1. Introduction

Thetaseries are the most classical examples of modular forms and still one of the most efficient tools to construct them. In some arithmetic situations (like Siegel modular forms over the integers) they are the only available tool. Today we deal with modulispaces of bundles over curves which are simpler but still pose interesting problems.

We have:

Curves C

$\Rightarrow$ Bundles $\mathcal{E}$ on C

$\Rightarrow$ Modulispace of Bundles $\mathcal{M}_G$

$\Rightarrow$ line-bundles $\mathcal{L}_c$ on $\mathcal{M}_G$.

2. Curves

We can define curves as either compact connected Riemann-surfaces or as smooth projective irreducible algebraic curves. The latter is more complicated but works over more general basefields. However for today the complex numbers suffice. Their most important is the genus g, "the number of handles".

On a curve a vectorbundle always means holomorphic (or algebraic) vectorbundle. It has as structuregroup the algebraic group GL_r. Later we need other groups, like the symplectic group SP_{2r} or spin-groups. A vectorbundle with structuregroup SP_{2r} is the same as a vectorbundle $\mathcal{E}$ of rank $2r$ together with a non-degenerate symplectic inner product $\mathcal{E} \times \mathcal{E} \to \mathcal{O}_C$.

Vectorbundles on curves $\mathcal{E}$ have a determinant $det(\mathcal{E})$ which is a line-bundle. It has a degree, and line-bundles of degree zero are parametrised by the Jacobian J, an abelian variety (complex torus) of dimension equal to the genus g of C.

3. Classical thetas

The Riemann thetafunction plays an important role in the theory of principally polarised abelian varieties. It is given by the explicit sum

$$\theta(\underline{z};Z)=\sum_{\underline{n}} exp(\pi i\underline{n}^t Z\underline{n}+\underline{n}^t\underline{z}).$$

(Z lies in the Siegel upper halfspace, etc.) For curves C of genus g the zeroset of θ on the jacobian J consists of line-bundles $\mathcal{L}$ of degree $g-1$ for which

$H^0(C,\mathcal{L})\neq 0$, or $H^1(C,\mathcal{L})\neq 0$ (both spaces have the same dimension by Riemann-Roch).

A modular interpretation:

Consider the tautological line-bundle $\mathcal{L}$ of degree $g-1$ on $C\times J$. The derived direct image $\mathbb{R}pr_{2,*}\mathcal{L}$ is (by EGA3) locally representable by a complex of vectorbundles

$\mathcal{F}^0\to\mathcal{F}^1$.

The line-bundle $det(\mathcal{F}^1)\otimes det(\mathcal{F}^0)^{\otimes -1}$ is globally defined on J, and is the inverse of the determinant of the cohomology of $\mathcal{L}$. Furthermore the differential of the complex defines a global section which (after suitable trivialisations over the complex numbers) can be identified with the theta-function.

4. Theta for GL_n and SL_n

The definition as determinant of cohomology generalises to vectorbundles of higher rank. Namely on the stack of vectorbundles $\mathcal{E}$ of slope $g-1$ we get a global section of the inverse of the determinant of cohomology, whose zeroset consists of bundles $\mathcal{E}$ for which $H^0(C,\mathcal{E})$ (or $H^1(C,\mathcal{E})$) does not vanish. If we restrict to vectorbundles of rank n with trivial determinant we can consider the thetafunctions for twists $\mathcal{E}\otimes\mathcal{L}$, where $\mathcal{L}$ is a line-bundle of degree $g-1$. The isomorphism-class of the determinant-bundle on the moduli-stack of SL_n-bundles does not depend on $\mathcal{L}$: Replacing $\mathcal{L}$ by $\mathcal{L}(x)$ for a point $x\in C$ changes the determinant of cohomology by the (trivial) determinant of the fibre of $\mathcal{E}$ in x, etc..

Thus all these thetas lie in a fixed space of sections. It turns out that this space has dimension n^g which is the minimum possible: Namely the moduli-stack of SL_n-bundles admits an action of $J[n]=H^1(C,\mu_n)$ by central twists. It extends to an action on the determinant-of-cohomology bundle if one passes to the usual central $\mathbb{G}_m$-extension (all important in classical theta-theory), and all representations of this group with the identity as central character are multiples of one irreducible representation of dimension n^g.

5. G-bundles

Suppose G is a simple, semisimple, and simply connected algebraic group, and we consider G-torsors on a curve C. These are classified by a stack $\mathcal{M}_G$ of relative dimension $(g-1)dim(G)$. Analogues of theta-functions should be sections of line-bundles on $\mathcal{M}_G$, so we first have to determine these. If $x \in C$ is a point any G-torsor becomes trivial on the complement $C^0 = C - \{x\}$. If we choose a local coordinate t near x we thus obtain a representation as double cosets

$$\mathcal{M}_G = \Gamma(C^0, G)\backslash LG/L^{\geq 0}G.$$

Here $\Gamma(C^0, G)$ is an ind-groupscheme, LG is the loopgroup (also an ind-groupscheme) with

$LG(R) = G(R((t)))$,

and $L^{\geq 0}$ are the holomorphic loops. The quotient

$\mathbb{D}_G = LG/L^{\geq 0}G$

is an inductive limit of projective algebraic varieties (Bruhat-cells) and also known as affine Grassmannian. It is known that $Pic(\mathbb{D}_G) = \mathbb{Z}$ is cyclic. Any of its elements is equivariant under a central $\mathbb{G}_m$-extension $\tilde{L}G$ of LG. In characteristic zero it can be constructed using the G-invariant bilinear form B on $\mathfrak{g}$ for which $B(\check{\alpha}, \check{\alpha}) = 2$ for a long root α. Finally $\Gamma(C^0, G)$ admits no characters so that the pullback from $Pic(\mathcal{M}_G)$ to $Pic(\mathbb{D}_G)$ is injective. Thus a line-bundle on $\mathcal{M}_G$ is determined by its "central charge".

6. Determinant of cohomology

For any G-module E we may pullback the determinant of cohomology bundle via the induced map from $\mathcal{M}_G$ to $\mathcal{M}_{SL(E)}$. The invariant of the pullback in $Pic(\mathbb{D}_G) = \mathbb{Z}$ is the ratio of the trace-form tr_E and B. It follows that for the groups SL_n and SP_{2n} the determinant of cohomology of the standard-representation generates the Picard-group of $\mathbb{D}_G$ and also of $\mathcal{M}_G$. For orthogonal representations the determinant of cohomology as well as its canonical section admit squareroots (the Pfaffians), and these for the standard representation generate the Picard-groups for $G = Spin(n)$. However for exceptional groups these constructions are not sufficient. For example for the group E_8 we can only construct a line-bundle with central charge 30, from the (orthogonal) adjoint representation.

7. Construction of line-bundles with central charge one (interlude especially for Luc)

Such a construction can be done as follows: For a Borel B of G consider sufficiently generic B-torsors, i.e. the degree of the associated T-torsors should correpond to a suffiently antidominant coweight. For example for $G = SL_n$ one considers flags

$$\mathcal{E}_0 = (0) \subset \mathcal{E}_1 \subset ... \subset \mathcal{E}_n = \mathcal{E}$$

such that the degrees $deg(\mathcal{E}_i/\mathcal{E}_{i-1})$ are strongly increasing. For such a T-torsor the twist $\mathcal{U}$ of the unipotent radical U of B is "very negative", and our space is an $H^1(C,\mathcal{U})$-fibrespace over J^l. Over the base the even form B defines a canonical polarisation and a symmetric line-bundle in this polarisation, thus by pullback a line-bundle on the stack of generic B-torsors. This stack covers $\mathcal{M}_G$, and one constructs a descent-datum to get a line-bundle on $\mathcal{M}_G$.

Remark: The Hitchin-fibration defines coverings $D \to C$ and abelian subvarieties $A \subset J^l_D$ mapping to $\mathcal{M}_G$. The pullback of a line-bundle of central charge one defines a polarisation on A which again can be described by the bilinear form B. This can be used to get upper bounds for the dimension of the space of global sections, even in (good) positive characteristics.

8. Application of the Verlinde-formula

The Verlinde-formula computes the dimension of spaces of global sections of line-bundles on $\mathcal{M}_G$, in characteristic zero. More generally suppose given points $\{x_1, ..., x_r\} \subset C$, and for each point x_i an irreducible representation E_i of G. The fibres at x_i of the universal G-torsor on $\mathcal{M}_G \times C$ define vector-bundles $\mathcal{E}_i$ on $\mathcal{M}_G$. If $\mathcal{L}_c$ denotes the line-bundle of central charge c, the Verlinde-formula computes the dimension of

$\Gamma(\mathcal{M}_G, \bigotimes_{i=1}^{r} \mathcal{E}_i \otimes \mathcal{L}_c)$.

It is known that this dimension does not depend on the choice of C nor on that of the x_i (projective connections), and that it vanishes unless all E_i are integrable. That means that the dual $\check{\theta}$ of the maximal root θ has only eigenvalues of size $\leq c$. There are (for fixed c) only finitely many integrable irreducible representations.

9. Degenerating the curve

For computing dimensions one degenerates C into a projective line with g nodes. A G-torsor on such a singular curve is the same as a G-torsor on its normalisation $\mathbb{P}^1$, together with identifications of the fibres in g pairs of points. The latter correspond to sections of a twist of G, considered as scheme with $G \times G$-action.

As the regular functions on G are the direct sum of tensorproducts $E \otimes E^v$, with E running over all irreducible representations of G, one gets that to obtain the dimension we may replace C by $\mathbb{P}^1$, add to the E_i g-pairs $\{E, E^v\}$, and sum over all such g-tuples of pairs. The sum is effectively finite because of the integrability condition.

Remark: This construction only covers G-bundles with "good reduction". One can show that our sections also extend to others, using rigid analysis, but this was deemed to be too difficult for the general public. An alternative is to reduce to Lie-algebras.

10. Verlinde for the projective line

Finally for the projective line everything is governed by the Verlinde-algebra $\mathcal{F}_c$. It is the vectorspace with basis the irreducible representations of E of level $\leq c$. It has an orthogonal form where the product of basiselements vanishes except that $< E, E^v >= 1$ is one. Furthermore we have a product defined by

$< E_1 \cdot E_2, E_3 >=$ dimension of space of global sections of $\mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3 \otimes \mathcal{L}_c$.

By various deformations one shows that this product is associative, and the trivial representation is a unit. Also if

$\kappa = \sum E \cdot E^v$

is the Casimir element our desired dimension is the inner product of $\kappa^g \cdot E_1 \cdot ... \cdot E_r$ with the unit. One can determine the structure of $\mathcal{F}_c$ and compute this explicitely.

11. The structure of the Verlinde-algebra

Let $\check{h} = \rho(\check{\theta}) + 1$ (dual Coxeter number). Associating $\lambda + \rho$ to an irreducible representation with highest weight λ identifies the set of integrable representations with elements μ in the weight-lattice such that $\mu(\check{\alpha}) > 0$ (α positive), $\mu(\check{\theta}) < c + \check{h}$. If we replace "$<$" by "$\leq$" we obtain a fundamental domain for an action of the affine Weylgroup W^{aff}. This group acts on $\mathbb{Z}[weights]$. Twist this action by the sign character and form the quotient. It has as basis the μ's above and thus can be identified with $\mathcal{F}_c$. If instead we only divide by the usual Weylgroup W we obtain the representation-ring $R(G)$. Fact:

The projection $R(G) \to \mathcal{F}_c$ is a homomorphism of rings.

This determines the multiplication on $\mathcal{F}_c$. The characters of $\mathcal{F}_c$ are identified with regular elements $x \in T$ (modulo W) such that $\alpha(x)^{c+\check{h}} = 1$ for long roots α.

12. Level one

We now specialise to the case of $c = 1$ and that there are only long roots, i.e. G is of type A, D, E. Then integrable representations are miniscule representations. They correspond to characters of the center Z of G. Furthermore the product in $\mathcal{F}_1$ corresponds to the product of characters, so $\mathcal{F}_1$ is the group-algebra of the dual Z^v. Also $\kappa = |Z|$ is a scalar and thus

$dim(\gamma(\mathcal{M}_G, \mathcal{L}_1) = |Z|^g$.

Especially for G of type E_8 there is a unique nontrivial section (from now on called "E_8-theta") which defines a divisor in $\mathcal{M}_G$. What is that divisor? A variant: Let X denote the E_8-lattice which its even unimodular form. For any principally polarised abelian variety A $A \otimes X \cong A^8$ is not only principally polarised but admits a canonical ample symmetric linebundle defining this principal polarisation. The zeroset of its unique global section is a canonical theta-divisor. Is there a geometric description for it. If $A = J$ is the Jacobian of a curve C $A \otimes X$ classifies T-torsors, T the maximal torus of the group E_8. The E_8-theta restricts to a multiple of this canonical theta. From the proof of the Verlinde-formula one derives that this restriction does not vanish for Mumford-curves (it is nonzero on the trivial torsor).

13. Spin-thetas?

For other G's as the space of sections admits an action of the central $\mathbb{G}_m$-extension of $H^1(C, Z)$ it is the unique irreducible extension of that central extension. This can be also shown in characteristic p if p is a good prime for G: Semicontinuity gives a lower bound for the dimension of the space of sections, and the Hitchin-fibration an upper bound. Restricting the E_8-theta gives canonical sections for E_6 and E_7. As the case of SL_n is wellknown this leaves type D. For spin-groups $Spin_{4n}$ the centre Z is isomorphic to $\mu_2 \times \mu_2$. For each subfactor μ_2 we get a maximal isotropic subgroup $H^1(C, \mu_2) \subset H^1(C, Z)$ and thus thetafunctions which are eigenfunctions for this subgroup (the eigenvalues depend on how we lift our subgroup into the central extension). For the diagonal μ_2 we get the previous Pfaffians for SO_{4n}-bundles, while the other possibilities are somehow related to the two spin-representations. For example $Spin_8$ embeds into E_8 and the restriction of the E_8-theta is a multiple of one of the "spin-thetas".

14. Examples of divisors in $\mathcal{M}_G$

a) nilpotent endomorphisms

Over $\mathcal{M}_G$ we construct a relative projective scheme which classifies isomorphism classes of triples $(P, \mathcal{L}, N)$, with

- P a G-torsor on C,
- $\mathcal{L}$ a line-bundle on the base,
- $N \in \Gamma(C, \mathfrak{g}^{ad,P} \otimes \omega_C) \otimes \mathcal{L}$ a nonzero (on each fibre) nilpotent section.

By the theory of Hitchin fibrations its image lies in a proper divisor.

b) parabolic reduction

Suppose $Q \subset G$ is a maximal proper parabolic, Q^0 its derived group, so $Q/Q^0 \cong \mathbb{G}_m$. Consider reductions of a G-torsor P on C to a Q-torsor, together with an injection of the the associated line-bundle into some fixed line-bundle $\mathcal{L}$ on C. This is again projective over $\mathcal{M}_G$, and sometimes (for suitable degree of $\mathcal{L}$) has the right dimension to define a divisor.

c) A common generalisation: Suppose Z is a projective scheme with a very ample line-bundle $\mathcal{M}$, and G operates on the tuple $(Z, \mathcal{M})$. For a fixed line-bundle $\mathcal{L}$ on C consider the relative projective scheme over $\mathcal{M}_G$ whose fibre over a G-torsor P consists of section $s : C \to Z^P$ and an injection $s^*(\mathcal{M}) \to \mathcal{L}$, up to $\mathbb{G}_m$-action. Under suitable condition its image defines a divisor in $\mathcal{M}_G$.

In case a) where N is supposed to lie in the minimal nilpotent conjugacy class the central charge is $(2h^\vee)^g$.