A PROBLEM IN CARTOGRAPHY

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1. Introduction. The central problem of mathematical cartography is the problem of representing a portion of the curved surface of the earth on a flat piece of paper without introducing any more distortion than is absolutely necessary. This note will propose a quantitative definition for the term "distortion," and then study the mathematical problem of choosing a method of mapping which minimizes distortion.

To simplify the problem we first replace the rather irregular surface of the earth by a perfect sphere.

DEFINITIONS. Let S be the sphere of radius r consisting of all points x in the 3-dimensional euclidean space with distance r from the origin, and let U be any nondegenerate subset of S. (By "nondegenerate" we mean that U must contain at least two distinct points.)

A map projection f on the domain U will mean a function which assigns to each point x of U some point $f(x)$ of the euclidean plane E.

Let $d_S(x, y)$ denote the geodesic distance between two points x and y of the sphere S. By definition, this is equal to the length of the shorter great circle arc joining x to y. The euclidean distance between two points a and b of the plane E will be denoted analogously by $d_E(a, b)$.

The scale of a map projection f with respect to a pair of distinct points x and y in the domain U is defined to be the ratio

$$d_E(f(x), f(y))/d_S(x, y).$$

Ideally we would like this scale to be the same for all pairs of points x and y in U, but this is not usually possible. So we must introduce the *minimum scale* σ_1, defined to be the infimum of the ratio $d_E(f(x), f(y))/d_S(x, y)$ as x and y vary over all pairs of distinct points in U, and the *maximum scale* σ_2, defined to be the supremum of the ratio $d_E(f(x), f(y))/d_S(x, y)$. In other words σ_1 and σ_2 are the

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"best" possible constants such that the inequality

$$\sigma_1 d_S(x, y) \leqq d_E(f(x), f(y)) \leqq \sigma_2 d_S(x, y)$$

holds for all points x and y in U.

To measure the extent to which scale fails to be constant we propose the following:

DEFINITION. *The distortion of the map projection f is the natural logarithm*

$$\delta = \log(\sigma_2/\sigma_1)$$

of the ratio of maximum scale to minimum scale.

Thus $0 \leqq \delta \leqq \infty$, where δ is finite if and only if both σ_1 and σ_2 are positive and finite numbers. If δ is finite, notice that the function f is continuous and one-to-one.

We would like to find a map projection f with no distortion at all ($\delta = 0$). Since this is not possible except in a few special and uninteresting cases (e.g., the case of a domain U consisting of only three points), the best we can actually do is to try to find a map projection for which δ is as small as possible.

DEFINITION. *A minimum distortion map projection f_0 on U will mean a map projection whose distortion δ_0 is less than or equal to the distortion of every other map projection on U.*

PRELIMINARY THEOREM. *For every nondegenerate set of points U on the sphere there exists a minimum distortion map projection f_0 with domain U.*

The proof of this theorem, which is quite elementary, will be deferred until Appendix A.

Unfortunately the proof will fail to suggest answers to a number of relevant questions: Is this minimum distortion map f_0 unique in some sense? Is f_0 differentiable (assuming that U is a nice enough set so that differentiability makes sense)? How can one actually construct f_0, or even a reasonable approximation to f_0? How can one estimate the minimum possible distortion δ_0 associated with a given set U?

This note will succeed in answering these questions only in one very special case, namely, the case of the region bounded by a circle on S.

Given a fixed point x_0 of S, let D_α denote the closed *disk* of geodesic radius $r\alpha$, consisting of all points x in S for which $d_S(x, x_0) \leqq r\alpha$. Here α can be any number in the interval $0 < \alpha < \pi$.

MAIN THEOREM. *There is one and, up to similarity transformations of the plane, only one minimum distortion map projection f_0 on the domain D_α. This map projection is infinitely differentiable, and has distortion δ_0 equal to* $\log(\alpha/\sin\alpha)$.

This minimum distortion projection f_0, known to cartographers as the "azimuthal equidistant projection," can be characterized by the fact that it preserves both distances and directions from the central point x_0. The explicit formula $\delta_0 = \log(\alpha/\sin\alpha)$ shows that the distortion δ_0 is small for small values of α, being asymptotically equal to

$$\alpha^2/6 \sim \tfrac{2}{3} \text{ area } D_\alpha/\text{area } S$$

as $\alpha \to 0$. However δ_0 tends to infinity as $\alpha \to \pi$.

This theorem will be proved in Section 2. The problem of estimating the δ_0 associated with a more general domain U is discussed in Section 3. There are two appendices, one proving that minimum distortion map projections exist, and a second discussing a corresponding problem for conformal map projections, following Chebyshef.

2. The azimuthal equidistant projection. Again let D_α denote a spherical disk of geodesic radius $r\alpha$ centered at x_0.

LEMMA 1. *The distortion δ for any map projection f with domain D_α satisfies* $\delta \geqq \log(\alpha/\sin\alpha)$.

Proof. We may assume that f has finite distortion. Hence the "Lipschitz inequality"

$$d_E(f(x), f(y)) \leqq \sigma_2 d_S(x, y) \tag{1}$$

is satisfied, where σ_2 is a finite constant, and it follows that f is continuous. Furthermore f is one-to-one.

Let C_α denote the boundary of the disk D_α. Clearly the image $f(C_\alpha)$ is a simple closed curve in the plane. We shall first prove:

ASSERTION A. *Every half-line emanating from the point $f(x_0)$ in the plane must intersect the simple closed curve $f(C_\alpha)$ at least once.*

Proof. The Jordan Curve Theorem asserts that the simple closed curve $f(C_\alpha)$ cuts the plane into two components

$$E - f(C_\alpha) = E_1 \cup E_2,$$

one of these components, say E_1, being bounded, and the second unbounded. But the bounded component E_1 is just the image, under the continuous one-to-one function f, of the interior of the disk D_α. This is proved, for example, in Newman [**10**, Theorem 12.2, p. 121]. In particular it follows that the point $f(x_0)$ must belong to the bounded component E_1. Hence every half-line emanating from x_0 must cross $f(C_\alpha)$, since otherwise it would lie completely within the bounded set E_1 which is impossible. This proves Assertion A.

Since the curve C_α on S has finite length $2\pi r \sin\alpha$, it follows easily from the Lipschitz inequality (1) that $f(C_\alpha)$ also has finite length L, where

$$L \leqq 2\pi\sigma_2 r \sin\alpha. \tag{2}$$

(The *length* of a not necessarily smooth curve is defined for example in [**6**, p. 36].)

Now let us make use of the inequality

$$d_E(f(x), f(y)) \geqq \sigma_1 d_S(x, y). \tag{3}$$

Since every point of C_α has geodesic distance exactly $r\alpha$ from x_0 it follows that every point of $f(C_\alpha)$ has euclidean distance $\geqq \sigma_1 r\alpha$ from $f(x_0)$.

Thus $f(C_\alpha)$ is a simple closed curve of finite length L which lies outside an open disk D^* of radius $\sigma_1 r\alpha$ in the plane, and cuts every half-line through the center of this disk.

ASSERTION B. *This implies that* $L \geqq 2\pi\sigma_1 r\alpha$, *where equality holds if and only if* $f(C_\alpha)$ *is precisely equal to the boundary of* D^*.

Proof. Cut $f(C_\alpha)$ by a straight line through the center of D^* and choose intersection points, say a and b, which lie on opposite sides of D^*. Let A be either one of the two arcs of $f(C_\alpha)$ from a to b. Introducing polar coordinates ρ and θ about the center of D^*, first assume that the arc A can be described, in terms of a parameter t, by piecewise smooth functions

$$\rho = \rho(t), \qquad \theta = \theta(t).$$

Then

$$\text{length } A = \int (\dot\rho^2 + \rho^2\dot\theta^2)^{1/2} dt \geqq \int \rho \left| \dot\theta \right| dt,$$

where the dot denotes differentiation. Since

$$\rho \geqq \sigma_1 r\alpha \quad \text{and} \quad \int \left| \dot\theta \right| dt \geqq \left| \int \dot\theta dt \right| \geqq \pi,$$

this proves that length $A \geqq \pi\sigma_1 r\alpha$, and therefore $L \geqq 2\pi\sigma_1 r\alpha$, as required.

If A is not piecewise smooth, then an extra step is needed. For each $\epsilon > 0$ it is possible to approximate A by a polygonal path A'_ϵ from a to b which lies outside the disk of radius $\sigma_1 r\alpha - \epsilon$ and satisfies

$$\text{length } A \geqq \text{length } A'_\epsilon \geqq \pi(\sigma_1 r\alpha - \epsilon).$$

Letting $\epsilon \to 0$, we obtain length $A \geqq \pi\sigma_1 r\alpha$, as before.

Now suppose that the length of A is precisely equal to $\pi\sigma_1 r\alpha$. Then any portion of A which has distance greater than $\sigma_1 r\alpha$ from the center of D^* must be a straight line segment. Otherwise, replacing some small portion of A by a straight line segment we could decrease its length, which is impossible.

Any maximal line segment A_0 which forms a part of A must lead from one of the end points a or b of A to a point of the circle bounding D^*. The only other possibility would be that both end points of A_0 lie on the circle, which is impossible. Thus A consists of a line segment (possibly degenerate) from a to the circle, followed by a circle arc, followed by a line segment to b. Elementary geometry now shows that the minimal length $\pi\sigma_1 r\alpha$ is achieved only if A is the semicircle. Hence L can equal $2\pi\sigma_1 r\alpha$ only if $f(C_\alpha)$ is the full circle. This completes the proof of Assertion B.

Combining Assertion B with the inequality (2) we obtain

$$2\pi\sigma_1 r\alpha \leqq 2\pi\sigma_2 r \sin \alpha$$

or

$$\alpha/\sin \alpha \leqq \sigma_2/\sigma_1$$

and hence $\log(\alpha/\sin \alpha) \leqq \delta$, which completes the proof of Lemma 1.

LEMMA 2. *If the distortion of f is precisely equal to* $\log(\alpha/\sin \alpha)$, *then f is an azimuthal equidistant projection.*

By definition this means that f carries each great circle passing through x_0 into a straight line in the plane, the angle between two great circles being equal to the angle between the corresponding straight lines, and that f carries each circle C centered at x_0 to a circle $f(C)$ centered at $f(x_0)$, the radius of $f(C)$ being proportional to the geodesic radius of C.

To differential geometers, this means that f is the inverse of the so called exponential map. It follows that f is infinitely differentiable, even at x_0. See for example [**9**, p. 147].

Proof of Lemma 2. If $\delta = \log(\alpha/\sin \alpha)$, then according to Assertion B the image $f(C_\alpha)$ must be precisely equal to the circle of radius

$$\sigma_1 r\alpha = \sigma_2 r \sin \alpha$$

centered at $f(x_0)$. Hence the image $f(D_\alpha)$ must be precisely the closed disk bounded by this circle. (Compare the proof of Assertion A.)

Now consider an arbitrary point x of D_α. Construct a great circle segment from x_0 through x to a point $\bar{x}$ on the boundary C_α of D_α. If c denotes the geodesic distance $d_S(x_0, x)$, note that x has geodesic distance precisely $r\alpha - c$ from $\bar{x}$, and geodesic distance strictly greater than $r\alpha - c$ from every other point of C_α. Hence, using inequality (3), the image $f(x)$ must

(a) have distance at least $\sigma_1 c$ from $f(x_0)$,

(b) have distance at least $\sigma_1(r\alpha - c)$ from $f(\bar{x})$, and

(c) have distance greater than $\sigma_1(r\alpha - c)$ from every other point of $f(C_\alpha)$.

Clearly there is one and only one point in the disk $f(D_\alpha)$ which satisfies these three conditions: namely, the point which lies at distance $\sigma_1 c$ along the line segment from $f(x_0)$ to $f(\bar{x})$. Thus the map projection f on D_α is completely determined by what is does to boundary points of D_α.

To complete the proof of Lemma 2 we need only verify that f carries the circle C_α to the circle $f(C_\alpha)$ by a similarity transformation which multiplies all lengths by the constant factor σ_2. Suppose that we cut C_α into two arcs A and A', so that

$$\text{length } A + \text{length } A' = \text{length } C_\alpha = 2\pi r \sin \alpha.$$

The Lipschitz inequality (1) implies that

$$\text{length } f(A) \leqq \sigma_2 \text{ length } A, \qquad \text{length } f(A') \leqq \sigma_2 \text{ length } A'. \tag{4}$$

But

$$\text{length } f(A) + \text{length } f(A') = \text{length } f(C_\alpha)$$

is precisely equal to σ_2 times the length $2\pi r \sin \alpha$ of C_α. So both of the inequalities (4) must actually be equalities. This proves Lemma 2.

Now we must prove the converse.

LEMMA 3. *The azimuthal equidistant projection on the disk D_α has distortion δ precisely equal to* $\log(\alpha/\sin \alpha)$.

Proof. Centering D_α at the north pole, we will use the longitude $0 \leqq \theta \leqq 2\pi$ and the colatitude $0 \leqq \gamma \leqq \alpha$ as coordinates. Suppose that f maps the point with colatitude γ and longitude θ to the point with cartesian coordinates $(r\gamma \cos \theta, r\gamma \sin \theta)$ in the plane. The length of any smooth curve $\gamma = \gamma(t)$, $\theta = \theta(t)$ in D_α is given by the integral

$$L = r \int (\dot{\gamma}^2 + \dot{\theta}^2 \sin^2 \gamma)^{1/2} dt,$$

and the length of the corresponding curve in $f(D_\alpha)$ is

$$L' = r \int (\dot{\gamma}^2 + \dot{\theta}^2 \gamma^2)^{1/2} dt.$$

But, since $\gamma/\sin \gamma$ is a monotone increasing function of γ, we have

$$\sin \gamma \leqq \gamma \leqq (\alpha/\sin \alpha) \sin \gamma,$$

from which it follows easily that

$$L \leqq L' \leqq (\alpha/\sin \alpha) L. \tag{5}$$

Starting from this inequality (5) we will prove that

$$d_S(x, y) \leqq d_E(f(x), f(y)) \leqq (\alpha/\sin \alpha) d_S(x, y)$$

for every x and y in D_α. Clearly this will imply that $\delta \leqq \log(\alpha/\sin \alpha)$ and hence, by Lemma 1, that $\delta = \log(\alpha/\sin \alpha)$.

Proof that $d_S(x, y) \leqq d_E(f(x), f(y))$. Join $f(x)$ to $f(y)$ within the convex set $f(D_\alpha)$ by a line segment of length L' precisely equal to $d_E(f(x), f(y))$. The corresponding curve in D_α will have length $L \geqq d_S(x, y)$. Since $L \leqq L'$, we obtain $d_S(x, y) \leqq d_E(f(x), f(y))$, as required.

Proof that $d_E(f(x), f(y)) \leqq (\alpha/\sin\alpha)d_S(x, y)$. First suppose that $\alpha \leqq \pi/2$, so that the disk D_α is "geodesically convex." Then the proof is quite analogous. Join x to y, *within* D_α, by a great circle segment A of length $L = d_S(x, y)$. Then $f(A)$ has length $L' \geqq d_E(f(x), f(y))$, so the inequality $L' \leqq (\alpha/\sin\alpha)L$ implies that $d_E(f(x), f(y)) \leqq (\alpha/\sin\alpha)d_S(x, y)$, as required.

If $\alpha > \pi/2$, so that the disk D_α is not geodesically convex, then a more complicated argument is necessary. Suppose that the shortest great circle arc from x to y does not lie completely within D_α, but rather crosses out of D_α at a boundary point $\bar{x}$, and then crosses back in at another boundary point $\bar{y}$. We shall show that

$$d_E(f(x), f(\bar{x})) \leqq (\alpha/\sin\alpha)d_S(x, \bar{x}), \tag{6}$$

$$d_E(f(\bar{x}), f(\bar{y})) \leqq (\alpha/\sin\alpha)d_S(\bar{x}, \bar{y}), \tag{7}$$

$$d_E(f(\bar{y}), f(y)) \leqq (\alpha/\sin\alpha)d_S(\bar{y}, y). \tag{8}$$

Adding these three inequalities, we shall clearly obtain the required inequality.

But (6) and (8) can be proved by the argument above. To prove (7) we introduce an auxiliary azimuthal equidistant projection g whose domain is the complementary disk $D'_{\pi-\alpha}$ centered at the south pole. Since $\pi - \alpha \leqq \pi/2$ we have

$$d_E(g(\bar{x}), g(\bar{y})) \leqq ((\pi - \alpha)/\sin(\pi - \alpha))d_S(\bar{x}, \bar{y}).$$

Multiplying this by $\alpha/(\pi - \alpha)$ we obtain the required inequality (7). This completes the proof of Lemma 3.

Clearly Lemmas 1, 2, and 3 imply the "Main Theorem" of Section 1.

3. Discussion. How can one estimate the minimum possible distortion δ_0 for map projections on a given set U? Here is a crude estimate. Define the *angular width* w of a set U as follows. Choose a smallest possible "lune" (figure bounded by two great semicircles) containing U, and let w be the angle at the vertex of this lune.

ASSERTION. *Any set with angular width* $w < \pi$ *possesses a map projection with distortion* $\delta \leqq \log\sec(w/2)$.

This is proved by rotating so that the lune is centered on the equator, and then using the latitude and longitude of x as the cartesian coordinates of $f(x)$. The computations are similar to those in the proof of Lemma 3.

It is conjectured that this estimate gives the right order of magnitude in the case of a small geodesically convex region, in the sense that δ_0 is greater than say one sixth of $\log\sec(w/2)$. But $\log\sec(w/2)$ is not a really good estimate for δ_0, except perhaps in the case of a long narrow region.

It would be more interesting to find a relation between δ_0 and area.

PROBLEM. *Among all geodesically convex regions of given area, does the disk* D_α *require the largest distortion?*

In other words, if $\text{area}(U)=\text{area}(D_\alpha)$ does it follow that U has a map projection with distortion $\delta \leqq \log(\alpha/\sin \alpha)$? If true this would imply the existence of map projection with smaller distortion than any which are actually known for many regions on the sphere. A test case which would be particularly interesting would be that of a small "rectangular" region on the sphere.

Slightly cruder is the following possible estimate.

PROBLEM. *Does every geodesically convex region U possess a map projection with distortion less than the normalized area,*

$$\delta < \text{area } U/\text{area } S?$$

As an example, for the continental United States with about 1.5 percent of the earth's area, does there exist a map projection with scale errors of no more than 1.5 percent (or perhaps $1.5+\epsilon$ to allow for the lack of geodesic convexity)? All standard map projections for the continental United States seem to have scale errors of at least 2.2 percent.

Appendix A. Minimum distortion projections always exist. We shall first prove the following. Let U be a subset of the sphere S and let $\overline{U}$ denote the topological closure of U.

LEMMA 4. *Any map projection f on U with distortion $\delta < \infty$ extends uniquely to a map projection $\bar{f}$ on $\overline{U}$ having the same distortion δ.*

Proof. The inequalities

$$\sigma_1 d_S(x, y) \leqq d_E(f(x), f(y)) \leqq \sigma_2 d_S(x, y)$$

show that f is uniformly continuous, and hence extends uniquely to a continuous function $\bar{f}$ on $\overline{U}$. (See [**3**, p. 55].) Clearly $\bar{f}$ will also satisfy these inequalities.

Now, given some fixed set U, consider all possible map projections f with domain U, and let δ_0 denote the infimum of the corresponding distortions $\delta(f)$. We must construct a map projection f_0 whose distortion is precisely equal to δ_0. We may assume that $\delta_0 < \infty$, since otherwise there is nothing to prove.

REMARK. Note that there exists a map projection with finite distortion on U if and only if the closure $\overline{U}$ is not the entire sphere. For if U is not everywhere dense on S then U is contained in some disk $D_{\pi-\epsilon}$ and hence possesses a map projection with distortion $\delta \leqq \log((\pi-\epsilon)/\sin(\pi-\epsilon)) < \infty$. But if $\overline{U}=S$ then a map projection with finite distortion on U would extend to a map projection with finite distortion on S, which is impossible since $S \supset D_\alpha$ for all α, or since S is not homeomorphic to any subset of E. (See for example [**10**, p. 122].)

Choose a sequence of map projections $\{f_1, f_2, f_3, \cdots\}$ on U so that the corresponding sequence $\{\delta_1, \delta_2, \delta_3, \cdots\}$ of distortions tends to the limit δ_0. We may assume that each f_i has been chosen so as to have maximum scale equal to 1, and so that the image $f_i(U)$ contains the origin.

Choose a countable dense subset

$$U' = \{x_1, x_2, x_3, \cdots\}$$

of U. Since the points $f_1(x_1), f_2(x_1), \cdots$ all have distance $\leqq \pi r$ from the origin, we can choose a convergent subsequence. That is there exists an infinite set I_1 of positive integers so that the sequence of points $f_i(x_1)$, where i tends to infinity through the set I_1, converges to some limit a_1 in E. Similarly we can find an infinite set $I_2 \subset I_1$ so that the limit

$$\lim\{f_i(x_2) \mid i \to \infty, i \in I_2\}$$

exists. Call this limit a_2. Continuing inductively we can define a function f from U' to the plane by $f(x_j) = a_j = \lim\{f_i(x_j) \mid i \to \infty, i \in I_j\}$. Since the inequalities

$$e^{-\delta_i} d_S(x, y) \leqq d_E(f_i(x), f_i(y)) \leqq d_S(x, y)$$

hold for all i, it follows, taking the limit as i tends to infinity through an appropriate I_j, that

$$e^{-\delta_0} d_S(x, y) \leqq d_E(f(x), f(y)) \leqq d_S(x, y)$$

for all x and y in U'. Thus f is a map projection on U' with distortion δ_0.

Now applying Lemma 4 we obtain the required map projection on U with distortion δ_0.

Appendix B. Conformal map projections. Recall that a map projection f, defined on an open set U, is called *conformal* (cartographers prefer the term "orthomorphic") if it is differentiable and preserves angles. (That is, f transforms any pair of curves in U, whose tangent vectors at a point of intersection span the angle α into a pair of curves in E, whose tangent vectors at the corresponding intersection point span the same angle α.)

It follows that f has a well defined *infinitesimal-scale* $\sigma(x)$ at each point x of U. By definition $\sigma(x)$ is the limit of the ratio $d_E(f(x), f(y))/d_S(x, y)$ as y tends to the limit x. (Compare [**1**, p. 74].)

We shall make use of the *Laplace-Beltrami operator* Δ, a second order partial differential operator which assigns to each twice differentiable real valued function g on a Riemannian manifold a new real valued function Δg. In euclidean space this is the familiar Laplace operator. We shall use Δ only on the sphere S of radius r. Using latitude λ and longitude θ as coordinates, the operator Δ on the sphere takes the form

$$r^2 \Delta g = g_{\lambda\lambda} - g_\lambda \tan \lambda + g_{\theta\theta} \sec^2 \lambda.$$

(Compare [**14**, p. 160]. The subscripts denote partial derivatives.)

Suppose now that U is a simply connected open subset of the sphere S.

LEMMA 5. *The infinitesimal-scale function $\sigma(x)$ associated with a conformal map projection f on U determines f up to an (orientation preserving or reversing)* *rigid motion of the plane. A given positive real valued function σ on U is the infinitesimal-scale function associated with some conformal f if and only if σ is twice differentiable and satisfies the differential equation $r^2\Delta \log \sigma = 1$.*

As an example, the function $\sigma(x) = \sec(\text{latitude } x)$ provides a solution to this equation $r^2\Delta \log \sigma = 1$, except at the north and south poles. The corresponding f turns out to be the familiar Mercator projection.

(Note that our differential equation cannot have any solution which is defined and smooth throughout the entire sphere, since the condition $\Delta \log \sigma > 0$ implies easily that σ cannot have any local maximum.)

Proof of Lemma 5. More generally, consider a smooth surface M provided with a Riemannian metric, expressed in terms of local coordinates u and v as $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. Let Δ denote the associated Laplace-Beltrami operator, and let K denote the Gaussian curvature of M. Consider a second Riemannian metric of the form $\sigma^2 ds^2$ on M, where σ is a positive twice differentiable function. Computation (using for example [**14**, pp. 113, 160]) shows that the Gaussian curvature K' associated with this new Riemannian metric is given by the formula $K' = (K - \Delta \log \sigma)/\sigma^2$.

If σ is the infinitesimal-scale function associated with a conformal mapping f from M to M', then clearly $K'(x)$ is just the Gaussian curvature of M' at $f(x)$. Thus if M' is the euclidean plane, with $K' \equiv 0$, we see that the differential equation

$$\Delta \log \sigma = K$$

must be satisfied. In particular, taking M to be the open subset U of S, with $K \equiv 1/r^2$, we obtain the required equation

$$r^2 \Delta \log \sigma = 1.$$

Conversely, given any solution σ to the differential equation $\Delta \log \sigma = K$, the Riemannian metric $\sigma^2 ds^2$ has curvature K' identically zero. Hence any sufficiently small connected open subset of M, with the metric $\sigma^2 ds^2$, can be mapped isometrically onto an open subset of the plane ([**14**, p. 145]). This isometry is unique up to rigid motions of the plane, since any isometry ϕ from one connected open subset of the plane to another extends to an isometry of the entire plane. (Assuming that ϕ preserves orientation, we can think of ϕ as a complex analytic function [**1**, p. 74] with $|d\phi/dz| \equiv 1$. Hence $d\phi/dz$ is constant and $\phi(z) = cz + c'$ with $|c| = 1$.)

Now if M is simply connected then a monodromy argument shows that these local isometries can be chosen so as to fit together to yield a smooth mapping f from all of M to E.

(Compare [**8**, p. 1297]. The "Monodromy Theorem" says that if we are given connected open sets U_α covering a simply connected manifold M, and for each U_α a collection F_α of functions from U_α to Y satisfying the following condition, then there exists a function from M to Y whose restriction to each U_α belongs to F_α. The condition is that for each f_α in F_α and each x in $U_\alpha \cap U_\beta$ there should exist one and only one f_β in F_β which coincides with f_α throughout some neighborhood of x. Compare [**1**, p. 285], [**12**, p. 67].)

In the large, this mapping f from M to E may not be one-to-one, but locally it carries M, with the metric $\sigma^2 ds^2$, isometrically to E. Hence it carries M with the original metric ds^2 conformally to E, the infinitesimal-scale function of f being precisely equal to σ. This completes the proof of Lemma 5.

Chebyshef [**2**] studied conformal map projections, using the ratio $\sup \sigma(x)/\inf \sigma(x)$ of maximum infinitesimal-scale to minimum infinitesimal-scale as a measure of distortion.

REMARK. If the domain U is geodesically convex, note that the maximum infinitesimal-scale $\sup \sigma(x)$ is equal to the maximum scale σ_2 of Section 1. (Compare the proof of Lemma 3.) Similarly, if f is one-to-one and $f(U)$ is convex, then $\inf \sigma(x) = \sigma_1$.

CHEBYSHEF THEOREM. *If U is a simply connected region bounded by a twice differentiable curve, then there exists one and, up to a similarity transformation of E, only one conformal map projection which minimizes this ratio* $\sup \sigma/\inf \sigma$. *This "best possible" conformal map projection is characterized by the property that its infinitesimal-scale function $\sigma(x)$ is constant along the boundary of U.*

This result has been available for more than a hundred years, but to my knowledge it has never been used by actual map makers.

Proof. Setting $g(x) = \log \sigma(x)$, first note that the differential equation $r^2 \Delta g = 1$ has a unique solution satisfying the boundary condition $g(x) = 0$ for $x \in bd(U)$. See for example [**5**, p. 288]. If h is any other function which is twice differentiable and satisfies the equation $r^2 \Delta h = 1$ throughout the interior of U, then we shall show that

$$\sup h - \inf h \geqq \sup g - \inf g, \tag{9}$$

where equality holds if and only if

$$h = g + \text{constant}.$$

(Note that $\sup g - \inf g$ is just the logarithm of the ratio $\sup \sigma/\inf \sigma$ which we want to minimize.) Clearly this will complete the proof.

Since $\Delta g > 0$, an easy argument shows that the function g cannot attain its maximum at any interior point of U. Since g must achieve a maximum at some point of the compact set $\overline{U}$, it follows that the maximum must be attained on $bd(U)$. Thus $\sup g(x) = 0$.

The difference $h - g$ satisfies the homogeneous equation $\Delta(h - g) = 0$, and so cannot achieve its maximum at an interior point of U unless $h - g = \text{constant}$. (See [**5**, p. 232].) Hence any sequence of points $x_1, x_2, \cdots$ for which

$$\lim_{i \to \infty} (h(x_i) - g(x_i)) = \sup(h - g)$$

must be a sequence tending to the boundary of U, unless $h-g=$constant.

Setting $c=\sup h$ (we may assume that c is finite since otherwise (9) would trivially be satisfied), we have

$$g(x_i) \to 0, \qquad h(x_i) \leqq c,$$

hence

$$\sup(h-g) = \lim(h(x_i) - g(x_i)) \leqq c,$$

or in other words

$$h(x) \leqq g(x) + c$$

for all x. Therefore $\inf h \leqq \inf g+c$, which proves (9).

If equality holds, then at the interior point x_0 of U where g achieves its minimum we have

$$h(x_0) = g(x_0) + c.$$

Thus $h-g$ achieves its maximum c at an interior point, and hence is constant. This completes the proof.

REMARK. The "best possible" conformal map projection f, although locally well behaved, may not be one-to-one in the large. However, if U is geodesically convex, then it can be shown that f is one-to-one and that $f(U)$ is also convex.

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