Eleventh exercise sheet "Algebra II" winter term 2024/5. Using Proposition 3.1.4 from the lecture one can show

Problem 1 (4 points). *Let* $|\cdot|_{1,2}$ *be two inequivalent and non-trivial valuations of a field* K*. Show that there is* $x \in K$ *with* $|x|_1 > 1$ *and* $|x|_2 < 1$*.*

Problem 2 (4 points). *Let* N *be a positive integer and* $|\cdot|_i$*,* $1 \le i \le N$*, be* N *pairwise inequivalent and non-trivial absolute values on a field* K*. Then there is* $x \in K$ *with* $|x|_1 > 1$ *and* $|x|_i < 1$ *for* $1 < i \le N$*.*

Problem 3 (4 points). *Under the assumptions of the previous problem, let* $\varepsilon \in (0, \infty)_{\mathbb{R}}$*. Show that there is* $x \in K$ *with* $|1 - x|_1 < \varepsilon$ *and* $|x|_i < \varepsilon$ *for* $2 \le i \le N$*.*

Problem 4 (4 points). *Let* N *be a positive integer and* $|\cdot|_i$*,* $1 \le i \le N$*, be* N *pairwise inequivalent and non-trivial absolute values on a field* K*. Let* K_i *be the completion of* K *with respect to* $|\cdot|_i$*. Show that the diagonal image of* K *in* $\prod_{i=1}^{N} K_i$ *is dense!*

For the following three problems, let K be complete with non-Archmedean and non-trivial absolute value $|\cdot|$. Let $\mathfrak{k} = K^o/K^{oo}$ denote the residue field of K.

Problem 5 (3 points). *Let* $P \in K^o[T]$ *be a polynomial with leading coefficient* 1*. If its residue class* $\overline{P} \in \mathfrak{k}[T]$ *is irreducible, show that* P *is irreducible as an element of* $K[T]$*.*

Problem 6 (3 points). *Let* $\mathfrak{k}'$ *be a finite separable field extension of* $\mathfrak{k}$*. Show that there are a finite separable field extension* L/K *with* $e = 1$ *(i. e.,* $|L^\times| = |K^\times|$ *and an isomorphism* $\mathfrak{k}' \xrightarrow{\lambda} \mathfrak{l} = L^o/L^{oo}$*!*

Problem 7 (3 points). *Let* $\mathfrak{k}' \xrightarrow{\lambda_1} \mathfrak{l}_1$ *be another solution to the previous problem. Show that there is a unique isomorphism* $L \xrightarrow{\iota} L_1$ *of extension fields of* $\mathfrak{k}$ *such that* $\lambda_1 = \bar{\iota}\lambda$*, where* $\mathfrak{l} \xrightarrow{\bar{\iota}} \mathfrak{l}_1$ *is the isomorphism of residue fields defined by* ι*.*

Twenty of the 25 points available from this exercise sheet are bonus points which are disregarded in the calculation of the 50%-limit for passing the exercises.

Solutions should be submitted to the tutor by e-mail before Friday January 17 24:00.