A subset of a topological space X will be called *connected* if it becomes a connected topological space when equipped with the induced topology. Obviously, $\{x\}$ is always a connected subset of X.

Problem 1 (6 points). *Show that every non-empty connected subset of X is contained in a unique $\subseteq$-maximal connected subset.*

The $\subseteq$-maximal elements of the set of non-empty connected subsets of X will be called the *connected components* of X. The set $\pi_0(X)$ of connected components of X will be equipped with the quotient topology for the map $X \xrightarrow{p} \pi_0(X)$ sending $x \in X$ to the unique connected component of X containing x. Thus, a subset $U \subseteq \pi_0(X)$ is open if and only if $p^{-1}U$ is an open subset of X.

Problem 2 (2 points). *Show that every connected component of X is a closed subset of X.*

Let a subset of X be called *clopen* if it is both closed and open. Let the set of *quasi-components* be the set of equivalence classes on X where $x \sim y$ if and only if the sets of clopen subsets containing x and of clopen subsets containing y coincide. The equivalence class $q(x)$ of x is easily seen to be the intersection of all clopen subsets of X containing x. In particular, all quasi-components are closed. The space QX of quasi-components will be equipped with the quotient topology for q.

Problem 3 (1 point). *Show that QX is Hausdorff.*

Problem 4 (2 points). *Show that every connected component of X is contained in a unique quasi-component.*

Because of this there is a unique continuous map $\pi_0 X \xrightarrow{r} QX$ such that $q = rp$. Obviously, the following conditions are equivalent:

- r is bijective.
- r is a homeomorphism.
- Every quasi-component is connected.

Problem 5 (4 points). *Let X be a quasicompact topological space with the following property:*

> *If Q is a quasi-component of X and $Q = A \cup B$ where A and B are disjoint closed subsets of X then there are disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.*

Show that every quasi-component of X is connected.

The existence of U and V is obvious when X is T_4. By a well-known result of general topology, a quasicompact space is T_4 if and only if it is T_2. Thus, for compact spaces the spaces of connected components and quasicomponents are canonically homeomorphic and compact (in particular, Hausdorff). That the same holds for spectral spaces follows from

Problem 6 (4 points)**.** *Let X be a topological space satisfying the equivalent conditions of Sheet 1 Problem 6. Show that Problem 5 can be applied to X.*

Solutions should be e-mailed to my institute e-mail address (my second name (franke) at math dot uni hyphen bonn dot de) before Monday November 4.