Problem sheet 10 Rigid analytic geometry Winter term 2024/25

In the following problems let $A$ be a Noetherian complete Tate ring, and let $(A^\sharp, s)$ be a pair of definition for $A$. Let $M$ and $N$ be finitely generated $A$"=modules.

**Problem 1** (3 points). *For a subset $U \subseteq M$, show that the following conditions are equivalent:*

- *For every finitely generated $A^\sharp$-submodule $X \subseteq M$, there is $n \in \mathbb{N}$ such that $s^n X \subseteq U$.*
- *There is a finitely generated $A^\sharp$-submodule $X \subseteq M$ with $M = \bigcup_{n=0}^\infty s^{-n} X$ and such that $X \subseteq U$.*

**Problem 2** (3 points). *Show that we have a unique structore of a nat-$A$-module on $M$ for which $U$ is a neighbourhood of zero if and only if it satisfies the equivalent conditions from the previous problem!*

If $M = A$, it follows from Proposition 2.2.2 of the lecture (Problem 10 of sheet 8) that the topology from the previous problem coincides with the topology on $A$ we started with.

From now on $M$ will always be assumed to be equipped with this topology and $N$ will be equipped with the anologous topology.

**Problem 3** (2 points). *If $T$ is an arbitrary nat-$A$-module, show that a morphism of $A$-modules $M \to T$ is automatically continuous!*

**Problem 4** (3 points). *If $\vec{m} = (m_i)_{i=1}^k$ generate $M$ as an $A$-module, show that $A^k \xrightarrow{\vec{m}} M$ is continuous and $M$ carries the quotient topology!*

**Problem 5** (4 points). *Show that $M$ is complete and has a countable neigbourhood base of $0$! Moreover, show that the topology we have fixed on $A$ is the only one with these properties!*

**Problem 6** (3 points). *If $N$ is a submodule of $M$, show that it is closed and its topology coincides with the one induced from $M$!*

**Problem 7** (2 points). *If $M \xrightarrow{f} N$ is a surjective morphisms of $A$-modules, show that it is open!*

**Problem 8** (5 points). *Let $X$ be a spectral space and $\mathcal{X} \subseteq X$ a subset which intersects every non-empty constructible subset of $X$. We equip $\mathcal{X}$ whith the ordinary topology for which*

$$\mathfrak{B} = \left\{ \Omega \cap \mathcal{X} \;\middle|\; \Omega \in \mathfrak{Qc}(X) \right\}$$

*is a topology base and with the* $G_+$*-topology obtained by forcing the elements of* $\mathfrak{B}$ *to be quasicompact. Construct a homeomorphism between* $X$ *and* $\mathcal{X}^*$*!*

Five of the 25 points from this sheet are bonus points.

Solutions should be e-mailed to my institute e-mail address (my second name (franke) at math dot uni hyphen bonn dot de) before Monday January 13.