V5B4 Selected Topics in PDE and Mathematical Models: *Introduction to Mathematical Hydrodynamics*

Dimitri COBB

Winter Term 2023-2024

1 Introduction

This course is primarily meant as an introduction to non-linear evolution Partial Differential Equations (evolution PDEs) at a Masters level. These equations describe the dynamics of a quantity which depends both on time and space (such as the pressure of a fluid, the temperature in a room, *etc.*), and are omnipresent in Physics. Our focus will be to understand the different methods and challenges naturally arising in PDEs that come from incompressible fluid dynamics.

The main goal of the lectures will be to present, through many examples, the type of questions that are studied in the field of PDEs, the mathematical methods developed for them, as well as to provide some insight on the behavior of the underlying physical systems.

After a short introduction where we will present typical questions for non-linear problems, the course will be divided into three parts. Firstly, we will discuss the question of existence and uniqueness of solutions with a given initial datum (the Cauchy problem) by means of "elementary" methods. In a second part, we will step away from Cauchy problem and investigate some of the qualitative properties of solutions, such as long time or far-field asymptotics, or the distribution of kinetic energy inside the fluid. The last and third part is dedicated to "non-elementary methods" in PDEs, where we will introduce tools from Harmonic Analysis.

2 Tentative Table of Contents

2.1 Existence and Uniqueness of Solutions

Introduction : Burgers equation. Strong solutions and finite time blow-up. Weak solutions and non-uniqueness.

Energy Methods I : incompressible Euler system, existence and uniqueness of strong H^k Sobolev solutions, Friedrichs scheme for constructing a solution.

Energy methods II : (Generalized) Surface Quasi-Geostrophic equations, existence and uniqueness of strong H^k solutions, global weak solutions (in some cases).

Navier-Stokes equations I : global Leray solutions, uniqueness in 2D, local strong solutions in 3D, further remarks on the 3D problem.

2.2 Qualitative Properties of Solutions

Navier-Stokes equations II : decay of kinetic energy and the Masuda-Kato theorem, asymptotic behavior at infinity $|x| \to +\infty$.

Energy distribution in 2D Euler : bounded Serfati solutions, local energy balance, algebraic time growth of solutions and Zelik's theorem.

2.3 Advanced methods (if time allows)

Global Uniqueness in 2D Euler : Vorticity and incompressibility, Singular Integral Operators, Yudovich solutions.

Ideal Magnetohydrodynamics : ideal MHD equations, vorticity form, Besov spaces and Littlewood-Paley analysis, lower bounds on the lifespan of solutions.

Transport equations : log-Lipschitz vector fields, loss of regularity of the solution, application to an Active Scalar Equation.

3 Prerequisites

- ODEs, Cauchy-Lipschitz theorem, multi-variable calculus.
- Integration Theory: Lebesgue spaces, Hölder inequalities, approximation by smooth functions.
- Basic Functional Analysis, Picard-Banach fixed point theorem, duality, weak convergence.
- Sobolev spaces H^k, Sobolev embeddings, compact embeddings.
- Fourier transform, Plancherel identity

4 Exam

Information on the oral exam will be given during the semester.

References and Suggested Readings

No reading is required prior or during the course. However, the following books and articles cover most of the topics of the lectures, and are full of other interesting content.

- H. Bahouri, J.-Y. Chemin and R. Danchin: *"Fourier analysis and nonlinear partial differential equations"*. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Scinences), Springer, Heidelberg, 2011.
- L. Brandolese: *Hexagonal structures in 2D Navier-Stokes flows.* Comm. Partial Differential Equations 47(2022), no.6, pp. 1070–1097.
- R. Danchin and F. Fanelli: *The well-posedness issue for the density-dependent Euler equations in endpoint Besov spaces.* J. Math. Pures Appl. (9), **96** (2011), n. 3, pp. 253–278.
- T. Gallay: *Infinite energy solutions of the two-dimensional Navier-Stokes equations.* (Lecture notes), Annales de la Faculté des Sciences de Toulouse **26** (2017), pp. 979–1027.
- P. G. Lemarié-Rieusset: *"Recent developments in the Navier-Stokes problem"*. Chapman & Hall/CRC Research Notes in Mathematics, n. 431. Chapman & Hall/CRC, Boca Raton, FL, 2002.
- A. J. Majda and A. L. Bertozzi: *"Vorticity and incompressible flow"*. Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002.
- E. M. Stein: *"Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals"*. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, **43**. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.