z \mapsto (z, z^{-1})$. The product map $S_{\mathrm{sc}} \times S_{\mathrm{sc}} \to S_{\mathrm{sc}}$ factors through the isogeny $S_{\mathrm{sc}} \times S_{\mathrm{sc}} \to U_S$. The dual of the resulting map $U_S \to S_{\mathrm{sc}}$ will be called $\Delta : \widehat{S}_{\mathrm{ad}} \to \widehat{U_S}$.

We may choose elements $g_1, g_2 \in G(\overline{F})$, $h_1, h_2 \in H(\overline{F})$ which give rise to the following diagram of $\overline{F}$-tori

$$\begin{array}{ccccc} T_1^H & \xrightarrow{\mathrm{Ad}(h_1)} & S^H & \xleftarrow{\mathrm{Ad}(h_2)} & T_2^H \\ \Big\downarrow{\scriptstyle \varphi_{\gamma_1^H,\gamma_1}} & & \Big\downarrow{\scriptstyle \varphi_S} & & \Big\downarrow{\scriptstyle \varphi_{\gamma_2^H,\gamma_2}} \\ T_1 & \xrightarrow{\mathrm{Ad}(g_1)} & S & \xleftarrow{\mathrm{Ad}(g_2)} & T_2 \end{array} \tag{2.2}$$

Then $\mathrm{Ad}(g_1, g_2)$ gives rise to an isomorphism of $\overline{F}$-tori $U_{1,2} \to U_S$. The homomorphism

$$Z(\widehat{H})/Z(\widehat{G}) \longrightarrow \widehat{S^H}/Z(\widehat{G}) \longrightarrow \widehat{S}_{\mathrm{ad}} \xrightarrow{\ \Delta\ } \widehat{U_S} \xrightarrow{\widehat{\mathrm{Ad}}(g_1,g_2)} \widehat{U_{1,2}}$$

is Γ-equivariant, and s_U is the image of s under it.

Let z_σ^{sc} be a 1-cochain $\Gamma \to G_{\mathrm{sc}}(\overline{F})$ which lifts z_σ, and choose lifts $p_{i,\mathrm{sc}} \in G_{\mathrm{sc}}(\overline{F})$ of p_i. The cochain

$$\Gamma \to U_{1,2}, \quad \sigma \mapsto [(p_{1,\mathrm{sc}}^{-1}z_\sigma\sigma(p_{1,\mathrm{sc}}))^{-1}, p_{2,\mathrm{sc}}^{-1}z_\sigma\sigma(p_{2,\mathrm{sc}})]$$

is in fact a 1-cocycle, and this 1-cocycle is $\mathrm{inv}\left(\frac{\gamma_1,\gamma_1'}{\gamma_2,\gamma_2'}\right)$.

We have now described both sides of Equation (2.1) and turn to its proof. Let E/F be a finite Galois-extension s.t. the tori T_1, T_2, S split over E and the elements $p_i, p_{i,\mathrm{sc}}, z_\sigma, z_\sigma^{\mathrm{sc}}, g_i, h_i$ are defined over E, and let Σ be its Galois-group.

Applying $H^{-1}_{\mathrm{Tate}}(\Sigma, X^*(-))$ to the maps of complexes

$$\begin{bmatrix} 1 \\ \uparrow \\ Z(\widehat{H}) \end{bmatrix} \longrightarrow \begin{bmatrix} \widehat{S}^H/Z(\widehat{H}) \\ \uparrow \\ \widehat{S}^H \end{bmatrix} \underset{\widehat{\varphi}_S^{-1}}{\longrightarrow} \begin{bmatrix} \widehat{S}^H/Z(\widehat{H}) \\ \uparrow \widehat{\varphi}_S \\ \widehat{S} \end{bmatrix}$$

we obtain

$$H^{-1}_{\mathrm{Tate}}(\Sigma, X_*(S^{H_{\mathrm{sc}}}) \to X_*(S)) \to H^{-1}_{\mathrm{Tate}}(\Sigma, X^*(Z(\widehat{H}))) \hookrightarrow X^*(Z(\widehat{H})^\Sigma)$$

In this way, $s \in Z(\widehat{H})^\Sigma$ gives rise to a character $\tilde{s}$ on $H^{-1}_{\mathrm{Tate}}(\Sigma, X_*(S^{H_{\mathrm{sc}}}) \to X_*(S))$, and via the inverse of the Tate-Nakayama-isomorphism

$$\mathrm{NT} : H^1(\Sigma, S^{H_{\mathrm{sc}}} \to S) \to H^{-1}_{\mathrm{Tate}}(\Sigma, X_*(S^{H_{\mathrm{sc}}}) \to X_*(S))$$

we obtain a character $\tilde{s} \circ \mathrm{NT}$ on $H^1(\Sigma, S^{H_{\mathrm{sc}}} \to S)$.

On the other hand we have the following diagram of complexes

$$\begin{array}{ccccccc}
\left[\begin{matrix} 1 \\ \downarrow \\ T_1 \times T_2 \end{matrix}\right] & \longrightarrow & \left[\begin{matrix} T_1^{H_{\mathrm{sc}}} \times T_2^{H_{\mathrm{sc}}} \\ \downarrow \\ T_1 \times T_2 \end{matrix}\right] & \xrightarrow[\mathrm{Ad}(g_1,g_2)]{\mathrm{Ad}(h_1,h_2)} & \left[\begin{matrix} S^{H_{\mathrm{sc}}} \times S^{H_{\mathrm{sc}}} \\ \downarrow \\ S \times S \end{matrix}\right] & \xrightarrow{\cdot} & \left[\begin{matrix} S^{H_{\mathrm{sc}}} \\ \downarrow \\ S \end{matrix}\right] \\
(2.3) & & & & & & \uparrow \\
\left[\begin{matrix} 1 \\ \downarrow \\ U_{1,2} \end{matrix}\right] & \longrightarrow & \left[\begin{matrix} T_1^{H_{\mathrm{sc}}} \times T_2^{H_{\mathrm{sc}}} \\ \downarrow \\ U_{1,2} \end{matrix}\right] & \xrightarrow[\mathrm{Ad}(g_1,g_2)]{\mathrm{Ad}(h_1,h_2)} & \left[\begin{matrix} S^{H_{\mathrm{sc}}} \times S^{H_{\mathrm{sc}}} \\ \downarrow \\ U_S \end{matrix}\right] & \xrightarrow{\cdot} & \left[\begin{matrix} S^{H_{\mathrm{sc}}} \\ \downarrow \\ S^{G_{\mathrm{sc}}} \end{matrix}\right]
\end{array}$$

We want to apply the functor $H^1(\Sigma, -)$ to this diagram to obtain

$$H^1(\Sigma, T_1 \times T_2) \to H^1(\Sigma, S^{H_{\mathrm{sc}}} \to S) \leftarrow H^1(\Sigma, U_{1,2})$$

For this, we need to show that the maps $\frac{\mathrm{Ad}(h_1,h_2)}{\mathrm{Ad}(g_1,g_2)}$ in both rows of the diagram translate the Σ-action on their source to a homotopically trivial twist of the Σ-action on their target, and then apply Proposition 2.4.1. We will show this for the top row, the argument for the bottom row being similar. Moreover it is enough to treat the case of the map

$$[T_1^{H_{\mathrm{sc}}} \to T_1] \to [S^{H_{\mathrm{sc}}} \to S]$$

It is clear that this map translates the Σ-action on the left hand side to a twist of the Σ-action on the right hand side by an element $w_\sigma \in Z^1(\Sigma, \Omega(S^H, H))$. For $s \in S$ put $k_\sigma(s) = w_\sigma(s_{\mathrm{sc}})s_{\mathrm{sc}}^{-1}$, where $s_{\mathrm{sc}} \in S^{H_{\mathrm{sc}}}$ is any element whose image in $S^{H_{\mathrm{ad}}}$ is the same as that of $\varphi_S^{-1}(s)$. This gives a well-defined homotopy from w_σ to 1. Moreover, since the map $S^{H_{\mathrm{sc}}} \to S$ has a finite kernel and connected cokernel, this homotopy is unique, which forces the collection $(k_\sigma)_\sigma$ to satisfy the conditions of Proposition 2.4.1.

Equation (2.1) will be proved once we establish the following statements:

1. The images in $H^1(\Sigma, S^{H_{\mathrm{sc}}} \to S)$ of $(\mathrm{inv}(\gamma_1, \gamma_1')^{-1}, \mathrm{inv}(\gamma_2, \gamma_2'))$ and $\mathrm{inv}(\frac{\gamma_1, \gamma_1'}{\gamma_2, \gamma_2'})$ coincide.
2. The pull-back of $\tilde{s} \circ \mathrm{NT}$ to $H^1(\Sigma, T_1 \times T_2)$ equals $\langle -, (\widehat{\varphi}_1^{-1}(s), \widehat{\varphi}_2^{-1}(s))\rangle$.
3. The pull-back of $\tilde{s} \circ \mathrm{NT}$ to $H^1(\Sigma, U_{1,2})$ equals $\langle -, s_U\rangle$.

To 1: This can be checked right away using the explicit formula of the map $\ddagger$ in terms of inhomogenous cochains.

To 2: For any torus T, let $[-,-]$ denote the canonical pairing $H^{-1}_{\mathrm{Tate}}(\Sigma, X_*(T)) \times \widehat{T}^\Sigma \to \mathbb{C}^\times$. By Lemma 2.3.2 we have $\langle a, b\rangle = [\mathrm{NT}(a), b]$. Using the functoriality of the isomorphism NT, which in the case of the maps $\frac{\mathrm{Ad}(h_1,h_2)}{\mathrm{Ad}(g_1,g_2)}$ is the content of Proposition 2.4.2, it is enough to show that the pull-back of $\tilde{s}$ under

$$H^{-1}_{\mathrm{Tate}}(\Sigma, X_*(T_1) \otimes X_*(T_2)) \to H^{-1}_{\mathrm{Tate}}(\Sigma, X_*(S^{H_{\mathrm{sc}}} \to X_*(S)))$$

equals $[-,\widehat{\varphi}_1^{-1}(s),\widehat{\varphi}_2^{-1}(s)]$. This follows from the fact that the compositon

$$\begin{bmatrix} 1 \\ \uparrow \\ \widehat{T}_1\times\widehat{T}_2 \end{bmatrix} \leftarrow \begin{bmatrix} \widehat{S}^H/Z(\widehat{H}) \\ \uparrow \\ \widehat{S} \end{bmatrix} \leftarrow \begin{bmatrix} 1 \\ \uparrow \\ Z(\widehat{H}) \end{bmatrix}$$

where the left map is the dual of the upper row of diagram (2.3) and the right map is the natural inclusion coincides with the map

$$Z(\widehat{H}) \xrightarrow{\Delta} Z(\widehat{H})\times Z(\widehat{H}) \xrightarrow{\widehat{\varphi}_1^{-1},\widehat{\varphi}_2^{-1}} \widehat{T}_1\times\widehat{T}_2$$

To 3: The same arguments as above reduce to showing that s_U is the image of s under the map

$$\begin{bmatrix} 1 \\ \uparrow \\ \widehat{U_{1,2}} \end{bmatrix} \leftarrow \begin{bmatrix} \widehat{S^H}_{\mathrm{ad}} \\ \uparrow \\ \widehat{S}_{\mathrm{ad}} \end{bmatrix} \leftarrow \begin{bmatrix} 1 \\ \uparrow \\ Z(\widehat{H})/Z(\widehat{G}) \end{bmatrix}$$

where the left map is the dual of the lower row of diagram (2.3) and the right map is the natural inclusion. This follows from the construction of s_U. □

3 Statement of the main result

We fix an unramified reductive group G over F, and a Borel pair (T_0,B_0) of G defined over F. Then Γ acts on $X^*(T_0)$ through a finite cyclic subgroup of $\mathrm{Aut}(X^*(T_0))$ generated by the image of Fi; we will denote by ϑ both this image as well as its dual in $\mathrm{Aut}(X_*(T_0))$. Let $(\widehat{G},\widehat{B}_0,\widehat{T}_0)$ be the dual datum to (G,B_0,T_0). If $\Omega(T_0,G)$ and $\Omega(\widehat{T}_0,\widehat{G})$ denote the corresponding Weyl-groups, then there is a natural isomorphism between them given by duality. We choose an L-group ${}^LG$ for G s.t. the Γ-action on $\widehat{G}$ preserves the pair $(\widehat{B}_0,\widehat{T}_0)$.

We also fix an endoscopic triple $(H,s,\widehat{\eta})$ for G s.t. H is unramified. We choose again a Borel pair (T_0^H,B_0^H) defined over F, let $(\widehat{H},\widehat{B}_0^H,\widehat{T}_0^H)$ be the dual datum to (H,B_0^H,T_0^H) and ${}^LH$ an L-group for H s.t. the Γ-action on $\widehat{H}$ preserves $(\widehat{B}_0^H,\widehat{T}_0^H)$.

We choose a hyperspecial point o in the apartment of T_0 and obtain an O_F-structure on G and $\mathfrak{g}$. Then G_o,G_{o^+} resp. $\mathfrak{g}_o,\mathfrak{g}_{o^+}$ will be the parahoric and its pro-unipotent radical of $G(O_{F^u})$ resp. $\mathfrak{g}(O_{F^u})$ associated to o. We also choose a hyperspecial point, denoted again by o, in the apartment of T_0^H and obtain the same structures on H and $\mathfrak{h}$.

Up to equivalence the map $\widehat{\eta}:\widehat{H}\to\widehat{G}$ may be chosen so that $\widehat{\eta}^{-1}(\widehat{B}_0,\widehat{T}_0)=(\widehat{B}_0^H,\widehat{T}_0^H)$. Then we have in particular an isomorphism of complex tori $\widehat{\eta}|_{T_0^H}:\widehat{T}_0^H\to\widehat{T}_0$. There exists an element $\omega\in Z^1(\Gamma,\Omega(\widehat{T}_0,\widehat{G}))$ s.t. $\omega(\sigma)\sigma\circ\widehat{\eta}|_{\widehat{T}_0^H}\circ\sigma^{-1}=\widehat{\eta}|_{\widehat{T}_0^H}$ for all $\sigma\in\Gamma$. Thus we dually obtain an isomorphism of F-tori $\eta:T_0^\omega\to T_0^H$, where T_0^ω denotes the twist of T_0 by ω. If T^H and T are maximal tori in H and G respectively, then we call an isomorphism $T^H\to T$ *admissible* if it is of the form $\mathrm{Ad}(g)\eta\mathrm{Ad}(h)$ for some $h\in H(\overline{F})$, $g\in G(\overline{F})$.

We would like to alert the reader that there are two important elements of $Z^1(\Gamma,\Omega(\widehat{T}_0,\widehat{G}))$ that we will be working with: one is the element ω from the preceding paragraph, which comes from the endoscopic triple, and the other is an element w, which will be defined in the following subsection, and comes from the Langlands parameter υ.

By [Hal93, Lemma 6.1] the map $\widehat{\eta} : \widehat{H} \to \widehat{G}$ can be extended to an L-embedding ${}^L\eta : {}^LH \to {}^LG$ in such a way, that the 1-cocycle

$$I_F \to {}^LH \to {}^LG \to \widehat{G}$$

is trivial. We choose such an extension. The extended triple $(H, s, {}^L\eta)$ is then unramified in the sense of [Hal93].

3.1 Review of the construction of DeBacker and Reeder

In this section we want to review the construction from [DR09] of the L-packet on G and its pure inner forms corresponding to a Langlands parameter $\upsilon : W_F \to {}^LG$ which is TRSELP in the sense of loc. cit. Our purpose is not to give the details of the construction, but rather to gather the necessary notation and properties needed in the subsequent sections.

Recall that υ is called TRSELP if it is trivial on the wild inertia, $\mathrm{Cent}(\upsilon(I_F), \widehat{G})$ is a maximal torus of $\widehat{G}$, and $Z(\widehat{G})^\Gamma$ is of finite index in $\mathrm{Cent}(\upsilon, \widehat{G})$. Note that then υ is automatically trivial on $\mathrm{SL}_2(\mathbb{C})$. Up to equivalence we may assume that $\upsilon(I_F) \subset \widehat{T}_0$. There is an element $w \in Z^1(\Gamma, \Omega(\widehat{T}_0, \widehat{G}))$ s.t.

$$\mathrm{Ad}(\upsilon(\sigma))|_{\widehat{T}_0} \quad = \quad w(\sigma)\sigma, \quad \forall \sigma \in W_F$$

Let T_0^w be the twist of T_0 by w. The ellipticity of υ implies that T_0^w/Z is anisotropic. Put $X = X_*(T_0^w)$. This is a $\mathbb{Z}[\Gamma]$-module, where the Γ-action comes from that on T_0^w. Let $\bar{X}$ be the quotient of X by the coroot-lattice, and X_Γ resp. $\bar{X}_\Gamma$ denote the Γ-coinvariants in X resp. $\bar{X}$. Let X_w be the preimage of $[X_\Gamma]_{\mathrm{tor}}$ in X. Write C_υ for the component group of the centralizer of υ in $\widehat{G}$. We have the following diagram

$$\begin{array}{ccccc}
\mathrm{Irr}(C_\upsilon) & = & \mathrm{Irr}(\pi_0(\widehat{T_0^w}^{\Gamma})) & \longrightarrow & \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma)) \\
 & & \uparrow \cong & & \uparrow \cong \\
X_w & \longrightarrow & [X_\Gamma]_{\mathrm{tor}} & \longrightarrow & [\bar{X}_\Gamma]_{\mathrm{tor}} \\
 & & \cong \downarrow \mathrm{DR}_{T_0^w} & & \mathrm{DR}_G \downarrow \cong \\
 & & H^1(\Gamma, T_0^w) & \longrightarrow & H^1(\Gamma, G)
\end{array} \tag{3.1}$$

The bottom square of it is [DR09, Lemma 2.6.1]. The top equality follows from

$$\mathrm{Cent}(\upsilon, \widehat{G}) = \widehat{T_0^w}^{\Gamma}$$

while the rest is given by the obvious restriction maps.

The map $X_w \to H^1(\Gamma, G)$ in this diagram will be denoted by r. For any $u \in H^1(\Gamma, G)$ we let $[r^{-1}(u)]$ be the image of $r^{-1}(u)$ in $[X_\Gamma]_{\mathrm{tor}}$. The map $X_w \to \mathrm{Irr}(C_\upsilon)$ will be denoted by $\lambda \mapsto \rho_\lambda$.

From the Langlands parameter υ DeBacker and Reeder construct (see [DR09, §4]) a Langlands parameter $\upsilon_T : W_F \to {}^L T_0^w$ (called φ_T in loc. cit.) which corresponds to a regular depth-zero character $\theta : T_0^w(F) \to \mathbb{C}^\times$ (both notations θ and χ_υ are used for this character in loc.cit). Moreover, given $\lambda \in X_w$, they construct the following objects

- An element $u_\lambda \in Z^1(\Gamma, G)$ (trivial on inertia). Let $(\psi_\lambda, u_\lambda) : G \to G^\lambda$ be the corresponding twist.
- A maximal torus $T_\lambda \subset G^\lambda$, together with an element $p_\lambda \in G^\lambda(F^u)$ s.t.
$$\mathrm{Ad}(p_\lambda)\psi_\lambda : T_0^w \to T_\lambda$$
is an isomorphism of F-tori
- A depth-zero supercuspidal representation π_λ of $G^\lambda(F)$.

Furthermore they show in the proof of [DR09, Thm 4.5.3] that for $\lambda, \mu \in X_w$ one has $\rho_\lambda = \rho_\mu$ if and only if $\psi_\lambda \circ \psi_\mu^{-1} : G^\mu \to G^\lambda$ is a trivial twist and the transfer of π_μ to G^λ with respect to one (hence any) strongly trivial twist equivalent to $\psi_\lambda \circ \psi_\mu^{-1}$ coincides with π_λ. Thus if we put
$$I = \{(G^\lambda, \psi_\lambda, u_\lambda) |\ \lambda \in X_w\}$$
and construct G^I as in Section 2, then for each $\rho \in \mathrm{Irr}(C_\upsilon)$ we obtain a conjugation-invariant function $\Theta_{\upsilon,\rho}$ on $G^I(F)$ by taking any $\lambda \in X_w$ s.t. $\rho_\lambda = \rho$ and extending the character of π_λ to a conjugation-invariant function on $G^I(F)$.

To simplify their stability calculations, DeBacker and Reeder rigidify their constructions in the following way. In every class of $H^1(\Gamma, G)$ they choose a specific representative $u \in Z^1(\Gamma, G)$, which again gives rise to a twist $(\psi, u) : G \to G^u$. For each $\lambda \in r^{-1}(u)$ they construct an element $q_\lambda \in G^u(F^u)$ s.t. the maximal torus $S_\lambda = \mathrm{Ad}(q_\lambda)\psi(T_0)$ is defined over F and
$$\mathrm{Ad}(q_\lambda)\psi : T_0^w \to S_\lambda$$
is an isomorphism over F. For any strongly regular semi-simple element $Q \in S_0(F)$ the map
$$\lambda \mapsto \mathrm{Ad}(q_\lambda)\psi \mathrm{Ad}(q_0^{-1})Q$$
is a bijection from $[r^{-1}(u)]$ to a set of representatives for the rational classes inside the stable class of Q in $G^u(F)$ ([DR09, Lem. 2.10.1]). In particular, the tori S_λ exhaust the stable class of T_0^w in G^u. It will be important for later to note that $p_0 = q_0 \in G(O_{F^u})$. For every $\rho \in \mathrm{Irr}(C_\upsilon)$ mapping to the class of u, they define a representation $\pi_u(\upsilon, \rho)$ on $G^u(F)$. It is equal to the transfer of π_λ via any strongly trivial twist $G^\lambda \to G^u$ equivalent to $\psi \circ \psi_\lambda^{-1}$, where λ is any element of $r^{-1}(u)$.

It is clear from the constructions that for any $\lambda \in r^{-1}(u)$, the twist $\psi_\lambda \circ \psi^{-1}$ defines an injection from the conjugacy classes in $G^u(F)$ to the conjugacy classes in $G^I(F)$ whose image consists of those conjugacy classes which meet $G^\mu(F)$ for $\mu \in r^{-1}(u)$. Moreover, this twist identifies the character of $\pi_u(\upsilon, \rho)$ with the function $\Theta_{\upsilon,\rho}$, where both are viewed as class functions.

The same construction can be applied to a TRSELP $\upsilon^H : W_F \to {}^L H$ and the corresponding objects will carry the superscript H.

3.2 The Whittaker character

We extend the chosen pair (T_0, B_0) of G to a splitting $(T_0, B_0, \{X_\alpha\})$ where each simple root vector X_α is chosen so that the homomorphism

$$\mathbb{G}_a \to G$$

determined by it is defined over O_{F^u} and the image of 1 under

$$\mathbb{G}_a(O_{F^u}) \to G(O_{F^u}) \to G(\overline{\mathbb{F}_q})$$

is non-trivial. Such a splitting is called admissible by [Hal93]. Moreover we require that $X_{\sigma(\alpha)} = \sigma(X_\alpha)$ for all $\sigma \in \Gamma$. Let N denote the unipotent radical of B_0.

Lemma 3.2.1. *There exists an additive character $\psi : F \to \mathbb{C}^\times$ which is non-trivial on O_F but trivial on πO_F, s.t. the representation $\pi_1(\upsilon, 1)$ is generic with respect to the character $N(F) \to \mathbb{C}^\times$ determined by ψ and the chosen splitting.*

Proof: The representation $\pi_1(\upsilon, 1)$ is the same as the representation π_0 defined in [DR09, §4.5]. By Lemmas 6.2.1 and 6.1.2 in loc. cit. it is generic with respect to a character $N(F) \to \mathbb{C}^\times$ which has depth zero at o. This character is generic and is thus given by the composition of the F-homomorphism

$$N \to \prod_{\alpha \in \Delta} \mathbb{G}_a \xrightarrow{\Sigma} \mathbb{G}_a$$

determined by the chosen splitting with an additive character

$$\psi : F \to \mathbb{C}^\times$$

The choice of the simple root vectors X_α ensures that the homomorphism $N \to \mathbb{G}_a$ is in fact defined over O_F and moreover maps $N(O_F)$ surjectively onto $\mathbb{G}_a(O_F)$. Since the character $N(F) \to \mathbb{C}^\times$ has depth zero at o we see that ψ is non-trivial on O_F and trivial on πO_F. □

From now on we fix an additive character $\psi : F \to \mathbb{C}^\times$ as in the above Lemma.

3.3 Definition of the unstable character

For $t \in \mathrm{Cent}(\upsilon, \widehat{G})$ we define on $G^I(F)$ the function

$$\Theta_\upsilon^t = \sum_{\rho \in \mathrm{Irr}(C_\upsilon)} [e_\rho \mathrm{tr}\, \rho(t)] \Theta_{\upsilon,\rho}$$

where for any $\lambda \in X_w$ with $\rho_\lambda = \rho$ we put $e_\rho = e(G^\lambda)$, the latter being the sign defined in [Kot83]. This is the t-unstable character corresponding to the packet $\Pi(\upsilon)$ defined in [DR09, §4.5].

We will also define the t-unstable character of the normalized L-packet $\Pi_u(\upsilon)$ defined in [DR09, §4.6] for the specific twists $(\psi, u) : G \to G^u$ considered there. This character is

$$\Theta_{\upsilon,u}^t := e(G^u) \sum_{\rho \in \mathrm{Irr}(C_\upsilon, u)} [\mathrm{tr}\, \rho(t)] \Theta_{\pi_u(\upsilon,\rho)}$$

where $\mathrm{Irr}(C_\upsilon, u)$ is the fiber over u of the map $\mathrm{Irr}(C_\upsilon) \to H^1(\Gamma, G)$ given in diagram (3.1). We will show in Lemma 6.1.1 that the map $H^1(\Gamma, G) \to \pi_0(Z(\widehat{G})^\Gamma)$ in diagram (3.1) is a particular normalization of the Kottwitz isomorphism, and so the set $\mathrm{Irr}(C_\upsilon, u)$ is the set of all irreducible representations of C_υ which transform under $\pi_0(Z(\widehat{G})^\Gamma)$ by the character corresponding to u via the Kottwitz isomorphism.

The restriction of Θ^1_υ to $G(F)$, which also equals $\Theta^1_{\upsilon,1}$, will be denoted by $\mathcal{S}\Theta_\upsilon$.

3.4 Statement of the main result

Before stating the main result, we need to impose some mild conditions on the residual characteristic of F. These restrictions come from the papers [DR09] and [Hal93]. To state them, let n_G denote the smallest dimension of a faithful representation of G, and n_H be the corresponding number for H. Let e be the ramification degree of $F/\mathbb{Q}_p$ and e_G be the maximum over the ramification degrees (again over $\mathbb{Q}_p$) of all splitting fields of maximal tori of G. The restrictions we impose are

- $q_F \geq |R(T_0, B_0)|$
- $p \geq (2+e)\max(n_G, n_H)$
- $p \geq 2 + e_G$

The first two items are imposed in [DR09, §12.4], while the third is imposed in the main result of [Hal93] – Theorem 10.18. From now on we assume that these restrictions hold.

Let $\upsilon^H : W_F \to {}^L H$ be a Langlands parameter for H, then $\upsilon = {}^L\eta \circ \upsilon^H$ is a Langlands parameter for G. We are interested in the situation in which both υ^H and υ are TRSELP. Then $(H, s, \widehat{\eta})$ is automatically an elliptic endoscopic triple for G. Up to equivalence we may assume that υ^H maps inertia into $\widehat{T}_0^H$, then υ maps inertia into $\widehat{T}_0$ by our choice of $\widehat{\eta}$. There are elements $w \in Z^1(\Gamma, \Omega(\widehat{T}_0, \widehat{G}))$, $w^H \in Z^1(\Gamma, \Omega(\widehat{T}_0^H, \widehat{H}))$ s.t.

$$\begin{aligned} \mathrm{Ad}(\upsilon(\sigma))|_{\widehat{T}_0} &= w(\sigma)\sigma, \quad \forall \sigma \in W_F \\ \mathrm{Ad}(\upsilon^H(\sigma))|_{\widehat{T}_0^H} &= w^H(\sigma)\sigma, \quad \forall \sigma \in W_F \end{aligned}$$

Let Δ_ψ be the Whittaker normalization [KS99, §5.3] of the absolute transfer factor for (G, H) with respect to the generic character on $N(F)$ determined by ψ and let Δ^I_ψ be its extension to G^I defined in Section 2.2. We will identify the element $s \in Z(\widehat{H})^\Gamma$ with its image in $\widehat{T}_0$ under $\widehat{\eta}$. Then from Section 3.3 we have the functions Θ^s_υ on $G^I(F)$ and $\mathcal{S}\Theta_{\upsilon^H}$ on $H(F)$. The main result of this paper is

Theorem 3.4.1. *For any strongly regular semi-simple element $\gamma \in G^I(F)$ the following equality holds*

$$\Theta^s_\upsilon(\gamma) = \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta^I_\psi(\gamma^H, \gamma) \frac{D^H(\gamma^H)^2}{D^G(\gamma)^2} \mathcal{S}\Theta_{\upsilon^H}(\gamma^H)$$

Recall that $D^G(\gamma) = |\det(1 - \mathrm{Ad}(\gamma)|\mathfrak{g}/\mathfrak{g}_\gamma)|^{1/2}$.

In terms of the normalized L-packets, this statement can be reformulated as follows. Let $(\varphi, u) : G \to G^u$ be a pure inner twist of the type considered in [DR09, §4.6] and let $\Delta_{\psi,u}$ be the normalization of the absolute transfer factor for (G^u, H) corresponding to Δ_ψ as in Section 2.2. Then

Theorem 3.4.2. *For any strongly regular semi-simple element $\gamma \in G^u(F)$ the following equality holds*

$$\Theta^s_{\upsilon,u}(\gamma) = \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_{\psi,u}(\gamma^H, \gamma) \frac{D(\gamma^H)^2}{D(\gamma)^2} \mathcal{S}\Theta_{\upsilon^H}(\gamma^H) \tag{3.2}$$

3.5 A consequence

Let as before $(\varphi, u) : G \to G^u$ be a twist and let $\upsilon : W_F \to {}^L G^u$ be a TRSELP. We identify $\widehat{G}^u$ and $\widehat{G}$. Then we have $Z(\widehat{G})^\Gamma \subset \mathrm{Cent}(\upsilon, \widehat{G})$ and if we choose $s \in Z(\widehat{G})^\Gamma$, then (G, s, id) is an extended endoscopic triple for G^u. In this situation we obtain the following corollary of Theorem 3.4.2, which was conjectured by Kottwitz in [Kot83].

Corollary 3.5.1. *Let $\gamma \in G(F)$ and $\gamma' \in G^u(F)$ be stably conjugate. Then*

$$\sum_{\pi \in \Pi_u(\upsilon)} \Theta_\pi(\gamma') = e(G^u) \mathcal{S}\Theta_\upsilon(\gamma)$$

Proof: We consider first the left hand side of the equality in Theorem 3.4.2. Since s belongs to $Z(\widehat{G})^\Gamma$, we have

$$\Theta^s_{\upsilon,u}(\gamma') = e(G^u)\langle u, s\rangle^{-1} \sum_{\pi \in \Pi_u(\upsilon)} \Theta_\pi(\gamma')$$

where $\langle\rangle : H^1(F, G) \times Z(\widehat{G})^\Gamma \to \mathbb{C}^\times$ is the pairing given by the Kottwitz-isomorphism (see Lemma 6.1.1). Turning to the right hand side of said equality, one sees that if $\gamma \in G(F)$,$\gamma' \in G^u(F)$ are stably conjugate, then the Whittaker normalization of the transfer factor in this situation is given by the simple formula

$$\Delta_\psi(\gamma, \gamma') = \langle \mathrm{inv}(\gamma, \gamma'), s\rangle^{-1}$$

If γ, γ' are not stably conjugate, then the transfer factor is zero. Thus the right hand side consists of a single summand and is equal to

$$\langle \mathrm{inv}(\gamma, \gamma'), s\rangle^{-1} \mathcal{S}\Theta_\upsilon(\gamma)$$

The statement now follows using Fact 2.1.2. □

Remark: This Corollary implies that the function Θ^1_υ on $G^I(F)$ defined in Section 3.3 is stable with respect to the general notion of stable conjugacy, developed in Section 2, for which stable classes span across multiple pure inner forms. Thus Θ^1_υ can be viewed as the stable character of the large L-packet Π_υ on $G^I(F)$.

In this section we only need the notation from the beginning of Section 3. Moreover, it is independent of the restrictions posed on p in Section 3.4. The only restriction we impose on p is $p > 2$, although this again is just for convenience and could be removed.

There are three signs which can be assigned to the pair of groups (G, H) and which we need to equate. The first one is

$$\epsilon(G, H) = (-1)^{r_G - r_H}$$

where r_G and r_H are the F-split ranks of G and H. This sign plays an important role in the character formulas of [DR09].

The second sign enters in the normalization of the geometric transfer factors. It is defined relative to an additive character $\psi : F \to \mathbb{C}^\times$ as the local ϵ-factor $\epsilon_L(V, \psi)$ where V is the virtual representation of Γ of degree 0 given by the difference of the Γ-representations $V_G := X^*(T_0) \otimes \mathbb{C}$ and $V_H := X^*(T_0^H) \otimes \mathbb{C}$.

The third goes back to Weil and plays a role in Waldspurger's work [Wal95] on the local trace formula for Lie algebras. To construct it, let $\psi : F \to \mathbb{C}^\times$ be an additive character and $B : \mathfrak{g}(F) \times \mathfrak{g}(F) \to F$ a non-degenerate, $\mathrm{Ad}(G(F))$-invariant, symmetric bilinear form. Following the exposition in [Wal95, VIII] we define for a lattice $r \subset \mathfrak{g}(F)$

$$\begin{aligned} I(r) &= \int_r \psi(B(x,x)/2)dx \\ \tilde{r} &= \{x \in \mathfrak{g}(F) | \forall y \in r \ \psi(B(x,y)) = 1\} \end{aligned}$$

It has been shown by Weil that the function

$$r \mapsto \frac{I(r)}{|I(r)|}$$

is constant when restricted to the set $\{r | \tilde{r} \subset 2r\}$. This constant is called $\gamma_\psi(B)$, or $\gamma_\psi(\mathfrak{g})$ when B is understood. Furthermore, in loc. cit. Waldspurger explains how to transfer B to a non-degenerate, $\mathrm{Ad}(H(F))$-invariant, symmetric bilinear form $B_\mathfrak{h}$ on $\mathfrak{h}(F)$, thereby obtaining $\gamma_\psi(B_\mathfrak{h})$. The second sign we are interested in is $\gamma_\psi(B)\gamma_\psi(B_\mathfrak{h})^{-1}$. (The word "sign" is not yet justified here, all we know is that both constants and hence their quotient are eight roots of unity. We will see however that in our case the quotient is a sign.)

We extend the bilinear form B to a symmetric bilinear form $\mathfrak{g}(\overline{F}) \times \mathfrak{g}(\overline{F}) \to \overline{F}$ in the obvious way and denote it by the same letter. As remarked in loc.cit., this extension is $\mathrm{Ad}(G(\overline{F})) \rtimes \Gamma$-invariant. It is clear that if $V \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ defined over some extension E of F, then the restriction of B to V defines a symmetric bilinear form $V(E) \times V(E) \to E$.

The purpose of this section is to prove the following

Proposition 4.0.2. *Let $\psi : F \to \mathbb{C}^\times$ be an additive character which is non-trivial on O_F and trivial on πO_F. Let B be a "good" bilinear form in the sense of [DR09, A.1]. Then*

$$\epsilon_L(V, \psi) = \epsilon(G, H) = \gamma_\psi(B)\gamma_\psi(B_\mathfrak{h})^{-1}$$

The proof is contained in the following lemmas.

Remark: We would like to point out that the second of these equalities is also proved in [KV2]. The proof given here is different from the one in loc. cit. and establishes a connection between the above signs and the number of symmetric orbits of Γ in $R(T^H, G)$. This number is an important invariant in endoscopy and thus the following lemmas may be of independent interest.

Recall from the introduction of Section 3 that ϑ is the automorphism of $X^*(T_0)$ (and dually of $X_*(T_0)$) given by the action of Fi, and $\omega \in Z^1(\Gamma, \Omega(T_0, G))$ is an element such that $\eta : T_0^\omega \to T_0^H$ is an isomorphism of F-tori.

Lemma 4.0.3.

$$\epsilon(G, H) = \det(\omega)$$

Proof: A similar argument is given in the proof of [DR09, Lemma 12.3.5], but we will present it here since our situation and notation are different. ϑ is a finite-order automorphism of the real vector space $X^*(T_0) \otimes \mathbb{R}$ and hence is diagonalizable over $\mathbb{C}$ with eigenvalues roots of unity, and all non-real eigenvalues come in conjugate pairs. Thus $\det(\vartheta) = (-1)^{\dim(V_G) - \dim(V_G^\vartheta)}$. In the same way $\det(\omega\vartheta) = (-1)^{\dim(V_H) - \dim(V_H^{\omega\vartheta})}$. But

$$\epsilon(G, H) = (-1)^{\dim(V_G^\Gamma) - \dim(V_H^\Gamma)} = (-1)^{\dim(V_G^\vartheta) - \dim(V_H^{\omega\vartheta})} = \det(\omega)$$

□

Lemma 4.0.4.

$$\epsilon_L(V, \psi) = \det(\omega)$$

Proof: The Γ representations V_G and V_H are unramified. Applying [Tat77, 3.4.6] and noting that the isomorphism of local class field theory used in loc. cit. is normalized so that Fi corresponds to π, we obtain

$$\begin{aligned} \epsilon_L(V_G - V_H, \psi) &= \det V_G(\mathrm{Fi}^{-1}) \det V_H(\mathrm{Fi}^{-1})^{-1} \\ &= \left[\frac{\det(\vartheta)}{\det(\omega\vartheta)}\right]^{-1} \\ &= \det(\omega) \end{aligned}$$

□

These two lemmas complete the proof of the first equality in Proposition 4.0.2. To continue with the second equality, we need to recall some notions from [LS87]. Let T be a maximal torus of G, and $\mathcal{O}$ be a Γ-orbit in $R(T, G)$, the set of roots of T in G. Then $-\mathcal{O}$ is also a Γ-orbit in $R(T, G)$ and either $\mathcal{O} = -\mathcal{O}$, in which case $\mathcal{O}$ is called a *symmetric* orbit, or $\mathcal{O} \cap -\mathcal{O} = \emptyset$, in which case $\mathcal{O}$ is called an *asymmetric* orbit. For $\alpha \in R(T, G)$ let Γ_α be the stabilizer of α and $\Gamma_{\pm\alpha}$ be the stabilizer of the set $\{\alpha, -\alpha\}$. Let F_α and $F_{\pm\alpha}$ be the fixed fields of Γ_α and $\Gamma_{\pm\alpha}$ in $\overline{F}$. Then $[\Gamma_\alpha; \Gamma_{\pm\alpha}]$ equals 2 if the orbit of α is symmetric and 1 if it is asymmetric. If T is unramified, then both F_α and $F_{\pm\alpha}$ lie in F^u.

For any Γ-invariant subset $S \subset R(T, G)$ we put

$$\mathfrak{g}_S = \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$$

This is clearly a vector subspace of $\mathfrak{g}$ defined over F.

Lemma 4.0.5. *Let T be a maximal torus of G stably conjugate to T_0^ω. Then*

$$\gamma_\psi(B)\gamma_\psi(B_{\mathfrak{h}})^{-1} = \prod_{\mathcal{O}} \gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}}(F)})$$

where $\mathcal{O}$ runs over the set of symmetric orbits of Γ in $R(T,G)$.

Proof: We consider the root decomposition of $\mathfrak{g}$ relative to T:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R(T,G)} \mathfrak{g}_\alpha$$

If we put $\mathfrak{g}_0 = \mathfrak{t}$ then the invariance of B implies that for all $\alpha, \beta \in R(T,G) \cup \{0\}$ such that $\alpha \neq -\beta$ the subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ of $\mathfrak{g}$ are orthogonal with respect to B. This means that if $\mathcal{O}_1, ..., \mathcal{O}_k$ are the orbits in $R(T,G)$ of the group $\Gamma \times \{\pm 1\}$, where $\{\pm 1\}$ acts by scalar multiplication, then

$$\mathfrak{g}(F) = \mathfrak{t}(F) \oplus \bigoplus_{i=1}^{k} \mathfrak{g}_{\mathcal{O}_i}(F)$$

is an orthogonal decomposition of $\mathfrak{g}(F)$. Thus $\gamma_\psi(B)$ factors as

$$\gamma_\psi(B) = \gamma_\psi(B|_{\mathfrak{t}(F)}) \prod_{i=1}^{k} \gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}_i}(F)})$$

Consider one of the orbits $\mathcal{O}_i$. Either Γ acts transitively on it, in which case it is a symmetric Γ-orbit, or it decomposes as a disjoint union of two asymmetric Γ-orbits. We assume that the latter is the case, and write $\mathcal{O}_i = \mathcal{O}_i' \sqcup -\mathcal{O}_i'$ where $\mathcal{O}_i'$ is one of the two Γ orbits in $\mathcal{O}_i$. Then $\mathfrak{g}_{\mathcal{O}_i} = \mathfrak{g}_{\mathcal{O}_i'} \oplus \mathfrak{g}_{-\mathcal{O}_i'}$ is a decomposition over F as a direct sum of isotropic spaces. Let $r_+ \subset \mathfrak{g}_{\mathcal{O}_i'}(F)$ and $r_- \subset \mathfrak{g}_{-\mathcal{O}_i'}(F)$ be large enough lattices. Then $\gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}_i}(F)})$ is by definition the complex sign of

$$\begin{aligned} &\int_{r_+ \oplus r_-} \psi(B(x+y, x+y)/2) d(x,y) \\ = &\int_{r_+} \int_{r_-} \psi(B(x,y)) dx dy \end{aligned}$$

For each $x \in r_+$ the map $y \mapsto \psi(B(x,y))$ is a character of the additive group r_-. Thus if r_+^0 is the subgroup of r_+ consisting of all x s.t. this character is trivial, the above integral is equal to the positive real constant $\mathrm{vol}(r_+^0, dx)\mathrm{vol}(r_-, dy)$. This shows $\gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}_i}(F)}) = 1$ and we conclude that

$$\gamma_\psi(B) = \gamma_\psi(B|_{\mathfrak{t}(F)}) \prod_{\mathcal{O}} \gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}_i}(F)})$$

where $\mathcal{O}$ runs over the set of symmetric Γ-orbits in $R(T,G)$.

We can apply the same reasoning to the Lie algebra $\mathfrak{h}$ with the bilinear form $B_{\mathfrak{h}}$ and the torus T_0^H. Since T_0^H is contained in a Borel defined over F, there are no symmetric orbits of Γ in $R(T_0^H, H)$ and we conclude

$$\gamma_\psi(B_{\mathfrak{h}}) = \gamma_\psi(B_{\mathfrak{h}}|_{\mathfrak{t}_0^H(F)})$$

But we have chosen the torus T so that there exists an admissible isomorphism $T_0^H \to T$ over F, and the bilinear form $B_{\mathfrak{h}}$ is constructed so that the differential of this admissible isomorphism identifies $B_{\mathfrak{h}}|_{\mathfrak{t}_0^H(F)}$ with $B|_{\mathfrak{t}(F)}$. Thus

$$\gamma_\psi(B_{\mathfrak{h}}) = \gamma_\psi(B|_{\mathfrak{t}(F)})$$

and the lemma now follows. □

Lemma 4.0.6. *Let $\mathcal{O}$ be a symmetric orbit of Γ in $R(T,G)$. Then*

$$\gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}}(F)}) = -1$$

Proof: Choose $\alpha \in \mathcal{O}$ and $\sigma_\alpha \in \Gamma_{\pm\alpha} \setminus \Gamma_\alpha$. We can choose a non-zero $E \in \mathfrak{g}_\alpha(F_\alpha) \cap [\mathfrak{g}_o \setminus \mathfrak{g}_{o^+}]$ and then we have $\sigma_\alpha(E) \in \mathfrak{g}_{-\alpha}(F_\alpha) \cap [\mathfrak{g}_o \setminus \mathfrak{g}_{o^+}]$. Then by [DR09, §A.1]

$$B(E, \sigma_\alpha(E)) \in O_{F_{\pm\alpha}}^\times$$

The map

$$\varphi : F_\alpha \to \mathfrak{g}_{\mathcal{O}}(F), \qquad \lambda \mapsto \sum_{\sigma \in \Gamma/\Gamma_\alpha} \sigma(\lambda E)$$

is an isomorphism of F-vector spaces and clearly $\gamma_\psi(B|_{\mathfrak{g}_{\mathcal{O}}(F)}) = \gamma_\psi(\varphi^* B)$. To compute the bilinear form $\varphi^* B : F_\alpha \times F_\alpha \to F$ we notice that if $\sigma_1, ..., \sigma_k$ are representatives for $\Gamma/\Gamma_{\pm\alpha}$, then

$$\mathfrak{g}_{\mathcal{O}} = \bigoplus_{i=1}^{k} (\mathfrak{g}_{\sigma_i(\alpha)} \oplus \mathfrak{g}_{\sigma_i(-\alpha)})$$

is an orthogonal sum of hyperbolic planes defined over $F_{\pm\alpha}$. A direct computation shows that

$$\varphi^* B(\lambda, \mu) = \mathrm{tr}_{F_{\pm\alpha}/F}\Big([\lambda\sigma_\alpha(\mu) + \mu\sigma_\alpha(\lambda)] B(E, \sigma_\alpha(E))\Big)$$

If we put

$$\begin{aligned} \psi'(x) &= \mathrm{tr}_{F_{\pm\alpha}/F}(B(E, \sigma_\alpha(E))x) \\ B'(\mu, \lambda) &= \lambda\sigma_\alpha(\mu) + \mu\sigma_\alpha(\lambda) \end{aligned}$$

then $\psi' : F_{\pm\alpha} \to \mathbb{C}^\times$ is an additive character and $B' : F_\alpha \times F_\alpha \to F_{\pm\alpha}$ is a non-degenerate $F_{\pm\alpha}$-bilinear form, and clearly $\gamma_\psi(\varphi^* B) = \gamma_{\psi'}(B')$.

We will now compute $\gamma_{\psi'}(B')$.

First we claim that ψ' is non-trivial on $O_{F_{\pm\alpha}}$ but trivial on $\pi O_{F_{\pm\alpha}}$. To see this, note that $\mathrm{tr}_{F_{\pm\alpha}/F}$ induces for each $i \in \mathbb{Z}$ a homomorphism of additive groups $\pi^i O_{F_{\pm\alpha}} \to \pi^i O_F$ which fits into the diagram

$$\begin{array}{ccc} \pi^i O_{F_{\pm\alpha}} & \longrightarrow & \pi^i O_F \\ {\scriptstyle \mathrm{mod}\ \pi^{i+1}} \downarrow & & \downarrow {\scriptstyle \mathrm{mod}\ \pi^{i+1}} \\ k_{F_{\pm\alpha}} & \xrightarrow{\mathrm{tr}} & k_F \end{array}$$

and thus $\mathrm{tr}_{F_{\pm\alpha}/F} : O_{F_{\pm\alpha}} \to O_F$ is surjective ([Ser79, V.§1.Lemma 2]). This together with $B(E, \sigma_\alpha(E)) \in O_{F_{\pm\alpha}}^\times$ implies the claim about ψ'.

Next we compute the dual of the $O_{F_{\pm\alpha}}$-lattice O_{F_α} with respect to $\psi' \circ B'$.

$$\begin{aligned} & \{x \in F_\alpha | \, \forall y \in O_{F_\alpha} : \psi'(B(x,y)) = 1\} \\ = \; & \{x \in F_\alpha | \, \forall y \in O_{F_\alpha} : B(x,y) \in \pi O_{F_{\pm\alpha}}\} \\ = \; & \pi\{x \in F_\alpha | \, \forall y \in O_{F_\alpha} : x\sigma_\alpha(y) + y\sigma_\alpha(x) \in O_{F_{\pm\alpha}}\} \\ = \; & \pi\{x \in F_\alpha | \, \forall y \in O_{F_\alpha} : xy + \sigma_\alpha(y)\sigma_\alpha(x) \in O_{F_{\pm\alpha}}\} \end{aligned}$$

Thus we are looking for π times the dual of O_{F_α} with respect to the bilinear form $(x,y) \mapsto \mathrm{tr}_{F_\alpha/F_{\pm\alpha}}(xy)$. This dual is the codifferent of $F_\alpha/F_{\pm\alpha}$, which equals O_{F_α} since $F_\alpha/F_{\pm\alpha}$ is an unramified extension.

We conclude that the lattice O_{F_α} has the property that it contains its dual with respect to $\psi' \circ B'$. Since we are imposing the restriction $p > 2$ and thus $O_{F_\alpha} = 2O_{F_\alpha}$. Then by definition, $\gamma_{\psi'}(B')$ is the complex sign of

$$I := \int_{O_{F_\alpha}} \psi'(N(x))dx$$

where $N : F_\alpha \to F_{\pm\alpha}$ is the norm map and dx is a Haar measure on the additive group F_α. Let $(\xi_k)_{k \in k_{F_\alpha}}$ be a set of representatives for $O_{F_\alpha}/\pi O_{F_\alpha}$. Then

$$I = \sum_{k \in k_{F_\alpha}} \int_{\pi O_{F_\alpha}} \psi'(N(\xi_k + x))dx$$

One computes immediately that $\psi'(N(\xi_k + x)) = \psi'(N(\xi_k))$ for all $x \in \pi O_{F_\alpha}$ since ψ' is trivial on $\pi O_{F_{\pm\alpha}}$. This leads to

$$I = \mathrm{vol}(\pi O_{F_\alpha}, dx) \sum_{k \in k_{F_\alpha}} \psi'(N(\xi_k))$$

The restriction of ψ' to $O_{F_{\pm\alpha}}$ factors through the natural projection $O_{F_{\pm\alpha}} \to k_{F_{\pm\alpha}}$, and the composition of N with this projection factors through the projection $O_{F_\alpha} \to k_{F_\alpha}$ and induces the norm map associated to the extension $k_{F_\alpha}/k_{F_{\pm\alpha}}$, which we also call N. Thus

$$\begin{aligned} I &= \mathrm{vol}(\pi O_{F_\alpha}, dx) \sum_{k \in k_{F_\alpha}} \psi'(N(k)) \\ &= \mathrm{vol}(\pi O_{F_\alpha}, dx) \left[1 + \sum_{k \in k_{F_\alpha}^\times} \psi'(N(k)) \right] \end{aligned}$$

Now $N : k_{F_\alpha}^\times \to k_{F_{\pm\alpha}}^\times$ is a surjective homomorphism, the cardinality of whose fibers we will call A. Then

$$\begin{aligned} I &= \mathrm{vol}(\pi O_{F_\alpha}, dx) \left[1 + A \sum_{k \in k_{F_{\pm\alpha}}^\times} \psi'(k) \right] \\ &= \mathrm{vol}(\pi O_{F_\alpha}, dx) \left[-(A-1) + A \sum_{k \in k_{F_{\pm\alpha}}} \psi'(k) \right] \\ &= -(A-1)\mathrm{vol}(\pi O_{F_\alpha}, dx) \end{aligned}$$

since ψ' is a non-trivial character on the additive group $k_{F_{\pm\alpha}}$. We conclude that I is a negative real number, and the lemma follows. □

Lemma 4.0.7.

$$\det(\omega) = (-1)^N$$

where N is the number of symmetric orbits of Γ in $R(T,G)$.

Proof: We choose a $g \in G(\overline{F})$ s.t. $\mathrm{Ad}(g) : T_0^\omega \to T$ is an isomorphism over F and use it to regard ω and ϑ as automorphisms of $R(T,G)$. Moreover put $B = \mathrm{Ad}(g)B_0$ and write $\alpha > 0$ if $\alpha \in R(T,B)$. Let

$$\begin{aligned} S &= \{\alpha \in R(T,G) | \alpha > 0 \wedge \omega\alpha < 0\} \\ S' &= \{\alpha \in R(T,G) | \alpha > 0 \wedge \omega\vartheta\alpha < 0\} \end{aligned}$$

Since ϑ preserves the set of positive roots in $R(T,G)$, it induces a bijection $S' \to S$. Thus

$$\det(\omega) = (-1)^{|S|} = (-1)^{|S'|}$$

Claim 1: The cardinality of S' is congruent mod 2 to the cardinality of the intersection of S' with the union of the symmetric orbits of Γ in $R(T,G)$.

Put $T = \omega\vartheta$ for short. Then Γ acts on $R(T,G)$ via the cyclic group $< T >$. Let $\mathcal{O}$ be an orbit. We claim that the sets

$$\begin{aligned} \mathcal{O}_+ &= \{\alpha \in \mathcal{O} | \ \alpha > 0 \wedge T\alpha < 0\} \\ \mathcal{O}_- &= \{\alpha \in \mathcal{O} | \ \alpha < 0 \wedge T\alpha > 0\} \end{aligned}$$

have the same cardinality. To see this, consider the directed graph in the vector space $X^*(T_{\mathrm{ad}}) \otimes \mathbb{R}$ whose vertices are given by $\mathcal{O}$ and whose edges are given by

$$\{(\alpha, T\alpha) | \ \alpha \in \mathcal{O}\}$$

Then $\mathcal{O}_+$ is in bijection with the set of edges which start in the positive half space of $X^*(T_{\mathrm{ad}}) \otimes \mathbb{R}$ and end in the negative, while $\mathcal{O}_-$ is in bijection with the set of edges which start in the negative half space and end in the positive. But our graph is a closed loop, so these sets must have the same cardinality.

If $\mathcal{O}$ is an asymmetric orbit, then $-\mathcal{O}$ is also one and is disjoint from $\mathcal{O}$, and multiplication by -1 gives a bijection $\mathcal{O}_- \to [-\mathcal{O}]_+$. We conclude that $S' \cap (\mathcal{O} \cup -\mathcal{O})$ has an even cardinality. This proves Claim 1.

Claim 2: Let $\mathcal{O}$ be a symmetric orbit. Then its intersection with S' has an odd cardinality.

The group $< T >$ acts on $\mathcal{O}/\{\pm 1\}$ and all elements of the latter set are of the form $\{\alpha, -\alpha\}$ with $\alpha \in \mathcal{O}$. We choose an element $A \in \mathcal{O}/\{\pm 1\}$, and let $n = |\mathcal{O}|/2 - 1$. Then $A, TA, ..., T^n A$ enumerates $\mathcal{O}/\{\pm 1\}$. For each $0 \leq i \leq n$ let α_i be the positive member of $T^i A$. Then for each such i one of two cases occurs: either $T\alpha_i = -\alpha_{i+1}$ and $T(-\alpha_i) = \alpha_{i+1}$, or $T\alpha_i = \alpha_{i+1}$ and $T(-\alpha_i) = -\alpha_{i+1}$ (where we adopt the convention $\alpha_{n+1} = \alpha_0$). The cardinality of $S' \cap \mathcal{O}$ is the number of $0 \leq i \leq n$ for which the first case occurs. Now let M be the number of $0 \leq i < n$ for which the first case occurs (note the sharp inequality!). If M is even, then $T^n\alpha_0 = \alpha_n$ and thus $T\alpha_n$ must equal $-\alpha_0$, for otherwise the set $\{\alpha_0, T\alpha_0, ..., T^n\alpha_0\}$ will be a T-invariant subset of $\mathcal{O}$, which is impossible. Thus $|S' \cap \mathcal{O}| = M + 1$ is an odd number. If conversely M is odd, then $T^n\alpha_0 = -\alpha_n$ and by the same reasoning $T(-\alpha_n) = (-\alpha_0)$. It follows then that $|S' \cap \mathcal{O}| = M$, again an odd number. This proves Claim 2.

The two claims together imply that $(-1)^{|S'|} = (-1)^N$ and this finishes the lemma. $\square$

The second equality in Proposition 4.0.2 now follows from these lemmas.

In this section we will prove Theorem 3.4.2 in the special case where γ is a strongly regular topologically semi-simple element. Such elements are quite special and for them the proof simplifies considerably. We hope that discussing this case first might help the reader see more clearly some of the phenomena that enter into the general proof. It is also worth mentioning that in this special case the proof is valid without any restriction on the field F beyond odd residual characteristic.

5.1 Preparatory lemmas

Before we can give the proof for topologically semi-simple elements, we need to technical results. The first one will be an integral part of the general proof as well. The second one is a simple special case of the more general and technical discussion of transfer factors needed for the general case.

Recall our situation as laid out in the introduction to Section 3: we have a twist $(\varphi, u) : G \to G^u$, an extended endoscopic triple $(H, s, {}^L\eta)$ for G and a TRSELP $\upsilon^H : W_F \to {}^LH$ from which we obtain a TRSELP $\upsilon : W_F \to {}^LG$ by composing υ^H and ${}^L\eta$. In Section 3 we remarked that $\widehat{\eta}$ provides an isomorphism of $\overline{F}$-tori $\eta : T_0 \to T_0^H$. There is an element $w \in Z^1(\Gamma, \Omega(T_0, G))$ and a regular depth-zero character $\theta : T_0^w(F) \to \mathbb{C}^\times$. We also have the corresponding objects for H, which will carry the superscript H.

Lemma 5.1.1. *The isomorphism*

$$T_0^w \xrightarrow{\eta} T_0^{H,w^H}$$

is defined over F. If $\gamma \in T_0^w(F)$ is topologically semi-simple and $z \in Z^\circ(F)$ then

$$\theta(\gamma) = \theta^H(\eta(\gamma)) \qquad \theta(z) = \theta^H(\eta(z))\lambda^G(z)$$

where $\lambda^G : Z^\circ(F) \to \mathbb{C}^\times$ is the character of [LS87, Lemma 4.4.A].

Proof: Recall that T_0^w is the torus whose complex dual is given by the complex torus $\widehat{T}_0$ with Γ-action

$$\sigma(t) = \mathrm{Ad}(\upsilon(\sigma))t$$

for all $\sigma \in W_F, t \in \widehat{T}_0(\mathbb{C})$ where the conjugation takes place in ${}^LG$. Analogously we have the torus T_0^{H,w^H} whose complex dual is the complex torus $\widehat{T}_0^H$ with Γ-action

$$\sigma(t) = \mathrm{Ad}(\upsilon^H(\sigma))t$$

for all $\sigma \in W_F, t \in \widehat{T}_0^H(\mathbb{C})$ where now the conjugation takes place in ${}^LH$. The statement that

$$\eta : T_0^w \to T_0^{H,w^H}$$

is defined over F is equivalent to the statement that the isomorphism of complex tori

$$\widehat{\eta} : \widehat{T}_0^H \to \widehat{T}_0$$

is equivariant with respect to the above actions. But

$$\begin{aligned}\widehat{\eta}(\mathrm{Ad}(\upsilon^H(\sigma))t) &= {}^L\eta(\mathrm{Ad}(\upsilon^H(\sigma))t)\\ &= \mathrm{Ad}({}^L\eta\upsilon^H(\sigma))^L\eta(t)\\ &= \mathrm{Ad}(\upsilon(\sigma))\widehat{\eta}(t)\end{aligned}$$

This proves the first assertion.

The restriction of θ to the maximal bounded subgroup of T_0^ω, to which γ belongs, is determined by the restriction of the Langlands parameter υ_T of θ to inertia, which by construction is simply given by the restriction to inertia of $\upsilon = {}^L\eta \circ \upsilon^H$. This restriction is the cocycle

$$I_F \xrightarrow{\upsilon^H} {}^LH \xrightarrow{{}^L\eta} {}^LG \longrightarrow \widehat{G}$$

which by construction lands in $\widehat{T}_0$. Since ${}^L\eta$ is trivial on inertia, we see that this is the same as the cocycle

$$I_F \xrightarrow{\upsilon^H} {}^LH \longrightarrow \widehat{H} \xrightarrow{\widehat{\eta}} \widehat{G}$$

which also lands in $\widehat{T}_0$ and equals the restriction to inertia of $\widehat{\eta} \circ \upsilon_{T^H}$. The latter is the cocycle determining the restriction of $\theta^H \circ \eta$ to the maximal bounded subgroup of T_0^ω. This proves the second assertion.

Let T be any torus of G coming from H. In [LS87, §3.5] Langlands and Shelstad construct an element $a \in H^1(W_F, \widehat{T})$. The character $\lambda^G(z)$ is then the restriction to $Z^\circ(F)$ of the character on $T(F)$ corresponding via the Langlands correspondence to a. The construction of a involves χ-data, but one sees easily that its image under

$$H^1(W_F, \widehat{T}) \to H^1(W_F, \widehat{Z^\circ})$$

is independent of that choice and is in fact represented by the cocycle

$$W_F \hookrightarrow {}^LH \xrightarrow{{}^L\eta} {}^LG \longrightarrow {}^L[Z^\circ] \longrightarrow \widehat{Z^\circ}$$

By construction of the Langlands parameter υ_T of θ, the restriction of θ to $Z^\circ(F)$ is given by the cocycle

$$W_F \xrightarrow{\upsilon^H} {}^LH \xrightarrow{{}^L\eta} {}^LG \longrightarrow {}^L[Z^\circ] \longrightarrow \widehat{Z^\circ}$$

while that of $\theta^H \circ \eta$ is given by the cocycle

$$W_F \xrightarrow{\upsilon^H} {}^LH \longrightarrow \widehat{H} \xrightarrow{\widehat{\eta}} \widehat{G} \longrightarrow \widehat{Z^\circ}$$

It is clear that of these three cocycles, the second one equals the product of the first and the third, which implies the final statement of the lemma. □

Recall the splitting $(T_0, B_0, \{X_\alpha\})$ and the additive character $\psi : F \to \mathbb{C}^\times$ we chose in Section 3.2. From ψ we obtain a generic character of the unipotent radical of B_0, and hence a set of Whittaker data for G. We set Δ_ψ to be the normalization of the absolute transfer factor provided by that Whittaker data.

Proposition 5.1.2. *Let $\gamma \in G(O_F)$ be a strongly regular topologically semi-simple element, and let $\gamma^H \in H(F)$ be any preimage of γ. Then*

$$\Delta_\psi(\gamma^H, \gamma) = \epsilon(G, H)$$

Proof: Let Δ_0 be the normalization of the absolute transfer factor provided by $(T_0, B_0, \{X_\alpha\})$ [LS87, 3.7]. Then we have

$$\Delta_\psi = \epsilon_L(V, \psi)\Delta_0$$

Using Proposition 4.0.2 it will be enough to show that $\Delta_0(\gamma^H, \gamma) = 1$ and for this we follow the strategy of [Hal93, II]. Let T be the centralizer of γ in G. It is a maximal torus of G, unramified according to [Kot86, 7.1]. Thus we may pick our a-data to consist of units in O_{F^u} and our χ-data to be unramified. Let T^H be the centralizer of γ^H in H and identify it with T using the isomorphism $\varphi_{\gamma^H,\gamma}$. With these choices in place, we may construct and discuss the individual parts of Δ_0.

The same argument as in the proof of [Hal93, 7.2] shows that $\Delta_I(\gamma^H, \gamma) = 1$. In summary, the reason is that this factor arises from the evaluation of a character of $H^1(\Gamma, T)$ on the splitting invariant of T, but with our choices this invariant lies in $H^1(\Gamma/I_F, T(O_{F^u}))$, which is trivial. Furthermore, as remarked in the proof of [Hal93, 8.1], $(\alpha(\gamma) - 1)$ and a_α are units for each $\alpha \in R(T, G)$, and hence $\Delta_{II}(\gamma^H, \gamma)$ and $\Delta_{IV}(\gamma^H, \gamma)$ are trivial. The factor $\Delta_{III_1}(\gamma^H, \gamma)$ is trivial due to the choice of identification of T^H and T. The factor $\Delta_{III_2}(\gamma^H, \gamma)$ is trivial, because it equals $\langle a, \gamma \rangle$ for a certain $a \in H^1(W_F, \widehat{T})$, but by [Hal93, 11.2] the character $\langle a, \ \rangle$ is unramified. □

5.2 Proof for topologically semi-simple elements

Recall that the construction of DeBacker-Reeder, reviewed in Section 3.1, provides an element $q_0 \in G$ and an unramified maximal torus $S_0 = \mathrm{Ad}(q_0)T_0 \subset G$ such that the isomorphism $\mathrm{Ad}(q_0) : T_0^w \to S_0$ is defined over F. Moreover it provides elements $q_\lambda \in G^u$ and unramified maximal tori $S_\lambda = \mathrm{Ad}(q_\lambda)T_0 \subset G^u$ for each $\lambda \in r^{-1}(u)$. The maximal tori S_λ exhaust up to $G^u(F)$-conjugacy all maximal tori in G^u which are stably conjugate to S_0. Thus a strongly regular semi-simple element $\gamma \in G^u(F)$ is stably conjugate to an element of $S_0(F)$ if and only if it is $G^u(F)$-conjugate to an element of some $S_\lambda(F)$. We also have the corresponding objects for H, which will carry the superscript H.

We now turn to the proof of Theorem 3.4.2. Fix a topologically semi-simple strongly regular element $\gamma \in G^u(F)$. If this element is not stably conjugate to an element of $S_0(F)$, then it is not $G^u(F)$-conjugate to an element of any $S_\lambda(F)$, and thus [DR09, 10.1.1] implies that the left hand side of the equality we are proving vanishes. But in that case no preimage $\gamma^H \in H(F)$ of γ can be stably conjugate to an element of $S_0^H(F)$ and for the same reason the right hand side vanishes. Thus we may assume, after possibly conjugating in $G^u(F)$, that γ belongs to some $S_\lambda(F)$.

Lemma 5.1.1 implies that

$$j : S_0^H \xrightarrow{\mathrm{Ad}(q_0^H)^{-1}} T_0^H \xrightarrow{\eta^{-1}} T_0 \xrightarrow{\mathrm{Ad}(q_0)} S_0$$

is an admissible isomorphism defined over F. Put $\gamma_0 = \mathrm{Ad}(q_0 q_\lambda^{-1})\gamma$, $\gamma_0^H = j^{-1}(\gamma_0)$, and $\lambda' = \mathrm{Ad}(q_0)\lambda \in X_*(S_0)$.

Consider the right hand side of the equation in Theorem 3.4.2. Using Proposition 5.1.2, Fact 2.2.3, and the fact that $D(\gamma) = D(\gamma^H) = 1$ due to the topological semi-simplicity of γ, we obtain

$$\epsilon(G,H)\sum_{\gamma^H}\langle\mathrm{inv}(\gamma_0,\gamma),\widehat{\varphi}_{\gamma_0,\gamma^H}(s)\rangle^{-1}\mathcal{S}\Theta_{\upsilon^H}(\gamma^H) \tag{5.1}$$

where γ^H runs over the stable conjugacy classes of preimages of γ in $H(F)$.

Using Equation (10) in [DR09, 2.8] one easily computes that $\mathrm{inv}(\gamma_0,\gamma)$ is the element of $H^1(\Gamma/I_F,S_0(F^u))$ which assigns the value $\lambda'(\pi)$ to Fi. Using Lemmas 2.3.2 and 2.3.3 we see that

$$\langle\mathrm{inv}(\gamma_0,\gamma),\widehat{\varphi}_{\gamma_0,\gamma^H}(s)\rangle^{-1}=\lambda'(\widehat{\varphi}_{\gamma_0,\gamma^H}(s))$$

Consider the summation set of (5.1). If γ^H is not stably conjugate to an element of $S_0^H(F)$, then $\mathcal{S}\Theta_{\upsilon^H}(\gamma^H)$ vanishes by the same argument we used above. Thus the summation set can be identified with a set of representatives for the $H(F)$-stable classes of preimages of γ_0 in $S_0^H(F)$. Such a set is in bijection with a set of representatives for the quotient $\Omega(S_0^H,H)(F)\backslash\Omega(S_0,G)(F)$, where we interpret the smaller Weyl-group as a subgroup of the larger one via the isomorphism j. This bijection sends a representative y to $y\gamma_0^H$. If $\gamma^H=y\gamma_0^H$, then

$$\lambda'(\widehat{\varphi}_{\gamma_0,\gamma^H}(s))=(y\lambda')(\widehat{j}^{-1}(s))$$

and hence (5.1) becomes

$$\epsilon(G,H)\sum_{y\in\Omega(S_0^H,H)(F)\backslash\Omega(S_0,G)(F)}(y\lambda')(\widehat{j}^{-1}(s))\mathcal{S}\Theta_{\upsilon^H}(y\gamma_0^H) \tag{5.2}$$

According to [DR09, 11.2,9.6.2,2.11.2] we have

$$\mathcal{S}\Theta_{\upsilon^H}(y\gamma_0^H)=\epsilon(H,[T_0^H]^{w^H})\sum_{z\in\Omega(S_0^H,H)(F)}\theta_0^H(zy\gamma_0^H)$$

Using that $(y\lambda')(\widehat{j}^{-1}(s))=(zy\lambda')(\widehat{j}^{-1}(s))$ for $z\in\Omega(S_0^H,H)$ and $y\in\Omega(S_0,G)$, as well as the fact that $[T_0^H]^{w^H}\cong T_0^w$ (Lemma 5.1.1) we can rewrite (5.2) as

$$\epsilon(G,T_0^w)\sum_{y\in\Omega(S_0,G)(F)}(y\lambda')(\widehat{j}^{-1}(s))\theta_0^H(y\gamma_0^H) \tag{5.3}$$

By Lemma 5.1.1 for every topologically semi-simple element $\gamma^H\in S_0^H(F)$ we have $\theta_0^H(\gamma^H)=\theta_0(j(\gamma^H))$ and with this (5.3) becomes

$$\epsilon(G,T_0^w)\sum_{y\in\Omega(S_0,G)(F)}(y\lambda')(\widehat{j}^{-1}(s))y_*^{-1}\theta_0(\gamma_0) \tag{5.4}$$

Using the isomorphism $\mathrm{Ad}(q_0):T_0^w\to S_0$ and the fact that it identifies $W_o^{w\theta}$ with $\Omega(S_0,G)(F)$ ([DR09, 2.11.2]) we can rewrite (5.4) as

$$\epsilon(G,T_0^w)\sum_{y\in W_o^{w\theta}}(y\lambda)(\widehat{\eta}(s))y_*^{-1}\theta(\delta_0) \tag{5.5}$$

where now $\delta_0=\mathrm{Ad}(q_0)^{-1}\gamma_0\in T_0^w(F)$.

Now we will consider the left hand side of Theorem 3.4.2 and show that it also equals (5.5). By definition, we have

$$\Theta^s_{\upsilon,u}(\gamma)=\epsilon(G,G^u)\sum_{\rho\in\mathrm{Irr}(C_\upsilon,u)}\mathrm{tr}\,\rho(\widehat{\eta}(s))\Theta_{\pi_u(\upsilon,\rho)}(\gamma) \tag{5.6}$$

According to Diagram (3.1) the sum can be rewritten as

$$\Theta^s_{\upsilon,u}(\gamma) = \epsilon(G, G^u) \sum_{[\mu]\in[r^{-1}(u)]} \mu(\widehat{\eta}(s))\Theta_{\pi_u(\upsilon,\rho_\mu)}(\gamma) \tag{5.7}$$

where we recall that the elements $[\mu]$ belong to $[X_\Gamma]_{\mathrm{tor}}$. According to [DR09, 10.1.1], the summand corresponding to $[\mu]$ vanishes unless the tori S_μ and S_λ are $G^u(F)$-conjugate, which according to [DR09, 2.11.1] happens precisely when $[\lambda]$ and $[\mu]$ lie in the same $W_o^{w\theta}$-orbit. Thus (5.7) becomes

$$\epsilon(G, G^u) \sum_{[\mu]\in W_o^{w\theta}\cdot[\lambda]} \mu(\widehat{\eta}(s))\Theta_{\pi_u(\upsilon,\rho_\mu)}(\gamma) \tag{5.8}$$

Now fix $[\mu] \in W_o^{w\theta} \cdot [\lambda]$ and let $z_\mu \in W_o^{w\theta}$ be any element with $z_\mu \cdot [\lambda] = [\mu]$. According to the proof of [DR09, 2.11.1] there exists $g \in G^u(F)$ with the properties that $\mathrm{Ad}(g)S_\mu = S_\lambda$ and the images of $q_\lambda^{-1} g q_\mu$ and z_μ^{-1} in $\Omega(T_0, G)$ coincide. Then, applying [DR09, 10.1.1] and [DR09, 9.6.2], and recalling that $W^{w\theta}_{o,\mu}$ is the stabilizer of $[\mu]$ for the action of $W_o^{w\theta}$ on X_Γ, we obtain

$$\begin{aligned}
\Theta_{\pi_u(\upsilon,\rho_\mu)}(\gamma) &= \epsilon(G^u, T_0^w) \sum_{z\in W^{w\theta}_{o,\mu}} (g q_\mu z)_*\theta(\gamma) \\
&= \epsilon(G^u, T_0^w) \sum_{z\in W^{w\theta}_{o,\mu}} (q_\lambda)_*(z_\mu^{-1} z)_*\theta(\gamma) \\
&= \epsilon(G^u, T_0^w) \sum_{z\in W^{w\theta}_{o,\mu}} (z_\mu^{-1} z)_*\theta(\delta_0) \\
&= \epsilon(G^u, T_0^w) \sum_{z\in W^{w\theta}_{o,\lambda}} (z z_\mu^{-1})_*\theta(\delta_0)
\end{aligned}$$

where the last equation comes from the fact that $\mathrm{Ad}(z_\mu^{-1})$ provides an automorphism of $W_o^{w\theta}$ which sends $W^{w\theta}_{o,\mu}$ to $W^{w\theta}_{o,\lambda}$.

We can now plug this formula into (5.8). Observing that the summation set of (5.8) is in bijection with the quotient $W_o^{w\theta}/W^{w\theta}_{o,\lambda}$, we obtain

$$\epsilon(G, T_0^w) \sum_{y\in W_o^{w\theta}} [y\lambda](\widehat{\eta}(s)) y_*^{-1}\theta(\delta_0) \tag{5.9}$$

which is the same as (5.5). This concludes the proof of Theorem 3.4.2 for topologically semi-simple elements.

6 A formula for the unstable character

The purpose of this section is to establish a reduction formula, similar to the ones in [DR09, §9,§10], for $\Theta^t_{\upsilon,u}$. Before we can do so, we need some cohomological facts.

6.1 Cohomological lemmas II

Recall the diagram (3.1). We call a_G the composition

$$H^1(\Gamma, G) \to \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma))$$

of the right vertical isomorphisms in this diagram. In [Kot86, Thm 1.2] Kottwitz defines another isomorphism

$$H^1(\Gamma, G) \to \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma))$$

which he calls α_G. This isomorphism can be normalized in two different ways, and the two normalizations differ by a sign.

Lemma 6.1.1. *Depending on the normalization of α_G, one has*

$$a_G = \pm \alpha_G$$

Proof: Assume first that $G = T$ is a torus. One normalization of the isomorphism α_G is then given by the composition

$$H^1(\Gamma, T) \to \mathrm{Irr}(H^1(\Gamma, X^*(T))) \to \mathrm{Irr}(\pi_0(\widehat{T}^\Gamma))$$

where the first map arises via the cup product pairing

$$H^1(\Gamma, X^*(T)) \otimes H^1(\Gamma, T) \to \mathbb{C}^\times$$

and the second map is the dual of the isomorphism $\pi_0(\widehat{T}^\Gamma) \to H^1(\Gamma, X_*(\widehat{T}))$ of Lemma 2.3.1.

Thus if we precompose α_G by TN then by Lemma 2.3.2 the resulting isomorphism

$$[X_*(T)_\Gamma]_{\mathrm{tor}} \to \mathrm{Irr}(\pi_0(\widehat{T}^\Gamma))$$

will be given by the standard pairing $\widehat{T} \times X^*(\widehat{T}) \to \mathbb{C}^\times$.

On the other hand if we precompose a_G by TN then by Lemma 2.3.4 the resulting isomorphism

$$[X_*(T)_\Gamma]_{\mathrm{tor}} \to \mathrm{Irr}(\pi_0(\widehat{T}^\Gamma))$$

will be given by the negative of the standard pairing $\widehat{T} \times X^*(\widehat{T}) \to \mathbb{C}^\times$.

This proves that in the case $G = T$ with our normalization of α_G we have $a_G = -\alpha_G$. For the general case let $T \subset G$ be an elliptic maximal torus and consider the commutative diagrams

$$\begin{array}{ccc} \mathrm{Irr}(\pi_0(\widehat{T}^\Gamma)) & \longrightarrow & \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma)) \\ {\scriptstyle a_T}\uparrow & & \uparrow{\scriptstyle a_G} \\ H^1(\Gamma, T) & \longrightarrow & H^1(\Gamma, G) \end{array} \qquad \begin{array}{ccc} \mathrm{Irr}(\pi_0(\widehat{T}^\Gamma)) & \longrightarrow & \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma)) \\ {\scriptstyle -\alpha_T}\uparrow & & \uparrow{\scriptstyle -\alpha_G} \\ H^1(\Gamma, T) & \longrightarrow & H^1(\Gamma, G) \end{array}$$

The fact that the right diagram commutes is part of the statement of [Kot86, Thm 1.2], while for the left diagram it follows from the construction. We just proved that the left vertical arrows in the two diagrams coincide. But since T is elliptic, the bottom horizontal maps are surjective by [Kot86, Lemma 10.1]. Thus the right vertical maps in the two diagrams must also coincide. $\square$

Lemma 6.1.2. *Let $Q_0 \in \mathrm{Lie}(S_0)(F)$ be a regular semi-simple element and $\lambda \in r^{-1}(u)$. Put $Q_\lambda := \mathrm{Ad}(q_\lambda q_0^{-1})Q_0$. Then*

1. *$Q_\lambda \in \mathrm{Lie}(S_\lambda)(F)$.*
2. *The image of λ under the map*

$$[X_\Gamma]_{\mathrm{tor}} \stackrel{\mathrm{DR}}{\longrightarrow} H^1(\Gamma, T_0^w) \stackrel{\mathrm{Ad}(q_0)}{\longrightarrow} H^1(\Gamma, S_0)$$

   equals $\mathrm{inv}(Q_0, Q_\lambda)$.

3. *The map $\lambda \mapsto Q_\lambda$ establishes a bijection from $[r^{-1}(u)]$ to a set of representatives for the conjugacy classes of elements in $\mathrm{Lie}(G^u)(F)$ stably conjugate to Q_0.*
4. *Let $t \in [\widehat{T}_0^w]^\Gamma$ and t_{q_0} be its image under the dual of $\mathrm{Ad}(q_0^{-1}) : S_0 \to T_0^w$. Then*

$$\langle \mathrm{inv}(Q_0, Q_\lambda), t_{q_0} \rangle^{-1} = \lambda(t)$$

Proof: Recall from [DR09, §2.8] the equations

$$q_\lambda^{-1} u \mathrm{Fi}_G(q_\lambda) = t_\lambda \dot{w} \qquad q_0^{-1} \mathrm{Fi}_G(q_0) = \dot{w}$$

where $t_\lambda = \lambda(\pi)$. The inner twist ψ is unramified, so $Q_\lambda \in \mathrm{Lie}(S_\lambda)(F^u)$. To prove that $Q_\lambda \in \mathrm{Lie}(S_\lambda)(F)$ it is enough to show that it is fixed by $\mathrm{Fi}_{\mathrm{G^u}} = \mathrm{Ad}(u) \circ \mathrm{Fi}_{\mathrm{G}}$.

$$\begin{aligned} \mathrm{Ad}(u)\mathrm{Fi}_G(Q_\lambda) &= \mathrm{Ad}(u)\mathrm{Fi}_G \mathrm{Ad}(q_\lambda q_0^{-1}) Q_0 \\ &= \mathrm{Ad}(u \mathrm{Fi}_G(q_\lambda q_0^{-1})) Q_0 \\ &= \mathrm{Ad}(q_\lambda \dot{w} \mathrm{Fi}_G(q_0^{-1}) Q_0 \\ &= \mathrm{Ad}(q_\lambda \dot{w} \dot{w}^{-1} q_0^{-1}) Q_0 \\ &= Q_\lambda \end{aligned}$$

This proves the first assertion.

By construction the element $\mathrm{inv}(Q_0, Q_\lambda)$ is given by the cocycle

$$\sigma \mapsto q_0 q_\lambda^{-1} u \sigma(q_\lambda q_0^{-1})$$

We compute the value of this cocycle at Fi

$$\begin{aligned} q_0 q_\lambda^{-1} u \mathrm{Fi}_G(q_\lambda q_0^{-1}) &= q_0 t_\lambda \dot{w} \mathrm{Fi}_G(q_0^{-1}) \\ &= \mathrm{Ad}(q_0)(t_\lambda \dot{w} (q_0^{-1} \mathrm{Fi}_G(q_0))^{-1}) \\ &= \mathrm{Ad}(q_0)(t_\lambda) \end{aligned}$$

This proves the second assertion.

The third assertion follows immediately from the second and Lemma 2.1.5 (or rather from its Lie-algebra analog, which is proved in exactly the same way).

Finally, by functoriality of the Tate-Nakayama pairing we have

$$\langle \mathrm{inv}(Q_0, Q_\lambda), t_{q_0} \rangle^{-1} = \langle \mathrm{Ad}(q_0^{-1}) \mathrm{inv}(Q_0, Q_\lambda), t \rangle^{-1}$$

By the second assertion and Lemma 2.3.4 the element $\mathrm{Ad}(q_0^{-1})(\mathrm{inv}(Q_0, Q_\lambda))^{-1}$ of $H^1(\Gamma, T_0^w)$ is the image of λ under the Tate-Nakayama isomorphism

$$H^{-1}_{\mathrm{Tate}}(\Gamma, X_*(T_0^w)) \to H^1(\Gamma, T_0^w)$$

and hence by Lemma 2.3.2 we have

$$\langle \mathrm{Ad}(q_0^{-1}) \mathrm{inv}(Q_0, Q_\lambda)^{-1}, t \rangle = \lambda(t)$$

□

6.2 A reduction formula for the unstable character

We now return to the computation of $\Theta^t_{\upsilon,u}$.

The map

$$[X_\Gamma]_{\rm tor} \to {\rm Irr}(C_\upsilon), \qquad \lambda \mapsto \rho_\lambda$$

identifies $[r^{-1}(u)]$ with ${\rm Irr}(C_\upsilon, u)$. Since it is given simply by restriction of characters, we have ${\rm tr}\,\rho_\lambda(t) = \lambda(t)$. Moreover $e(G^u) = \epsilon(G, G^u)$, so

$$\Theta^t_{\upsilon,u} = \epsilon(G, G^u) \sum_{\lambda \in [r^{-1}(u)]} \lambda(t)\Theta_{\pi_u(\upsilon,\rho_\lambda)}$$

Our first goal is to use the results of [DR09, §9,§10] to derive a formula for $\Theta_{\pi_u(\upsilon,\rho_\lambda)}$ which is suitable for our purposes. Recall that there is a regular depth-zero character $\theta : T^w_0(F) \to \mathbb{C}^\times$ determined by the Langlands parameter υ.

Lemma 6.2.1. *Let $\lambda \in r^{-1}(u)$, $\theta_\lambda = {\rm Ad}(q_\lambda)_*\theta$, and $Q_\lambda \in {\rm Lie}(S_\lambda)(F)$ be any fixed regular semi-simple element. Then for any $\gamma \in G^u_{\rm sr}(F)_0$ and any $z \in Z(F)$ we have*

$$\Theta_{\pi_u(\upsilon,\rho_\lambda)}(z\gamma) = \epsilon(G^u, A_G)\theta(z) \sum_Q R(G^u_{\gamma_s}, S_Q, 1)(\gamma_u)[\varphi_{Q_\lambda,Q}]_*\theta_\lambda(\gamma_s)$$

where $S_Q = {\rm Cent}(Q, G^u)$ and the sum runs over any set of representatives for the $G^u_{\gamma_s}(F)$-conjugacy classes inside the $G^u(F)$-conjugacy class of Q_λ.

Proof: By [DR09, Lemmas 9.3.1,9.6.2] we know

$$\Theta_{\pi_u(\upsilon,\rho_\lambda)}(z\gamma) = \epsilon(G^u, S_\lambda)\theta(z)R(G^u, S_\lambda, \theta_\lambda)(\gamma)$$

We will apply [DR09, Lemma 10.0.4] to the last factor, but first we want to study the indexing set of the sum appearing in the formula of that lemma. This indexing set is

$$Y := \{(S', \theta') \in {\rm Ad}(G^u(F))(S_\lambda, \theta_\lambda) |\, \gamma_s \in S'\}/{\rm Ad}(G^u_{\gamma_s}(F))$$

First we claim that the map

$$\begin{array}{rcl} {\rm Ad}(G^u(F))Q_\lambda & \to & {\rm Ad}(G^u(F))(S_\lambda, \theta_\lambda) \\ Q & \mapsto & \varphi_{Q_\lambda,Q}(S_\lambda, \theta_\lambda) \end{array}$$

is a bijection. It is clearly well-defined, and is moreover surjective because if ${\rm Ad}(g)(S_\lambda, \theta_\lambda)$ belongs to the right hand side, then ${\rm Ad}(g)Q_\lambda$ belongs to the left hand side and is a preimage. For the injectivity let

$$(S', \theta') = \varphi_{Q_\lambda,Q}(S_\lambda, \theta_\lambda) = \varphi_{Q_\lambda,Q'}(S_\lambda, \theta_\lambda)$$

Then $\varphi_{Q',Q} \in \Omega(S', G^u)$ and $\varphi_{Q',Q}\theta' = \theta'$. Since θ' is regular, $\varphi_{Q',Q} = 1$ and thus $Q' = Q$.

This proves the claimed bijectivity. Moreover, since $\varphi_{Q_\lambda,Q}(S_\lambda) = S_Q$ and

$$\gamma_s \in S_Q \Rightarrow S_Q \subset G_{\gamma_s} \Rightarrow Q \in {\rm Lie}(S_Q) \subset {\rm Lie}(G^u_{\gamma_s}) \Rightarrow \gamma_s \in S_Q$$

we see that our bijection restricts to the bijection

$$\begin{array}{rcl} \mathrm{Ad}(G^u(F))Q_\lambda \cap \mathrm{Lie}(G^u_{\gamma_s})(F) & \to & \{(S',\theta') \in \mathrm{Ad}(G^u(F))(S_\lambda,\theta_\lambda) | \, \gamma_s \in S'\} \\ Q & \mapsto & \varphi_{Q_\lambda,Q}(S_\lambda,\theta_\lambda) \end{array}$$

Both sides of this bijection carry a natural action of $G^u_{\gamma_s}(F)$ and the bijection is equivariant with respect to these actions. Thus if we put

$$Y' := [\mathrm{Ad}(G^u(F))Q_\lambda \cap \mathrm{Lie}(G^u_{\gamma_s})(F)]/\mathrm{Ad}(G^u_{\gamma_s}(F))$$

we obtain a bijection

$$Y' \to Y$$

Applying now [DR09, Lemma 10.0.4] we obtain

$$\Theta_{\rho_\lambda}(z\gamma) = \epsilon(G^u,S_\lambda)\theta(z) \sum_{[Q]\in Y'} R(G^u_{\gamma_s},S_Q,1)(\gamma_u)[\varphi_{Q_\lambda,Q}]_*\theta_\lambda(\gamma_s)$$

To complete the lemma, we only need to observe that since S_λ/Z is anisotropic, the maximal split subtorus of S_λ is A_G and thus $\epsilon(G^u,S_\lambda) = \epsilon(G^u,A_G)$. □

We are now ready to establish the reduction formula for the t-unstable character.

Proposition 6.2.2. *Let $Q_0 \in \mathrm{Lie}(S_0)(F)$ be a regular semi-simple element, $\theta_0 = \mathrm{Ad}(q_0)_*\theta$, and t_{q_0} be the image of t under the dual of $\mathrm{Ad}(q_0^{-1})$. Then for any $\gamma \in G^u_{\mathrm{sr}}(F)_0$ and any $z \in Z(F)$ the value of $\Theta^t_{\upsilon,u}(z\gamma)$ is given by*

$$\epsilon(G,A_G)\theta(z)\sum_P[\varphi_{Q_0,P}]_*\theta_0(\gamma_s)\sum_Q\langle\mathrm{inv}(Q_0,Q),t_{q_0}\rangle^{-1}R(G^u_{\gamma_s},S_Q,1)(\gamma_u)$$

where P runs over a set of representatives for the $G^u_{\gamma_s}$-stable classes of elements of $\mathrm{Lie}(G^u_{\gamma_s})(F)$ which are G^u-stable conjugate to Q_0, and Q runs over a set of representatives for the $G^u_{\gamma_s}(F)$-conjugacy classes inside the $G^u_{\gamma_s}$-stable class of P.

Proof: For each $\lambda \in r^{-1}(u)$ put $\theta_\lambda = \mathrm{Ad}(q_\lambda)_*\theta$ and $Q_\lambda = \mathrm{Ad}(q_\lambda q_0^{-1})Q_0$. Then by Lemma 6.1.2 we know that $Q_\lambda \in \mathrm{Lie}(S_\lambda)(F)$ is regular semi-simple, and so applying Lemma 6.2.1 and using the transitivity of the sign $\epsilon(\cdot,\cdot)$ we obtain

$$\Theta^t_{\upsilon,u} = \epsilon(G,A_G)\theta(z)\sum_{\lambda\in[r^{-1}(u)]}\lambda(t)\sum_Q R(G^u_{\gamma_s},S_Q,1)(\gamma_u)[\varphi_{Q_\lambda,Q}]_*\theta_\lambda(\gamma_s)$$

where the sum runs over the $G^u_{\gamma_s}(F)$-conjugacy classes inside the intersection of the $G^u(F)$-conjugacy class of Q_λ with $\mathrm{Lie}(G^u_{\gamma_s})(F)$. We obviously have

$$[\varphi_{Q_\lambda,Q}]_*\theta_\lambda = [\varphi_{Q_\lambda,Q}]_*[\varphi_{Q_0,Q_\lambda}]_*\theta_0 = [\varphi_{Q_0,Q}]_*\theta_0$$

and thus

$$\Theta^t_{\upsilon,u} = \epsilon(G,A_G)\theta(z)\sum_{\lambda\in[r^{-1}(u)]}\lambda(t)\sum_Q R(G^u_{\gamma_s},S_Q,1)(\gamma_u)[\varphi_{Q_0,Q}]_*\theta_0(\gamma_s)$$

Applying again Lemma 6.1.2 we obtain

$$\Theta^t_{\upsilon,u} = \epsilon(G,A_G)\theta(z)\sum_{Q'}\langle\mathrm{inv}(Q_0,Q'),t_{q_0}\rangle^{-1}\sum_Q R(G^u_{\gamma_s},S_Q,1)(\gamma_u)[\varphi_{Q_0,Q}]_*\theta_0(\gamma_s)$$

where now Q' runs over a set of representatives for the $G^u(F)$-classes inside the G^u-stable class of Q_0, and Q runs over a set of representatives for the $G^u_{\gamma_s}(F)$-classes inside the intersection of the $G^u(F)$-class of Q' with $\mathrm{Lie}(G^u_{\gamma_s})(F)$.

For any Q' in the first summation set and Q in the second, we have

$$\mathrm{inv}(Q_0, Q') = \mathrm{inv}(Q_0, Q)$$

because Q' and Q are $G^u(F)$-conjugate. Thus we obtain

$$\Theta^t_{\upsilon,u} = \epsilon(G, A_G)\theta(z) \sum_Q \langle \mathrm{inv}(Q_0, Q), t_{q_0}\rangle^{-1} R(G^u_{\gamma_s}, S_Q, 1)(\gamma_u)[\varphi_{Q_0,Q}]_* \theta_0(\gamma_s)$$

where now Q runs over a set of representatives for the $G^u_{\gamma_s}(F)$-classes inside the intersection of the G^u-stable class of Q_0 with $\mathrm{Lie}(G^u_{\gamma_s})(F)$. The sum over Q can be written again as an iterated sum, where we first sum over $G^u_{\gamma_s}$-stable classes inside the intersection of the G^u-stable class of Q_0 with $\mathrm{Lie}(G^u_{\gamma_s})(F)$, and then over $G^u_{\gamma_s}(F)$-classes inside each $G^u_{\gamma_s}$-stable class. More precisely we have

$$\Theta^t_{\upsilon,u} = \epsilon(G, A_G)\theta(z) \sum_P \sum_Q \langle \mathrm{inv}(Q_0, Q), t_{q_0}\rangle^{-1} R(G^u_{\gamma_s}, S_Q, 1)(\gamma_u)[\varphi_{Q_0,Q}]_* \theta_0(\gamma_s)$$

where P runs over a set of representatives for the $G^u_{\gamma_s}$-stable classes of elements of $\mathrm{Lie}(G^u_{\gamma_s})(F)$ which are G^u-stable conjugate to Q_0, and Q runs over a set of representatives for the $G^u_{\gamma_s}(F)$-conjugacy classes inside the $G^u_{\gamma_s}$-stable class of P.

Now consider two elements P, Q in the above iterated summation set. By assumption they are $G^u_{\gamma_s}$-conjugate, thus $\varphi_{P,Q} = \mathrm{Ad}(g)$ with $g \in G^u_{\gamma_s}$. Since $\gamma_s \in S_Q$ the expression $\varphi_{Q,P}(\gamma_s)$ is defined and we conclude that it equals γ_s. Thus

$$\begin{aligned} [\varphi_{Q_0,Q}]_* \theta_0(\gamma_s) &= [\varphi_{P,Q}]_* [\varphi_{Q_0,P}]_* \theta_0(\gamma_s) \\ &= [\varphi_{Q_0,P}]_* \theta_0(\varphi^{-1}_{P,Q}(\gamma_s)) \\ &= [\varphi_{Q_0,P}]_* \theta_0(\gamma_s) \end{aligned}$$

Rearranging terms again we arrive at the desired formula for $\Theta^t_{\upsilon,u}$. □

7 Character identities

In this section we use all the notation established in the previous sections, in particular all parts of Section 3. Our goal is to prove Theorem 3.4.2.

7.1 Beginning of the proof of Theorem 3.4.2

Lemma 7.1.1. *Let D be a diagonalizable group defined over F and split over F^u. Then*

$$D(F) = D_s(F) \cdot \left[D(F) \cap \bigcap_{\chi \in X^*(D)} [\ker(\upsilon \circ \chi)] \right]$$

where D_s is the maximal split subtorus of D.

Proof: For any $x \in D(F^u)$ the map

$$\lambda_x : X^*(D) \to \mathbb{Z}, \qquad \chi \mapsto v(\chi(x))$$

is $\mathbb{Z}$-linear. Sending x to λ_x defines a homomorphism

$$D(F^u) \to X_*(D)$$

which is Γ-equivariant because of the Γ-invariance of $v : [F^u]^\times \to \mathbb{Z}$. A right inverse of this homomorphism is given by evaluation at π.

Now let $x \in D(F)$. Then $\lambda_x \in X_*(D)$ is Γ-fixed, and so its image $y = \pi^{\lambda_x} \in D(F^u)$ under the right inverse lies in $D_s(F)$. Thus $x = y \cdot (xy^{-1})$ is the desired decomposition. □

Lemma 7.1.2. *Assume that γ does not belong to $Z(F)G^u(F)_0$. Then both sides of* (3.2) *vanish.*

Proof: The left hand side vanishes by [DR09, Lemma 9.3.1]. Turning to the right hand side, assume by way of contradiction that some γ^H in the summation set of (3.2) lies in $A_H(F)H(F)_0$, and write $\gamma^H = zx$. The admissible isomorphism $\varphi_{\gamma^H,\gamma}$ sends x into $G^u(F)_0$ and because H is elliptic it maps $A_H(F)$ to $A_G(F)$. Thus $\gamma = \varphi_{\gamma^H,\gamma}(\gamma_H) \in A_G(F)G^u(F)_0$ contradicting the assumption of the lemma. We conclude that all γ^H occurring in the summation set of (3.2) lie outside of $A_H(F)H(F)_0$. But by the previous lemma, $Z_H(F)H(F)_0 = A_H(F)H(F)_0$, because the set

$$Z_H(F) \cap \bigcap_{\chi \in X^*(Z_H)} [\ker(v_F \circ \chi)]$$

lies in the maximal bounded subgroup of $T_0^H(F)$. By [DR09, Lemma 9.3.1] the left hand side of (3.2) also vanishes. □

Corollary 7.1.3. *If equation* (3.2) *holds for all strongly regular semi-simple $\gamma \in G^u(F)_0$, then it holds for all strongly-regular semi-simple $\gamma \in G^u(F)$.*

Proof: By Lemma 7.1.2 equation (3.2) holds trivially if γ does not belong to $Z(F)G^u(F)_0$. By Lemma 7.1.1 we have $Z(F)G^u(F)_0 = A_G(F)G^u(F)_0$, so it is enough to prove equation (3.2) for strongly regular semi-simple elements $\gamma = z\gamma'$ with $z \in A_G(F)$ and $\gamma' \in G^u(F)_0$. By Proposition 6.2.2 we know the behavior of the unstable character under central translations, namely $\Theta^s_{\upsilon,u}(z\gamma) = \theta(z)\Theta^s_{\upsilon,u}(\gamma)$ and thus using the assumption of the corollary we have

$$\begin{aligned}
\Theta^s_{\upsilon,u}(z\gamma) &= \theta(z) \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_{\psi,u}(\gamma^H,\gamma)\frac{D(\gamma^H)^2}{D(\gamma)^2}\mathcal{S}\Theta_{\upsilon^H}(\gamma^H) \\
&= \theta^H(\eta(z))\lambda^G(z) \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_{\psi,u}(\gamma^H,\gamma)\frac{D(\gamma^H)^2}{D(\gamma)^2}\mathcal{S}\Theta_{\upsilon^H}(\gamma^H)
\end{aligned}$$

where for the second equality we have invoked Lemma 5.1.1. Using [LS87, Lemma 4.4.A] and the obvious invariance of the terms $D(\gamma)$ and $D^H(\gamma^H)$ under central translations this can be written as

$$\begin{aligned}
&= \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_{\psi,u}(\eta(z)\gamma^H, z\gamma)\frac{D(\eta(z)\gamma^H)^2}{D(z\gamma)^2}\mathcal{S}\Theta_{\upsilon^H}(\eta(z)\gamma^H) \\
&= \sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_{\psi,u}(\gamma^H, z\gamma)\frac{D(\gamma^H)^2}{D(z\gamma)^2}\mathcal{S}\Theta_{\upsilon^H}(\gamma^H)
\end{aligned}$$

which was to be shown. □

7.2 A reduction formula for the endoscopic lift of the stable character

Lemma 7.2.1. *Let J be an unramified F-group and $y \in J(F)$ be a topologically semi-simple element belonging to a hyperspecial maximal compact subgroup. Let γ be an element of either $J(F)$ or $\mathrm{Lie}(J)(F)$ for which $\mathrm{Cent}(\gamma, J) \subset J_y$. Then the finite group $J^y(F)/J_y(F)$ acts simply transitively on the set of J_y-stable classes inside the J^y-stable class of γ.*

Proof: Clearly $J^y(F)$ acts on the J^y-stable class of γ, and $J_y(F)$ preserves each J_y-stable class inside, so that we obtain an action of $J^y(F)/J_y(F)$ on the set of J_y-stable classes inside the J^y-stable class of γ.

Consider the sequence

$$1 \to J_y(F) \to J^y(F) \to \pi_0(J^y)(F) \to H^1(F, J_y) \to H^1(F, J^y)$$

By [Kot86, Prop 7.1] the last arrow has trivial kernel, which implies that the third arrow is surjective, so that we have a short exact sequence

$$1 \to J_y(F) \to J^y(F) \to \pi_0(J^y)(F) \to 1$$

Let γ' be J^y-stably conjugate to γ, and pick $g \in J^y(\overline{F})$ s.t. $\mathrm{Ad}(g)\gamma = \gamma'$ and $g^{-1}\sigma(g) \in J_\gamma \subset J_y$ for any $\sigma \in \Gamma$. Then the image $\bar{g} \in \pi_0(J^y)$ of g belongs to $\pi_0(J^y)(F)$. Let $h \in J^y(F)$ be a preimage of $\bar{g}$. Then $\mathrm{Ad}(h)\gamma$ and γ' are stably conjugate by $gh^{-1} \in J_y(\overline{F})$. This proves transitivity.

To show simplicity, let γ' by J_y-stably conjugate to γ and pick $h \in J_y(\overline{F})$ s.t. $\mathrm{Ad}(h)\gamma = \gamma'$. If $g \in J^y(F)$ is also an element s.t. $\gamma' = \mathrm{Ad}(g)\gamma$, then $h^{-1}g \in \mathrm{Cent}(\gamma, J) \subset J_y$ so $g \in J_y(\overline{F}) \cap J^y(F) = J_y(F)$. □

Remark: The same proof shows that under the same assumptions, the finite group $J^y(F)/J_y(F)$ acts simply transitively on the set of $\mathrm{Ad}J_y(\overline{F})$-orbits in $\mathrm{Ad}J^y(\overline{F})\gamma \cap J_y(F)$.

Recall from Section 3.1 the maximal tori $S_0 \subset G$ and $S_\lambda \subset G^u$ for $\lambda \in r^{-1}(u)$.

Lemma 7.2.2. *Let $\gamma' \in G^u(F)$ be a strongly-regular semi-simple element. Assume that for some $\lambda \in r^{-1}(u)$ we have $\gamma'_s \in S_\lambda(F)$. Then there exists a $\gamma \in G(F)$ stably conjugate to γ' s.t. $\gamma_s \in S_0(F)$.*

Proof: By construction we know that $\mathrm{Ad}(q_0 q_\lambda^{-1}) : S_\lambda \to S_0$ is an isomorphism over F. Put $t = \mathrm{Ad}(q_0 q_\lambda^{-1})\gamma'_s$. Then t, being topologically semi-simple, belongs to the maximal bounded subgroup of $S_0(F)$ and thus lies in $G(O_F)$. It follows form [Kot86, Prop. 7.1] that G_t is quasi-split. The map $\mathrm{Ad}(q_0 q_\lambda^{-1}) : G^u_{\gamma_s} \to G_t$ is a twist and so there exists an $i \in G_t(\overline{F})$ s.t. if $T' = \mathrm{Cent}(\gamma', G^u_{\gamma'_s})$ and $f = \mathrm{Ad}(i q_0 q_\lambda^{-1})$ then the torus $T := f(T')$ and the isomorphism $f : T' \to T$ are defined over F. Put $\gamma = f(\gamma')$. By construction $\gamma_s = t$ and f is a (ψ, u)-equivalent twist, so γ is the element we want. □

Remark: The same proof can be applied to an element $\gamma^H \in H(F)$ and the trivial twist $(\mathrm{id}, 1) : H \to H$.

Lemma 7.2.3. *Let $\gamma \in G(F)_0$ and $\gamma^H \in H(F)_0$ be a pair of related strongly G-regular elements s.t. $\gamma_s \in S_0(F)$ and $\gamma^H_s \in S^H_0(F)$. Then the admissible isomorphism*

$\varphi_{\gamma^H,\gamma}$ makes $H_{\gamma_s^H}$ into an endoscopic group for G_{γ_s}. If Δ_0 and Δ_0' denote the transfer factors for (G, H) and $(G_{\gamma_s}, H_{\gamma_s^H}, \varphi_{\gamma^H,\gamma})$ normalized with respect to admissible splittings (in the sense of [Hal93, §7]) then one has

$$\Delta_0(\gamma^H, \gamma) = \Delta_0'(\gamma_u^H, \gamma_u)$$

Proof: By [Kot86, Prop. 7.1] both groups $H_{\gamma_s^H}$ and G_{γ_s} are unramified, so they fall in the framework of [Hal93] and one has the normalization Δ_0' of the transfer factor with respect to an admissible splitting. We want to apply [Hal93, Thm. 10.18] to conclude

$$\Delta_0(\gamma^H, \gamma) = \Delta_0'(\gamma^H, \gamma)$$

This theorem has two requirements. One is $p > e_G + 1$, which is given in the statement of the theorem, and which we are assuming. The other one is $\gamma \in G(O_F)$ and $\gamma^H \in H(O_F)$, which is a general requirement for the whole section 10 in loc. cit. However, tracing through the arguments of that section one sees that up to the proof of Thm. 10.18, the only property of γ and γ^H which is used is that fact that they are compact and so have a topological Jordan decomposition, together with the fact that homomorphisms preserve the topological Jordan decomposition and the knowledge of its explicit form for elements of extensions of F. In the proof of Thm. 10.18 the elements γ^H and γ are replaced by high powers of themselves, let's call them γ'^H and γ', which are very close (and can be made arbitrarily close) to γ_s^H and γ_s. Then Lemma 8.1. of loc. cit. is involved for the pair (γ'^H, γ'). For that Lemma it is essential that $\gamma'^H \in H(O_F)$ and $\gamma' \in G(O_F)$. But from our assumptions it follows that $\gamma_s^H \in H(O_F)$ and $\gamma_s \in G(O_F)$, and since these groups are open, and the elements γ'^H and γ' can be made arbitrarily close to γ_s^H and γ_s, Lemma 8.1 can be invoked.

Thus we conclude that

$$\Delta_0(\gamma^H, \gamma) = \Delta_0'(\gamma^H, \gamma)$$

By [LS90, §3.5] there exists a character $\lambda : Z_{G_{\gamma_s}}(F) \to \mathbb{C}^\times$ s.t. for all strongly regular elements $z^H \in H_{\gamma_s^H}(F)$ and $z \in G_{\gamma_s}(F)$ one has

$$\Delta_0'(z^H \gamma_s^H, z\gamma_s) = \lambda(\gamma_s)\Delta_0'(z^H, z)$$

Thus

$$\frac{\Delta_0'(\gamma^H, \gamma)}{\Delta_0'(\gamma_u^H, \gamma_u)} = \lambda(\gamma_s) = \frac{\Delta_0'(z^H \gamma_s^H, z\gamma_s)}{\Delta_0'(z^H, z)}$$

We choose z to lie in an unramified torus $T \subset G_{\gamma_s}$. Then

$$\frac{\Delta_0'(z^H \gamma_s^H, z\gamma_s)}{\Delta_0'(z^H, z)} = \langle a, \gamma_s \rangle$$

where $\langle a, \cdot \rangle$ is a character $T(F) \to \mathbb{C}^\times$ constructed in [LS87, §3.5]. By [Hal93, Lemma 11.2] this character is unramified, and thus takes trivial value at γ_s. It follows that

$$\Delta_0'(\gamma^H, \gamma) = \Delta_0'(\gamma_u^H, \gamma_u)$$

and the proof is complete. □

Lemma 7.2.4. *For $\gamma \in G(F)_0$ the expression*

$$\sum_{\gamma^H \in H_{\mathrm{sr}}(F)/\mathrm{st}} \Delta_0(\gamma^H,\gamma)\frac{D(\gamma^H)^2}{D(\gamma)^2}\mathcal{S}\Theta_{\upsilon^H}(\gamma^H) \tag{7.1}$$

is equal to

$$\sum_y \sum_\xi |H^y(F)/H_y(F)|^{-1} \sum_{z \in H_y(F)_{\mathrm{sr}}/\mathrm{st}} \Delta_{0,y,\xi}(z,\gamma_u)\frac{D_{H_y}(z)^2}{D_{G_{\gamma_s}}(\gamma_u)^2}\mathcal{S}\Theta_{\upsilon^H}(yz) \tag{7.2}$$

where y runs over a subset of $S_0^H(F)$ consisting of representatives for the stable classes of preimages of γ_s which lie in $S_0^H(F)$, ξ runs over the (H_y, G_{γ_s})-equivalence classes of admissible embeddings mapping y to γ_s, and $\Delta_{0,y,\xi}$ is the absolute transfer factor for (H_y, G_{γ_s}, ξ) normalized with respect to an admissible splitting.

Proof: The sum of the first expression runs over the set of stable classes of strongly-regular semi-simple elements in $H(F)$. If $\gamma^H \in H(F)$ is strongly-regular semi-simple, but γ_s^H does not lie in a torus which is stably conjugate to S_0^H, then by Proposition 6.2.2 we have $\mathcal{S}\Theta_{\upsilon^H}(\gamma^H) = 0$. Moreover if γ_s^H is not a preimage of γ_s, then γ^H is not a preimage of γ and so $\Delta_0(\gamma^H,\gamma) = 0$. Thus if $\Gamma^H \subset H(F)$ is the set of strongly-regular semi-simple elements γ^H for which γ_s^H is a preimage of γ_s and lies in a torus stably conjugate to S_0^H, then we may restrict the summation in the first expression to Γ^H/st. Let $Y \subset S_0^H(F)$ be a set of representatives for the stable classes of those elements of $S_0^H(F)$ which occur as the topologically semi-simple parts of elements in Γ^H. We claim that we have a surjective map

$$p : \Gamma^H/\mathrm{st} \to Y$$

By Lemma 7.2.2 and the remark thereafter every stable class $\mathcal{C} \subset \Gamma^H$ has a representative γ^H s.t. $\gamma_s^H \in S_0^H(F)$. By construction there exists $y \in Y$ stably conjugate to γ_s^H. By [Kot86, Prop. 7.1] there exists $h \in H(O_F)$ s.t. $\mathrm{Ad}(h)\gamma_s^H = y$. But then $\mathrm{Ad}(h)\gamma^H \in \mathcal{C}$. We see that there are elements in $\mathcal{C}$ whose topologically semi-simple parts lie in Y. If $\gamma^H, \gamma'^H \in \mathcal{C}$ are two such elements, then their stable conjugacy implies the stable conjugacy of their topologically semi-simple parts, but by construction of Y this means that their topologically semi-simple parts are actually equal. Thus we may define $p(\mathcal{C})$ by choosing any $\gamma^H \in \mathcal{C}$ with $\gamma_s^H \in Y$ and sending it to γ_s^H.

Next we claim that for every $y \in Y$ we have a surjective map

$$[H_y(F)]_{(H,y)-\mathrm{sr},\mathrm{tu}}/\mathrm{st} \to p^{-1}(y), \qquad z \mapsto yz$$

whose fibers are torsors under $H^y(F)/H_y(F)$. Here $[H_y(F)]_{(H,y)-\mathrm{sr},\mathrm{tu}}$ denotes the set of topologically unipotent elements $z \in H_y(F)$ for which yz is H-strongly regular, and st is stable conjugacy in H_y. It is immediate that this map is well-defined and surjective. We claim that each fiber constitutes a single H^y-stable class. If z, z' lie in the same fiber, then there exists $h \in H(\overline{F})$ s.t. $\mathrm{Ad}h(yz) = yz'$. But then $\mathrm{Ad}h(y) = y$, so $h \in H^y(\overline{F})$, and $\mathrm{Ad}h(z) = z'$, which shows that z, z' lie in the same H^y-stable class. Conversely if z, z' lie in the same H^y-stable class then they clearly lie in the same fiber. From Lemma 7.2.1 it now follows that the fibers are torsors under $H^y(F)/H_y(F)$.

We conclude that expression (7.1) is equal to

$$\sum_{y \in Y} |H^y(F)/H_y(F)|^{-1} \sum_{z \in [H_y(F)]_{(H,y)-\mathrm{sr},\mathrm{tu}}/\mathrm{st}} \Delta_0(yz,\gamma)\frac{D(yz)^2}{D(\gamma)^2}\mathcal{S}\Theta_{\upsilon^H}(yz) \tag{7.3}$$

Consider y, z contributing to the above expression. If (yz, γ) is not a pair of (G, H)-related elements, then $\Delta_0(yz, \gamma) = 0$. Now assume that (yz, γ) is a related pair. Then (z, γ_u) is a pair of $(G_{\gamma_s}, H_y, \varphi_{yz,\gamma})$-related elements, and from Lemma 7.2.3 we know that

$$\Delta_0(yz, \gamma) = \Delta_{0,y,\varphi_{yz,\gamma}}(z, \gamma_u)$$

Moreover, if ξ is a (G, H)-admissible embedding carrying y to γ_s but not equivalent to $\varphi_{yz,\gamma}$, the pair (z, γ_u) is not (G_{γ_s}, H_y, ξ)-related, and thus $\Delta_{0,y,\xi}(z, \gamma_u) = 0$. It follows that

$$\Delta_0(yz, \gamma) = \sum_{\xi} \Delta_{0,y,\xi}(z, \gamma_u)$$

where ξ runs over the set of (G_{γ_s}, H_y)-equivalence classes of (G, H)-admissible embeddings carrying y to γ_s. As in the proof of [Hal93, Lem. 8.1] we have

$$D(\gamma) = D_{G_{\gamma_s}}(\gamma_u) \qquad \text{and} \qquad D(yz) = D_{H_y}(z)$$

Thus expression (7.3) equals

$$\sum_{y \in Y} \sum_{\xi} |H^y(F)/H_y(F)|^{-1} \sum_{z} \Delta_{0,y,\xi}(z, \gamma_u) \frac{D_{H_y}(z)^2}{D_{G_{\gamma_s}}(\gamma_u)^2} \mathcal{S}\Theta_{\upsilon^H}(yz)$$

where z runs as in (7.3). Finally, note that every $z \in H_y(F)_{\mathrm{sr}}$ which is a (G_{γ_s}, H_y, ξ)-preimage of γ_u automatically belongs to the set $[H_y(F)]_{(H,y)-\mathrm{sr,tu}}$. Hence we may extend the summation over z to all of $H_y(F)_{\mathrm{sr}}$. Also if $y \in S_0^H(F)$ is a preimage of γ_s but does not belong to Y, then $H_y(F)$ does not contain a (G_{γ_s}, H_y, ξ)-preimage of γ_u for any ξ, and thus the terms $\Delta_{0,y,\xi}(z, \gamma_u)$ vanish for all ξ and z. Hence we may add to Y representatives for the stable classes of such elements without changing the value of the sum. But then the expression we obtain is (7.2). □

Corollary 7.2.5. *If $\gamma \in G^u(F)_0$ is a strongly regular semi-simple element which does not have a stable conjugate $\gamma' \in G(F)_0$ with $\gamma'_s \in S_0(F)$, then both sides of Equation* (3.2) *vanish.*

Proof: Consider first the left hand side. In view of Proposition 6.2.2, it vanishes unless γ_s lies in the centralizer of some $Q \in \mathrm{Lie}(G^u)(F)$ stably conjugate to Q_0. But such a Q is then rationally conjugate to Q_λ for some $\lambda \in r^{-1}(u)$ and hence replacing γ by a rational conjugate we may assume $\gamma_s \in S_\lambda$. Thus, by Lemma 7.2.2, the non-vanishing of the left hand side of Equation (3.2) implies the existence of γ' as claimed.

We now turn to the right hand side. Let $\tilde{\gamma} \in G(F)_0$ be any stable conjugate of γ. By Lemma 7.2.4, the right hand side of Equation (3.2) vanishes at $\tilde{\gamma}$ unless there exists a triple (y, ξ, z) s.t. $y \in S_0^H(F)$ is a preimage of $\tilde{\gamma}_s$, ξ is a (G, H)-admissible embedding s.t. $\xi(y) = \tilde{\gamma}_s$, and $z \in H_y(F)$ is a $(G_{\tilde{\gamma}_s}, H_y, \xi)$-preimage of $\tilde{\gamma}_u$. By Lemma 5.1.1 the map

$$S_0^H \xrightarrow{\mathrm{Ad}(q_0^H)^{-1}} T_0^{H,w^H} \xrightarrow{\eta^{-1}} T_0^w \xrightarrow{\mathrm{Ad}(q_0)} S_0$$

is an admissible isomorphism defined over F. Let y' be the image of y under this isomorphism. Then $G_{y'}$ is quasi-split by [Kot86, Prop. 7.1] and so there exists $z' \in G_{y'}(F)$ which is an image of z. But then $\gamma' = y'z'$ is a stable conjugate of $\tilde{\gamma}$, hence of γ. Thus the non-vanishing of the right hand side of Equation (3.2) at $\tilde{\gamma}$ implies the existence of γ' as claimed. But since γ and $\tilde{\gamma}$ are stably conjugate, the non-vanishing of said expression at γ is equivalent to its non-vanishing at $\tilde{\gamma}$, since the value at $\tilde{\gamma}$ differs from the value at γ by a non-zero multiplicative factor. □

7.3 Lemmas about transfer factors

In this section G' is an unramified F-group and $(H', s, {}^L\eta)$ is an unramified extended endoscopic triple for G'. Let (T_0', B_0') be a Borel pair of G' over F. We choose hyperspecial points in the buildings of G' and H', s.t. the one for G' lies in the apartment of T_0'. We also choose an admissible splitting $(T_0', B_0', \{X_\alpha'\})$ for G' in the sense of [Hal93, §7]. Then we have the transfer factors normalized with respect to that splitting both on the group level ([LS87, §3.7]), as well as on the Lie algebra level ([Kot99]). We will call both these transfer factors Δ_0, as there will be no possibility of confusion between the two.

Lemma 7.3.1. *For any semi-simple strongly regular topologically unipotent $\gamma^H \in H'(F)$ and $\gamma \in G'(F)$, we have*

$$\Delta_0(\gamma^H, \gamma)\frac{D(\gamma^H)^2}{D(\gamma)^2} = \Delta_0(\log(\gamma^H), \log(\gamma))\frac{D(\log(\gamma^H))}{D(\log(\gamma))}$$

Proof: We choose a positive integer m with the property that the sequences

$$\gamma_k = \gamma^{p^{km}} \qquad \gamma_k^H = [\gamma^H]^{p^{km}}$$

converge to 1 (cf. [DR09, §7]), and put

$$X^H = \log(\gamma^H), X = \log(\gamma), X_k^H = \log(\gamma_k^H), X_k = \log(\gamma_k)$$

As argued in [Hal93, §10] we have

$$\Delta_0(\gamma_{2k}^H, \gamma_{2k})\frac{D(\gamma_{2k}^H)^2}{D(\gamma_{2k})^2} = |p^{km}|^{-N}\Delta_0(\gamma^H, \gamma)$$

where N is the number of roots in G' outside H'. By the same arguments one also has

$$\Delta_0(X_{2k}^H, X_{2k})\frac{D(X_{2k}^H)}{D(X_{2k})} = |p^{km}|^{-N}\Delta_0(X^H, X)\frac{D(X^H)}{D(X)}$$

Thus it will be enough to show the equality claimed in the lemma with γ^H, γ replaced by $\gamma_{2k}^H, \gamma_{2k}$ for some k which we may freely choose.

As argued in [Wal97, §2.3], there exists a positive integer K s.t. for all $k > K$

$$\Delta_0(\gamma_{2k}^H, \gamma_{2k})\frac{D(\gamma_{2k}^H)}{D(\gamma_{2k})} = \Delta_0(X_{2k}^H, X_{2k})$$

We now claim that, after potentially increasing K, we have

$$D(\gamma_{2k}^H) = D(X_{2k}^H) \qquad D(\gamma_{2k}) = D(X_{2k})$$

For this it is enough to show that if $T \subset G'$ is a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}'$ and $Y \in \mathfrak{t}(F)$ is small enough then for all roots $\alpha \in R(T, G')$ we have

$$|\alpha(\exp(Y)) - 1| = |d\alpha(Y)|$$

First

$$|\alpha(\exp(Y))-1| = |\exp(d\alpha(Y))-1| = |d\alpha(Y)+\sum_{k>1}\frac{d\alpha(Y)^k}{k!}|$$

Let E/F be the extension splitting T, and let v_E be the unique valuation on E extending that on F (here we deviate from our usual notation). Putting $u=d\alpha(Y)$, we have by a computation similar to the proof of [DR09, B.1.1]

$$v_E(\frac{u^k}{k!}) = kv_E(u)-eA(k) > kv_E(u)-(k-1)$$

where as in loc. cit. $A(k)=\sum_{i>0}\lfloor\frac{k}{p^i}\rfloor$ and e is the ramification degree of $F/\mathbb{Q}_p$. Thus if $v_E(u)\geq 1$ then for all $k>1$

$$v_E(\frac{u^k}{k!}) > v_E(u)$$

from which follows

$$|u+\sum_{k>1}\frac{u^k}{k!}| = |u|$$

This finishes the proof of the claim about D and the lemma follows. □

Lemma 7.3.2. *Let $S\subset G'$ be a maximal torus defined over O_F. Let $Q\in\mathrm{Lie}(S)(O_F)$ be a semi-simple element whose image in $G'(k_F)$ is regular, and let Q^H be any preimage of Q in $\mathfrak{h}'(F)$. Then*

$$\Delta_0(Q^H,Q)=1$$

Proof: For $\alpha\in R(S,G')$ let $a_\alpha=d\alpha(Q)$. As Kottwitz observes in [Kot99], this defines a-data for $R(S,G')$ and with respect to that a-data, $\Delta_{II}(Q^H,Q)=1$. To show that $\Delta_I(Q^H,Q)=1$ we adapt the argument of [Hal93, Lem. 7.2]. The assumption on Q implies that $a_\alpha\in O_{F^u}^\times$. Then as in loc. cit. we see that the cocycle $m(\sigma_S)$ constructed in [LS87, §2.3] takes values in $G'(O_{F^u})$. Since the torus T_0' is also defined over O_{F^u}, there exists $g\in G'(O_{F^u})$ s.t. $S=\mathrm{Ad}(g)T_0'$. Thus the cocycle $\mathrm{Ad}(g)^{-1}m(\sigma_S)$ of Γ in $S(\overline{F})$ takes values in $S(O_{F^u})$ and is thus cohomologically trivial. But $\Delta_I(Q^H,Q)$ is the value of a character on $H^1(\Gamma,S)$ at that cocycle. □

Lemma 7.3.3. *Let $\gamma^H\in H'(F)$ and $\gamma\in G'(F)$ be semi-simple, strongly regular, and topologically unipotent. Then γ is an image of γ^H if and only if $\log(\gamma)$ is an image of $\log(\gamma^H)$.*

Proof: We define $\gamma_k^H,\gamma_k,X^H,X,X_k^H,X_k$ as in the proof of Lemma 7.3.1. It is clear that γ is an image of γ^H if and only if γ_k is an image of γ_k^H for some (then any) k. The same holds for the X's. This reduces the proof to the case where the elements are near the identity, in which case it is clear. □

7.4 Completion of the proof of Theorem 3.4.2

By Corollaries 7.1.3 and 7.2.5 it is enough to prove Equation (3.2) for all strongly regular semi-simple elements $\gamma\in G^u(F)_0$ which have a stable conjugate $\gamma'\in G(F)_0$ s.t. $\gamma_s'\in S_0(F)$. Fix such a pair γ,γ' and consider the value at γ of the right hand side of Equation (3.2):

$$\sum_{\gamma^H\in H_{\rm sr}(F)/{\rm st}} \Delta_{\psi,u}(\gamma^H,\gamma)\frac{D(\gamma^H)^2}{D(\gamma)^2}\mathcal{S}\Theta_{\upsilon^H}(\gamma^H) \tag{7.4}$$

By construction of $\Delta_{\psi,u}$ we have

$$\Delta_{\psi,u}(\gamma^H,\gamma)=\epsilon_L(V,\psi)\Delta_0(\gamma^H,\gamma')\langle \mathrm{inv}(\gamma',\gamma),\widehat{\varphi}_{\gamma',\gamma^H}(s)\rangle^{-1}$$

where Δ_0 is the absolute transfer factor for (G,H) normalized with respect to the splitting chosen in section 3.2. By Lemma 5.1.1 the map

$$S_0^H \xrightarrow{\mathrm{Ad}(q_0^H)^{-1}} T_0^{H,w^H} \xrightarrow{\eta^{-1}} T_0^w \xrightarrow{\mathrm{Ad}(q_0)} S_0$$

is an admissible isomorphism defined over F. We fix $Q_0\in\mathrm{Lie}(S_0)(F)$ satisfying the requirements of the element X_S in [DR09, Lemma 12.4.3], and let Q_0^H be the preimage of Q_0 under this embedding. Then Q_0^H also satisfies the same requirements.

We now apply Lemma 7.2.4 and Proposition 6.2.2 to conclude that (7.4) equals

$$\begin{aligned}&\epsilon_L(V,\psi)\epsilon(H,A_H)\sum_y\sum_\xi |H^y(F)/H_y(F)|^{-1}\sum_{P^H}[\varphi_{Q_0^H,P^H}]_*\theta_0^H(y)\\ &\langle\mathrm{inv}(\gamma',\gamma),\widehat{\varphi}_{\gamma',\gamma^H}(s)\rangle^{-1}\sum_{z\in H_y(F)_{\rm sr}/{\rm st}}\Delta_{0,y,\xi}(z,\gamma'_u)\frac{D_{H_y}(z)^2}{D_{G_{\gamma'_s}}(\gamma'_u)^2}\\ &\sum_{Q^H}R(H_y,S_{Q^H},1)(z)\end{aligned} \tag{7.5}$$

Let us recall the summation sets. y runs over a set $Y\subset S_0^H(F)$ representing the stable classes of preimages of γ'_s which intersect $S_0^H(F)$, ξ runs over the $(G_{\gamma'_s},H_y)$-equivalence classes of (G,H)-admissible embeddings which map y to γ'_s, P^H runs over a set of representatives for the H_y-stable classes of elements of $\mathrm{Lie}(H_y)(F)$ which are H-stably conjugate to Q_0^H, z runs over the stable classes of strongly regular elements in $H_y(F)$, and Q^H runs over a set of representatives for the $H_y(F)$-classes inside the H_y-stable class of P^H.

Consider a triple (y,ξ,P^H). Since $G_{\gamma'_s}$ is quasi-split, there exists an $(G_{\gamma'_s},H_y,\xi)$-image $P'\in\mathrm{Lie}(G_{\gamma'_s})(F)$ of P^H, unique up to stable conjugacy. We claim that the map

$$p:(y,\xi,P^H)\mapsto P'$$

is a surjection from the set of triples (y,ξ,P^H) occurring in (7.5) to the set of $G_{\gamma'_s}$-stable classes of elements of $\mathrm{Lie}(G_{\gamma'_s})(F)$ stably conjugate to Q_0, and moreover that the fiber of this surjection through (y,ξ,P^H) is a torsor under $H^y(F)/H_y(F)$ for the action of this group by conjugation on all factors of the triple (the first factor is of course fixed by this action).

To see surjectivity, choose P' in the target of p. Let $\tilde y=\varphi_{P',Q_0^H}(\gamma'_s)$. There exists a $y\in Y$ stably conjugate to $\tilde y$. By [Kot86, Prop. 7.1] there exists $h\in H(O_F)$ s.t. $\mathrm{Ad}(h)\tilde y=y$. Put $P^H=\mathrm{Ad}(h)Q_0^H$. Then $(y,\varphi_{P^H,P'},P^H)$ is a preimage of P' under p.

Now let (y,ξ,P^H) be an element in the source of p and let P' be its image. We claim that the map

$$\tilde p:\tilde P^H\mapsto(y,\varphi_{\tilde P^H,P'},\tilde P^H)$$

is an $H^y(F)/H_y(F)$-equivariant bijection from the set of H_y-stable classes inside the H^y-stable class of P^H to the fiber of p through (y,ξ,P^H). Once this has been shown, the claim about the fibers of p will follow from Lemma 7.2.1.

Indeed, let $\tilde{P}^H$ be H^y-stably conjugate to P^H. Then $\varphi_{\tilde{P}^H,P^H}(y)=y$ and moreover since P' is a $(G_{\gamma'_s},H_y,\xi)$-image of P^H we have $\varphi_{P^H,P'}(y)=\gamma'_s$. This implies $\varphi_{\tilde{P}^H,P'}(y)=\gamma'_s$ and we see that $(y,\varphi_{\tilde{P}^H,P'},\tilde{P}^H)$ belongs to the target of the proposed map $\tilde{p}$. If $\tilde{P}^H$ is replaced by an H_y-stable conjugate, then $\varphi_{\tilde{P}^H,P'}$ remains within its equivalence class. We see that $\tilde{p}$ is a well-defined and $H^y(F)/H_y(F)$-equivariant map as claimed. It is clearly injective. To show surjectivity, let $(\tilde{y},\tilde{\xi},\tilde{P}^H)\in p^{-1}(P')$. By definition of the map p, we must have that $\tilde{\xi}$ and $\varphi_{\tilde{P}^H,P'}$ are $(G_{\gamma'_s},H_{\tilde{y}})$-equivalent and $\tilde{y}=\varphi_{P',\tilde{P}^H}(\gamma'_s)$ and so we only have to show that $\tilde{P}^H$ and P^H are H^y-stably conjugate. We have $\varphi_{P^H,P'}(y)=\gamma'_s=\varphi_{\tilde{P}^H,P'}(\tilde{y})$. But recall that P^H and $\tilde{P}^H$ are H-stably conjugate. Thus $\varphi_{P^H,\tilde{P}^H}$ is defined and since $\varphi_{\tilde{P}^H,P'}=\varphi_{P^H,P'}\circ\varphi_{\tilde{P}^H,P^H}$ we have $\varphi_{P^H,\tilde{P}^H}(y)=\tilde{y}$. But Y contains only one element per stable class, which forces $y=\tilde{y}$, and so $\varphi_{P^H,\tilde{P}^H}(y)=y$, i.e. P^H and $\tilde{P}^H$ are H^y-stably conjugate. This conclude the proof of the claim about the map p.

Consider a triple (y,ξ,P^H) contributing to (7.5) and let P' be its image under p. We focus on the part of (7.5) given by

$$\langle \mathrm{inv}(\gamma',\gamma),\widehat{\varphi}_{\gamma',\gamma^H}(s)\rangle^{-1}\sum_{z\in H_y(F)_{\mathrm{sr}}/\mathrm{st}}\Delta_{0,y,\xi}(z,\gamma'_u)\frac{D_{H_y}(z)^2}{D_{G_{\gamma'_s}}(\gamma'_u)^2}$$
$$\sum_{Q^H}R(H_y,S_{Q^H},1)(z) \tag{7.6}$$

The map $\varphi_{\gamma',\gamma}$ defines an inner twist $G_{\gamma'_s}\to G^u_{\gamma_s}$ and maps γ'_u to γ_u. From this it follows that $D_{G_{\gamma'_s}(\gamma'_u)}=D_{G^u_{\gamma_s}(\gamma_u)}$, and $\mathrm{inv}(\gamma',\gamma)=\mathrm{inv}(\gamma'_u,\gamma_u)=\mathrm{inv}(X',X)$, where $X'=\log(\gamma'_u)$, $X=\log(\gamma_u)$. All z which are preimages of γ'_u are topologically unipotent, so we may restrict the sum over z to the topologically unipotent elements. Put $Z=\log(z)$. We will use Lemma 7.3.1 with $G'=G_{\gamma'_s}$ and $H'=H_y$. By [Kot86, Prop. 7.1] these groups are unramified and come with fixed hyperspecial maximal compact subgroups. We see that (7.6) equals

$$\langle \mathrm{inv}(X',X),\widehat{\varphi}_{\gamma',\gamma^H}(s)\rangle^{-1}\sum_{Z\in\mathfrak{h}_y(F)_{\mathrm{sr}}/\mathrm{st}}\Delta_{0,y,\xi}(Z,X')\frac{D_{\mathfrak{h}_y}(Z)}{D_{\mathfrak{g}_{\gamma'_s}}(X')}$$
$$\sum_{Q^H}R(H_y,S_{Q^H},1)(z) \tag{7.7}$$

The function

$$\Delta_{0,y,\varphi_{\gamma',\gamma}\circ\xi}(Z,X):=\Delta_{0,y,\xi}(Z,X')\langle \mathrm{inv}(X',X),\widehat{\varphi}_{\gamma',\gamma^H}(s)\rangle^{-1}$$

is a transfer factor for $(\mathfrak{g}^u_{\gamma_s},\mathfrak{h}_y,\varphi_{\gamma',\gamma}\circ\xi)$. Applying [DR09, Lem. 12.4.3] we conclude that (7.7) equals

$$\sum_{Z\in\mathfrak{h}_y(F)_{\mathrm{sr}}/\mathrm{st}}\Delta_{0,y,\varphi_{\gamma',\gamma}\circ\xi}(Z,X)\frac{D_{\mathfrak{h}_y}(Z)}{D_{\mathfrak{g}_{\gamma_s}}(X)}\sum_{Q^H}\epsilon(H_y,A_{H_y})\widehat{\mu}^{H_y}_{Q^H}(Z) \tag{7.8}$$

Here $\widehat{\mu}^{H_y}_{Q^H}$ is the Fourier transform (with respect to the transfer to $\mathfrak{h}_y$ of the bilinear form B and the character ψ) of the orbital integral at Q^H on $\mathfrak{h}_y(F)$.

We will now apply [Wal97, Conj. 1.2], which is now a theorem due to the work of [Wal97], [Wal06], [HCL07] and [Ngo08]. According to it, (7.8) equals

$$\gamma_\psi(\mathfrak{g}^u_{\gamma_s})\gamma_\psi(\mathfrak{h}_y)^{-1}\epsilon(H_y,A_{H_y})\sum_Q \Delta_{0,y,\varphi_{\gamma',\gamma}\circ\xi}(P^H,Q)\widehat{\mu}_Q^{G^u_{\gamma_s}}(X) \tag{7.9}$$

where Q runs over a set of representatives for the conjugacy classes of regular semi-simple elements in $\mathfrak{g}^u_{\gamma_s}(F)$.

For a moment we consider the signs in (7.9). The group H_y contains S_0^H, which is an elliptic maximal torus of H. Thus the inclusion $Z_H \to Z_{H_y}$ restricts to an isomorphism $A_H \to A_{H_y}$. The group $G_{\gamma_s'}$ contains S_0, which is an elliptic maximal torus of G, and again we get an isomorphism $A_G \to A_{G_{\gamma_s'}}$. The group $G^u_{\gamma_s}$ is an inner twist of $G_{\gamma_s'}$ and so we have an isomorphism $A_{G_{\gamma_s'}} \to A_{G^u_{\gamma_s}}$. Finally since H is elliptic for G, the natural inclusion $Z_G \to Z_H$ restricts to an isomorphism $A_G \to A_H$. All in all this gives an isomorphism $A_{H_y} \to A_{G^u_{\gamma_s}}$. Using this and the transitivity of the sign $\epsilon(\cdot,\cdot)$ we conclude

$$\epsilon(H_y,A_{H_y}) = \epsilon(H_y,G_{\gamma_s'})\epsilon(G_{\gamma_s'},G^u_{\gamma_s})\epsilon(G^u_{\gamma_s},A_{G^u_{\gamma_s}})$$

From [DR09, §12.3] we know

$$\epsilon(G_{\gamma_s'},G^u_{\gamma_s}) = \gamma_\psi(\mathfrak{g}_{\gamma_s'})\gamma_\psi(\mathfrak{g}^u_{\gamma_s})^{-1}$$

while from Proposition 4.0.2 we know

$$\epsilon(H_y,G_{\gamma_s'}) = \gamma_\psi(\mathfrak{h}_y)\gamma_\psi(\mathfrak{g}_{\gamma_s'})^{-1}$$

It follows that (7.9) equals

$$\sum_Q \Delta_{0,y,\varphi_{\gamma',\gamma}\circ\xi}(P^H,Q)\epsilon(G^u_{\gamma_s},A_{G^u_{\gamma_s}})\widehat{\mu}_Q^{G^u_{\gamma_s}}(X) \tag{7.10}$$

where Q runs over the same set as in (7.9).

Now there is a natural injection from the set of $G^u_{\gamma_s}$-stable classes of regular semi-simple elements in $\mathfrak{g}^u_{\gamma_s}(F)$ to the set of $G_{\gamma_s'}$-stable classes of regular semi-simple elements in $\mathfrak{g}_{\gamma_s'}(F)$. Since elliptic tori transfer to inner forms, this injection restricts to a bijection between those stable classes which are stably conjugate to Q_0 under G^u resp. G. Let $P \in \mathfrak{g}_{\gamma_s}(F)$ be an element whose class corresponds to that of P' under this bijection. Then (7.10) equals

$$\sum_Q \Delta_{0,y,\xi}(P^H,Q_0)\langle \mathrm{inv}(Q_0,Q),s_{q_0}\rangle^{-1}\epsilon(G^u_{\gamma_s},A_{G^u_{\gamma_s}})\widehat{\mu}_Q^{G^u_{\gamma_s}}(X) \tag{7.11}$$

where Q runs over the set of $G^u_{\gamma_s}(F)$-conjugacy classes inside the $G^u_{\gamma_s}$-stable class of P.

The torus $S_0 \subset G_{\gamma_s'}$ and the element Q_0 satisfy the requirements of Lemma 7.3.2. Moreover the element Q satisfies the requirements of [DR09, Lem 12.4.3] on the element X_S. Thus (7.11) equals

$$\sum_Q \langle \mathrm{inv}(Q_0,Q),s_{q_0}\rangle^{-1} R(G^u_{\gamma_s},S_Q,1)(\gamma_u) \tag{7.12}$$

where Q runs over the same set as in (7.11).

To recapitulate, if we compose the map p with the bijection $P' \mapsto P$ discussed in the preceding paragraph, we obtain a map

$$(y, \xi, P^H) \mapsto P$$

which is a surjection on the set of $G^u_{\gamma_s}$-stable classes of elements of $\mathfrak{g}^u_{\gamma_s}(F)$ which are stably conjugate to Q_0, and the fiber of that surjection through (y, ξ, P^H) is a torsor under $H^y(F)/H_y(F)$. This of course follows from the corresponding property of the map p. Moreover, if a triple (y, ξ, P^H) maps to P, then its contribution to (7.5) equals (7.12).

Before we apply this to the expression (7.5), we need to note that if (y, ξ, P^H) maps to P, then since $\varphi_{P,P^H}(\gamma_s) = y$ we have

$$\begin{aligned}
[\varphi_{Q_0^H, P^H}]_* \theta_0^H(y) &= [\varphi_{P^H, P}]_* [\varphi_{Q_0^H, P^H}]_* \theta_0^H(\gamma_s) \\
&= [\varphi_{Q_0, P}]_* [\varphi_{Q_0^H, Q_0}]_* \theta_0^H(\gamma_s) \\
&= [\varphi_{Q_0, P}]_* \theta_0(\gamma_s)
\end{aligned}$$

where the last equality follows from Lemma 5.1.1.

With this in mind, we see that (7.5) equals

$$\begin{aligned}
\epsilon_L(V, \psi)\epsilon(H, A_H) \quad \cdot \quad & \sum_P [\varphi_{Q_0, P}]_* \theta_0(\gamma_s) \\
& \sum_Q \langle \mathrm{inv}(Q_0, Q), s_{q_0} \rangle^{-1} R(G^u_{\gamma_s}, S_Q, 1)(\gamma_u) \qquad (7.13)
\end{aligned}$$

where P runs over a set of representatives for the $G^u_{\gamma_s}$-stable classes of elements in $\mathfrak{g}^u_{\gamma_s}(F)$ which are G^u-stably conjugate to Q_0, and Q runs over a set of representatives for the $G^u_{\gamma_s}(F)$-conjugacy classes inside the $G^u_{\gamma_s}$-stable class of P.

Again using the transitivity of $\epsilon(\cdot, \cdot)$ and the isomorphism $A_G \cong A_H$ we can write

$$\epsilon(H, A_H) = \epsilon(H, G)\epsilon(G, A_G)$$

and thus using Proposition 4.0.2 we see that (7.13) equals

$$\epsilon(G, A_G) \sum_P [\varphi_{Q_0, P}]_* \theta_0(\gamma_s) \sum_Q \langle \mathrm{inv}(Q_0, Q), s_{q_0} \rangle^{-1} R(G^u_{\gamma_s}, S_Q, 1)(\gamma_u) \qquad (7.14)$$

with both sums as in (7.13). By Proposition 6.2.2 this is the left hand side of Equation (3.2). This completes the proof of Theorem 3.4.2. $\square$

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