Taking roots

1. ... of sections

X = variety $/k=\bar{k}$, char $k = 0$, $L \in \mathrm{Pic}(X)$, $m \in \mathbb{Z}_{>0}$
and $0 \neq s \in H^0(X, L^m)$ (on any component)

Prop: Then there exists a variety X' (not necessarily irreducible) and a finite, flat morphism

$$\pi: X' \to X \quad \text{s.t.}$$

- $\pi_* \mathcal{O}_{X'} \simeq \bigoplus_{n=0}^{m-1} L^{-n}$
- $\exists\ s' \in H^0(X', \pi^* L)$: $s'^m = \pi^* s \in H^0(X', \pi^* L)$

$$Z(s') \underset{\pi}{\xrightarrow{\sim}} Z(s)$$

Moreover, if X is normal then so is X'.
and if X and $Z(s)$ are smooth, then X' and $Z(s')$ are smooth.

Proof: Construct the affine bundle

$$p: \mathbb{L} \to X \quad \text{as} \quad \underline{\mathrm{Spec}} \bigoplus_{n=0}^{\infty} L^{-n}$$

Then $p_* \mathcal{O}_{\mathbb{L}} = \bigoplus_{n=0}^{\infty} L^{-n}$.

The line bundle $p^* L$ has a canonical section $t \in H^0(\mathbb{L}, p^* L)$ (trivializing on the complement of the zero section).

The section t can be described in many (equivalent) ways:

- pointwise: $a \in \mathbb{L}$, $x = p(a) \Rightarrow \nexists\ (p^* L)(a) = L(x) = p^{-1}(x)$

Then $t(a) := a$

- $$\begin{aligned} H^0(\mathbb{L}, p^*L) &= H^0(X, p_*\mathcal{O}_{\mathbb{L}} \otimes L) \\ &= H^0(X, \bigoplus_{n=0}^{\infty} L^{-n} \otimes L) \\ &= H^0(X, L \oplus \mathcal{O} \oplus L^{-1} \oplus \dots) \end{aligned}$$

$$t \quad \hat{=} \quad 1 \in \mathcal{O}$$

- Since p is affine, giving t as $t : \mathcal{O}_{\mathbb{L}} \to p^*L$ is equivalent to giving its direct image $p_*t : p_*\mathcal{O}_{\mathbb{L}} \to p_*\mathcal{O}_{\mathbb{L}} \otimes L$. The latter is simply the inclusion
$$\bigoplus_{n=0}^{\infty} L^{-n} \hookrightarrow \bigoplus_{n=-1}^{\infty} L^{-n}$$

- Over a point $x \in X$: $L \hat{=} V$ = one-dimensional vector space

$\mathbb{L} \hat{=} \operatorname{Spec} \bigoplus S^n V^*$, $H^0(\mathbb{L}, p^*L) = \bigoplus S^n V^* \otimes V \supset V^* \otimes V$

Then $t \quad \hat{=} \quad \mathrm{id} \in V^* \otimes V$.

Now define X' as the zero set

$$Z(t^m - p^* s) \quad \text{of} \quad t^m - p^* s \in H^0(\mathbb{L}, p^* L^m)$$

and set $s' := t_{|X'} \in H^0(X', p^* L^m{}_{|X'})$

$$= H^0(X', \pi^* L^m) \quad \text{with } \pi = p_{|X'} .$$

Clearly, $s' = \pi^* s^m$.

- $\pi : X' \to X$ is *proper*:

  Locally $\mathbb{L} = X \times \mathbb{A}^1$, t = coordinate function on $\mathbb{A}^1$. Then $X' = Z(t^m - s)$.

  Compactify $\mathbb{L} = X \times \mathbb{A}^1 \subset X \times \mathbb{P}^1$. s.t. $t \mathrel{\hat{=}} z_0$.
  Then $\overline{X'} = Z(z_0^m - s z_1^m) \subset X \times \mathbb{P}^1$ is projective over X. As $z_1 = 0 \Rightarrow z_0 = 0$ on $\overline{X'}$ one has in fact $\overline{X}' = X'$ and thus $X' \to X$ proper

- $\pi : X' \to X$ is *finite*, by the same argument.

- By construction $Z(s') = X' \cap \underbrace{Z(t)}_{\text{zero section} \simeq X}$

  Hence, viewed as a subvariety of $Z(t) \simeq X$ one has that $Z(s')$ is defined by $p^* s_{|Z(t)} \mathrel{\hat{=}} s$. Thus $\underline{Z(s') \simeq Z(s)}$

- By construction: $X' \subset \mathbb{L}$ hypersurface with ideal sheaf p^*L^{-m}, i.e. there is a short exact sequence:

$$0 \to p^*L^{-m} \xrightarrow[\cdot(t^m - p^*s)]{} \mathcal{O}_{\mathbb{L}} \longrightarrow \mathcal{O}_{X'} \to 0$$

Taking direct image under the affine(!) morphism p yields:

$$0 \to \bigoplus_{n=0}^{\infty} L^{-n} \otimes L^{-m} \longrightarrow \bigoplus_{n=0}^{\infty} L^{-n} \longrightarrow \pi_* \mathcal{O}_{X'} \to 0$$

Since t corresponds to the natural inclusion, the first map is given by

$$\begin{array}{ccc} & & L^0 \\ & \overset{\cdot(-s)}{\nearrow} & \oplus \\ & & \vdots \\ L^{-m} & \xrightarrow{id} & \oplus \\ \oplus & & L^{-m} \\ L^{-m-1} & & \\ \oplus & \vdots & \\ \vdots & & \end{array}$$

i.e. $\alpha \longmapsto \alpha - s \cdot \alpha$

Thus, $\pi_* \mathcal{O}_{X'} = L^0 \oplus \ldots \oplus L^{-m-1}$

- The above description of $\pi_* \mathcal{O}_{X'}$ proves also <u>flatness</u> of $\pi : X' \to X$

- In order to prove that "X normal" implies "X' normal" it suffices to prove

$$H^0(\pi^{-1}(U), \mathcal{O}_{X'}) \twoheadrightarrow H^0(\pi^{-1}(U'), \mathcal{O}_{X'})$$

whenever $\operatorname{codim}(U \setminus U') \geq 2$.

But $H^0(\pi^{-1}(U), \mathcal{O}_{X'}) = H^0(U, \bigoplus_{n=0}^{m-1} L^{-n})$

and $H^0(\pi^{-1}(U'), \mathcal{O}_{X'}) = H^0(U', \bigoplus_{n=0}^{m-1} L^{-n})$.

If X is normal, then $H^0(U, L^{-n}) \twoheadrightarrow H^0(U', L^{-n})$

for $\operatorname{codim}(U \setminus U') \geq 2$.

- Eventually, assume $X, Z(s)$ smooth.
Locally X' is given as the fibre of

$$f: X \times \mathbb{A}^1 \to \mathbb{A}^1 \quad \text{with } f = t^m - s$$

and $df = mt^{m-1}dt - ds$

Thus, df surjective as long as $t \neq 0$.
For $t = 0$, e.g. $f|_{Z(s')}$, the ds is non-trivial,
as s is the local equation for the smooth $Z(s)$.

Therefore, X' smooth and for the smoothness of $Z(s')$
use $Z(s') \simeq Z(s)$. $\square$

Remark: • The construction also works for $Z(s) = \emptyset$,
i.e. s is a trivializing section of L^m.
Warning: If m is not minimal with $L^m \simeq \mathcal{O}$,
then X' is not connected

- If $Z(s) + \sum D_i$ is nc, then also $Z(s') + \sum \pi^* D_i$ nc.
Indeed if $z_1, \dots z_k, \dots z_{n-1}, z_n = s$ with $D_i = (z_i)$
is a local coordinate system on X, then
$z_1 \dots z_k \dots, z_n' = t$ is one on X'!

- Note that on X' one has, using

$$\pi^* s_{|X'} = t^m_{|X'}, \text{ that}$$

$$\therefore \pi^* \mathcal{O}(Z(s)) \simeq \pi^* L^m \simeq \mathcal{O}(m\, Z(s'))$$

- Suppose X, $Z(s)$ smooth. Then

$$\omega_{X'} = \pi^* \omega_X \otimes \pi^* L^{m-1} \quad \text{or}$$

$$K_{X'} = \pi^* K_X + (m-1)\, Z(s')$$

Consider the differential $\pi^* \Omega_X \xrightarrow{\rho} \Omega_{X'}$

In $x \in Z(s)$: Ω_X spanned by $dz_1, \dots dz_{n-1}, dz_n = ds$ and in y with $\pi(y) = x$: $\Omega_{X'}$ spanned by $dz_1 \dots dz_{n-1}, dt$

As $\pi^* s = t^m$, one finds that ρ is given by

$$dz_1 \longmapsto dz_1$$

$$\vdots$$

$$dz_{n-1} \longmapsto dz_{n-1}$$

$$dz_n \longmapsto m t^{m-1} dt .$$

Thus, $\det \rho : \pi^* K_X \to K_{X'}$ is multiplication with $m\, t^{m-1}$.

2. ... of line bundles (à la Bloch, Gieseker)

X = variety $/ k = \bar{k}$, char $k = 0$, $L \in \mathrm{Pic}(X)$, $m \geq 1$

Prop: There exists a variety X', a finite flat morphism $\pi : X' \to X$, $L' \in \mathrm{Pic}(X')$ such that

$$\pi^* L \simeq L'^{m}$$

If X is smooth, then one can choose X' smooth.
If $\sum D_i$ is a NC divisor, then we can construct
X', π, L' such that $\sum \pi^* D_i$ is NC as well. $(*)$

Proof: Suppose $X \subset \mathbb{P}^r$ and assume we know already how to construct X' for $\mathcal{O}(1)_{|X}$.
Then go in two steps: Write $L = \mathcal{O}(1)_{|X} \otimes \mathcal{O}(-1)'^{*}_{|X}$

$$\underbrace{X'' \xrightarrow{\pi_2}}_{\text{for } \pi_1^* \mathcal{O}(1)'} \underbrace{X' \xrightarrow{\pi_1} X}_{\text{for } \mathcal{O}(1)_{|X}}$$

(Here, $\mathcal{O}(1)$, $\mathcal{O}(1)'$ with respect to two different embeddings $X \subset \mathbb{P}^r$, $X \subset \mathbb{P}^{r'}$.)
Thus there exist $L_1 \in \mathrm{Pic}(X')$ with $L_1^m \simeq \pi_1^* \mathcal{O}(1)_{|X}$
and $L_2 \in \mathrm{Pic}(X'')$ with $L_2^m \cong \pi_2^*(\pi_1^* \mathcal{O}(1)')$
Then $(\pi_2^* L_1 \otimes L_2^*)^m = (\pi_1 \circ \pi_2)^* L$.

Thus, it remains to treat the case $X \subset \mathbb{P}^r$, $L = \mathcal{O}(1)|_X$.

For $X = \mathbb{P}^r$ no problem: Consider

$$\pi : \mathbb{P}^r \to \mathbb{P}^r \qquad [z_0 : \dots : z_r] \mapsto [z_0^m : \dots : z_r^m].$$

Clearly, $\pi^* \mathcal{O}(1) = \mathcal{O}(m)$.

Then take $X' := \pi^{-1}(X)$, $L' := \mathcal{O}(1)|_{X'}$.

In order to ensure $\circledast$, one has to choose π more carefully.

For any $g \in Gl(r+1)$ consider

$$\pi_g : \mathbb{P}^r \xrightarrow{\pi} \mathbb{P}^r \xrightarrow{g} \mathbb{P}^r$$

and define $X'_g := \pi_g^{-1}(X)$.

<u>Claim</u>: For g generic and X smooth, one also has X'_g smooth

$$\begin{tikzcd} Gl(r+1) & Gl(r+1) \times \mathbb{P}^r \arrow[l] \arrow[d, "\tilde{\pi}"] & \supset \mathcal{X} := \tilde{\pi}^{-1}(X) \arrow[d] \arrow[ll, bend right, "\psi"'] \\ & \mathbb{P}^r & \supset X \end{tikzcd}$$

$\tilde{\pi}(g, z) = \pi_g(z) = g\,\pi(z)$.

Check: $\tilde{\pi}$ is smooth.

Thus, if X is smooth, then also $\mathcal{X}$ smooth.

The generic fibre of $\psi : \mathcal{X} \to Gl(r+1)$ must also be smooth, but $\psi^{-1}(g) = X'_g$

The nc divisor ΣD_i is treated in the same manner. Simply choose $g \in Gl(r+1)$ generic for X, $D_1, \dots D_r$, and all possible intersections

$\square$

<u>Remark</u>: In this context there is no natural formula that compare the canonical bundles of X and X!

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3. ... of everything ("Kawamata covering")

Consider a smooth variety X with a nc divisor $D = \Sigma D_i$. Choose $m_i \in \mathbb{Z}_{>0}$.

<u>Proposition</u>: There exists a smooth variety X', a finite, flat morphism $\pi : X' \to X$ s.t.

$$\pi^* D_i = m_i \cdot D_i' \quad \text{for certain divisors } D_i' \subset X'$$

s.t. $\Sigma D_i'$ is again nc.

<u>Proof</u>: Proceeding by recursion it is enough to treat the case $m_2 = \dots = m_r = 1$

First use Bloch–Gieseker to take a root of $L_1 := \mathcal{O}(D_1)$.
Thus, $\exists\ \pi_1 : X_1 \to X$ with $\pi_1^* \mathcal{O}(D_1) \simeq L_1'^{\,m}$, X_1 smooth, $\Sigma \pi^* D_i$ nc.

Consider $s \in H^0(X_1, L_1'^{\,m})$ with $Z(s) = \pi_1^* D_1$

Taking the root of s yields

$\pi_2 : X_2 \longrightarrow X_1$ with $\pi_2^* s = t^{m_1}$ for

some $t \in H^0(X_2, \pi_2^* L_1)$ and such that

$Z(s) + \sum_{i=2}^{k} \pi_1^* D_i$ nc implies $Z(t) + \sum_{i=2}^{k} \pi_2^* \pi_1^* D_i$ nc

$\square$