On the large-scale geometry of flat surfaces

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Klaus Dankwart

aus

Böblingen

Bonn 2010

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Gutachter: Prof. Dr. Ursula Hamenstädt
2. Gutachter: Prof. Dr. Werner Ballmann

Tag der Promotion:

Acknowledgement: I am very grateful to Professor Hamenstädt for continuous encouragement and support since my second year at university. Without her, I would not have been able to write this thesis. Moreover, I would like to thank Sebastian Hensel and Emanuel Nipper for countless discussions and much more. All of them and especially Heike Backer were a source of good mood inside the working group.
I am also grateful to Professor Hubert for inviting me to Marseille and for introducing me to the Arnoux-Yoccoz diffeomorphism. Special thanks go to the Bonn international graduate school in mathematics for financial support which made this project possible.
Finally, I am grateful to my family and my friends for supporting me in every non-mathematical way one can image. Especially Inga who managed to handle all my moods and to cheer me up during bad times.

Contents

1 Introduction

In the studies of closed Riemann surfaces of genus $g \geq 2$ the uniformization Theorem plays a crucial role since it brings conformal and hyperbolic structures in one-to-one correspondence. However, there exists a family of so-called flat metrics in the same conformal class.

Each flat metric arises from a choice of charts away from a finite set of points so that the transition functions are half-translations. Outside the marked points one can pull back the flat metric to the surface. Due to the fact that the euclidean metric has been studied for more than 2500 years, this concept is a natural one. However, according to the Gauss-Bonnet Theorem the flat metric cannot be extended to the whole surface. In the isolated points the metric is of cone type with cone angle $k\pi, k \geq 3$.

The flat metric defines a volume element on the surface. After scaling each chart one again obtains an atlas of charts so that the transition functions are half-translations. Therefore, one can define a flat metric with the scaled geometrical properties and a scaled volume element. In hyperbolic geometry one normalizes the metric on a closed surface X to curvature -1 which is equivalent to scaling the metric to total area $2\pi\chi(X)$. Since the flat metric is singular euclidean we cannot determine a normalization by curvature. That is why we normalize each metric to total area 1.

For each flat metric on a Riemann surface there exists a hyperbolic metric in the same conformal class. Unfortunately the correspondence between hyperbolic and flat metric is hard to determine. It is generally impossible to decide whether two flat metrics are in the same conformal class.

It is the goal of this work to investigate the geometry and dynamics of flat metrics and to compare the structure of such metrics with the corresponding hyperbolic metrics. Since a flat metric is locally euclidean, the local properties of flat metrics are well understood whereas their behavior on large scales is less evident. It can be more easily investigated on the universal cover with the lifted metric.

As the hyperbolic and the flat metric on the universal cover are quasi-isometric, the flat metric shares various properties from coarse geometry with metrics of negative curvature. For example, the growth rate of metric balls is exponential, and geodesic triangles are uniformly thin. We can therefore compute the classic invariants of spaces of non-positive curvature, i.e. volume entropy and metric boundaries of the universal cover. One can define the Hausdorff dimension of the boundary. Due to the construction of Patterson-Sullivan measures, volume entropy and Hausdorff dimension are closely related.

On a closed hyperbolic surface, the volume entropy is always 1 and the Hausdorff di-

mension of the boundary of the universal cover is 1 as well.
This does not hold in the case of flat surfaces. The moduli space of flat metrics $\mathcal{Q}_g$ is the set of all closed flat surfaces of genus g and area 1, compare [Vee90]. We show that volume entropy and Hausdorff dimension of the boundary of the universal cover, under appropriate choice of the boundary metric, continuously depend on the point in $\mathcal{Q}_g$.

Theorem (Theorem 4.1, Corollary 4.2)**.** *The volume entropy and the Hausdorff dimension of the boundary are bounded from below by a positive constant.*
A sequence of flat surfaces diverges in $\mathcal{Q}_g$ if and only if the volume entropy and the Hausdorff dimension of the boundary tend to infinity.

Finite sheeted branched coverings form an important concept in the theory of Riemann surfaces. Since the investigated surfaces are endowed with a flat metric, we claim compatibility of the covering with the metric. That means that covering space and base space are flat surfaces and away from the branch points, the covering map is a local isometry.

Theorem (Theorem 4.2)**.** *Let $\pi : T \to S$ be a branched flat finite-sheeted covering. The volume entropy $e(\tilde{T}, \Gamma_T)$ of $\tilde{T}$ is bounded by the inequality*

$$e(\tilde{T}, \Gamma_T) \leq (a(S) + b(T))(e(\tilde{S}, \Gamma_S) + 1)$$

where $b(T)$ is logarithmic in the combinatorics of the covering and inverse proportional in the distance between the two closest branch points in T.
The same holds for the Hausdorff dimension of the Gromov boundary

Moreover, we construct a family of examples which show that the bounds are asymptotically sharp.
As another topic we investigate the asymptotic behavior of geodesic rays of flat metrics. For each flat surface $S = (X, d_q)$ there is a unique hyperbolic metric σ on the Riemann surface X in the same conformal class as d_q. As the hyperbolic metric is Riemannian, we can define the geodesic flow g_t on the unit tangent bundle of X. g_t acts ergodically with respect to the Lebesgue measure of σ. Let $v \in T^1X$ be a point in the unit tangent bundle. The flow $g : [i, j] \to T^1X, t \mapsto g_t(v)$ in direction of this point defines a geodesic arc in the unit tangent bundle. The arc projects to a geodesic arc $c_{i,j}$ on the surface. We straighten $c_{i,j}$ with respect to the flat metric. In each homotopy class of arcs with fixed endpoints there exists a unique length minimizing geodesic representative for the flat metric. The length of this representative is called the flat length of the homotopy class.

We define the flat length of the homotopy class $[c_{i,j}]$ as a function $F_{i,j} : T^1X \to \mathbb{R}_+$. The family $F_{i,j}$ forms a subadditive process. According to the Theorem of Kingman $T^{-1}F_{0,T}$ converges towards a constant function F a.e.

Theorem (Theorem 5.2)**.** *The volume entropy and the constant F are related.*

$$e(\tilde{S}, \Gamma_S) \geq F^{-1}$$

Furthermore, there is a unique length minimizing geodesic representative for the hyperbolic metric in any free homotopy class of closed curves. Such geodesic representatives also exist for the flat metric. The flat length as well as the hyperbolic length of each free homotopy class is defined as the length of the corresponding geodesic representative.
[Raf07] compared these quantities for each free homotopy class in his work. The hyperbolic metric of the surface admits a thick-thin decomposition. The hyperbolically thin part of the surface is a disjoint union of annuli. Let Y be a component of the thick part. Rafi defined the function $\lambda(Y)$ so that the following holds: Let $[\alpha]$ be a free homotopy class of closed curves which can be realized in Y and which do not contain a multiple of some boundary component of Y. Roughly speaking, the quotient of flat length and hyperbolic length of $[\alpha]$ is comparable to $\lambda(Y)$.

Theorem (Theorem 5.3)**.** *Let $S = (X, d_q)$ be a closed flat surface of genus $g \geq 2$. Let σ be the hyperbolic metric on X which is in same conformal class as the flat metric. Denote by $(X_>, X_<)$ the thick-thin decomposition of (X, σ). Let Y be a connected component of $X_>$ and denote by $\lambda(Y)$ the Rafi constant of Y. There exists a constant $A := A(\chi(X)) > 0$ which only depends on the topology of X such that*

$$F \geq A\lambda(Y)$$

In addition we define a geodesic flow on a flat surface S. Each locally geodesic segment which terminates at a cone point admits a one-parameter family of possible geodesic extensions. Therefore, a definition similar to the one for Riemannian metrics on the unit tangent bundle cannot be given.
That is why we have to make use of the universal cover $\tilde{S}$. We choose a metric on the boundary of $\tilde{S}$. Let $\mathcal{G}\tilde{S}$ be the set of all parametrized bi-infinite geodesics in $\tilde{S}$. The geodesic flow g_t acts as a reparametrization $g_t\alpha(s) := \alpha(t+s)$ on $\mathcal{G}\tilde{S}$. Each parametrized bi-infinite geodesic converges in positive and negative direction towards distinct limit points on the boundary. Therefore, we can project $\mathcal{G}\tilde{S}$ on $\partial\tilde{S} \times \partial\tilde{S} - \triangle$. The group Γ of deck transformations acts equivariantly on both sets.
The geodesic flow g_t acts on the fibers of the projection. We call a point in $\partial\tilde{S} \times \partial\tilde{S} - \triangle$

non-exceptional if any two geodesics with the same image arise from each other via reparametrization. The points in the complement are called exceptional. The set of exceptional points is countable and Γ-invariant. For any non-exceptional point we fix a geodesic in the fiber. Using the g_t action we define a $\mathbb{R}$-parametrization of the fiber in $\mathcal{G}\tilde{S}$ and pull back Lebesgue measure ℓ from the real line. There exists a standard technique for constructing an appropriate Γ-invariant measure $\tilde{\nu}$ on $\partial\tilde{S} \times \partial\tilde{S} - \triangle$. $\tilde{\nu}$ is absolutely continuous with respect to the square of Hausdorff measure on $\partial\tilde{S}$. $\tilde{\nu}$ is an atom-free Radon measure. We define the product measure $\tilde{\mu} = \tilde{\nu} \times \ell$ on $G\tilde{S}$ which is Γ- and g_t-invariant. The fibers of the exceptional points form a measure 0 set. $\tilde{\mu}$ descends to a positive finite quotient measure μ on the quotient space $\mathcal{G}\tilde{S}/\Gamma$.
Since the action of the geodesic flow g_t on $\mathcal{G}\tilde{S}$ commutes with the Γ-action on $\mathcal{G}\tilde{S}$, the geodesic flow is properly defined as a μ- invariant action on $\mathcal{G}\tilde{S}/\Gamma$. We show that g_t acts ergodically with respect to μ.
Finally, we investigate typical behavior. Let c be a locally geodesic arc S. We extend c as much as possible in positive and negative direction with the property that the extension is unique. Let c_{ext} be the extended arc which might be infinite. We estimate the frequency F of a μ-typical geodesic passing through c.

Theorem (Theorem 5.4)**.** *There is a constant $C(S)$ which depends on the geometry of S but not on c such that the following holds:*
A typical geodesic passes through c with a frequency F which is bounded from above and below by

$$C(S)^{-1}exp(-e(\tilde{S},\Gamma_S)l(c_{ext})) \leq F \leq C(S)exp(-e(\tilde{S},\Gamma_S)l(c_{ext}))$$

Finally we deal with a different object on a flat surface, the group of orientation preserving affine diffeomorphisms. Away from the singularities, each diffeomorphism descends to a differentiable mapping $U \subset \mathbb{R}^2 \to \mathbb{R}^2$ with constant derivative which we interpret as a matrix $A \in GL_+(2,\mathbb{R})$. As the transition functions are half-translations, A is independent of the choice of charts up to multiplication with $\pm id$. Therefore, there is a well-defined map of each affine diffeomorphism to its projectivized differential in $PGL_+(2,\mathbb{R}) = PSL(2,\mathbb{R})$. The image of the group of affine diffeomorphisms is the so-called Veech group which is a non-cocompact fuchsian group. In case the Veech group is a lattice, it exhibits dynamical properties on the underlying flat surface, see [MT02] [HS06] for instance. For a typical flat surface the Veech group is trivial. However, there are some well-studied examples of flat surfaces whose Veech groups are arithmetic lattices or triangle groups, see [HS06]. It is still an open question which fuchsian groups may appear as Veech groups of flat surfaces. For instance it is unknown whether there exists an infinite cyclic Veech group consisting of hyperbolic elements. There is a standard technique of finding Veech groups. For a flat surface S with large Veech group and a finite set of points there exists a finite sheeted covering branching over his set. The affine group of the covering surface is commensurable to the group of those affine diffeomorphisms on S whose periodic points contain the chosen set. Therefore, understanding the periodic points is one way to investigate the behavior of the affine diffeomorphism and might emerge as a tool of finding new Veech groups.

We investigate one of the most prominent examples of flat surfaces with non-trivial Veech group, the family of Arnoux Yoccoz surfaces in all genera with a distinguished affine diffeomorphism Φ. The Arnoux-Yoccoz diffeomorphism Φ is the only explicit example of affine pseudo-Anosov diffeomorphisms where it is known that the whole Veech group does not contain parabolic elements. The flat surface arises from a distinguished polygon $F \subset \mathbb{R}^2$ with an appropriate identification of sides. F turns out to be a so-called Markov partition for Φ. Hence we consider Φ as a mapping in coordinates of $F \subset \mathbb{R}^2$. The expansion factor α of Φ is a pisot number i.e. an algebraic number with all complex conjugates having absolute value less than 1.

We investigate periodic points using symbolic dynamics. Real numbers can be coded in terms of so-called α-expansions, a technique similar to continued fraction expansions. We code the vertical coordinate in the standard α-expansion and the horizontal one in a slight variation called a generalized α-expansion. This expansion gives rise to a mapping from $F \to \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{Z}}$. The isomorphism is the concatenation.

Φ commutes with the right-shift on the bi-infinite words. Therefore, the canonical candidates for periodic points under Φ are the preimages of purely periodic sequences. The standard α-expansion is well understood. It is injective. After choosing an appropriate word metric, the coding map is Lipschitz. The image domain also is well-known and there exist various results concerning periodic sequences.

Unfortunately these properties do not hold for the generalized α-expansion of the horizontal coordinate. Therefore, we construct a finer coding, the Markov-expansion. The Markov-expansion is again injective and Lipschitz. Since it is a refinement, there is a canonical projection from the Markov expansion to the generalized α-expansion. However, after coding the vertical coordinate in terms of α-expansion and the horizontal one in terms of Markov-expansion, it is difficult to determine the action of Φ. Investigation of the interplay between both expansions of the horizontal coordinate leads to an explicit description of the periodic points for the Arnoux Yoccoz-diffeomorphism.

There are already well-known descriptions of periodic points and their distributions for general pseudo-Anosov mappings, see [Bow71].

Our methods allow an explicit computation of periodic points. Furthermore, using the pisot properties of α, we can show that the coordinates of periodic points meet number theoretical conditions.

Theorem (Corollary 7.1)**.** *For all but a finite set of rational points y there is a periodic point in F with vertical coordinate y. On the other hand, there is no such periodic point for y an algebraic integer.*

The number theoretical properties might turn out to be useful for determining certain subgroups of the Veech groups when using branched covering constructions. For example [HLM09] showed that in the case of genus $g = 3$ the Veech group is not virtually cyclic. In their work they explicitly found a second pseudo-Anosov element Ψ.

Theorem (Proposition 7.3)**.** *There exist points which are periodic for Φ but not periodic for the conjugate of Φ with Ψ.*

Thus we construct Veech groups which still contain the original pseudo-Anosov up to finite index but still have infinite index in the original Veech group.
The thesis is organized as follows. Section 2 provides background material for spaces of non-positive curvature, i.e. for $Cat(0)$ and Gromov δ-hyperbolic spaces. Standard results and tools needed in the later context are mentioned. The visual boundary with the Gromov metric is defined. We relate the Hausdorff dimension of this metric and its Hausdorff measure to the volume entropy, using techniques from Patterson-Sullivan theory. Readers who are familiar with these concepts can skip this section. Section 3 deals with the geometry of flat metrics. The basic facts are introduced. We mainly work in the universal cover endowed with the lifted metric and we show that it is a metric space of non-positive curvature. Except for section 3.4 the results are well-known.
In section 4 we study how the variation of a flat metric influences volume entropy and Hausdorff dimension in $\mathcal{Q}_g$. Furthermore, we investigate how the quantities behave under branched coverings. Section 5 deals with asymptotic behavior of geodesics on flat surfaces. The asymptotic quotient of flat length and hyperbolic length of geodesic arcs is investigated. Moreover, we construct the geodesic flow on flat surfaces. In the next section we estimate the quantities for a family of examples which is a special kind of so-called square tiled surfaces.
In Section 7 we introduce concepts from symbolic dynamics and compute periodic points of the Arnoux-Yoccoz diffeomorphism.

2 General constructions for spaces of non-positive curvature

2.1 Metric spaces and geodesics

Let (X, d) be a metric space. X is proper if and only if closed metric balls of finite radius are compact. The distance between any two sets $U, V \subset X$ is defined as

$$d(U, V) := \inf_{u \in U, v \in V} d(u, v)$$

Let $f : Y \to X$ be some map and $U \subset X$ some set.
We define the distance

$$d(f, U) := d(im(f), U)$$

A geodesic joining the points x, y in X is a mapping $[x, y] : I = [a, b] \subset \mathbb{R} \to X$ such that $[x, y](a) = x, [x, y](b) = y$ and $d([x, y](s), [x, y](t)) = |s - t|$ for all $s, t \in I$.
A geodesic ray is a map $c : I = [a, \infty) \to X$ so that $d(c(s), c(t)) = |s - t|$ for all $s, t \in I$.
A geodesic line is a map $c : \mathbb{R} \to X$ such that $d(c(t), c(s)) = |t - s|$.
The space X is called geodesic if between any two points $x, y \in X$ there exists a connecting geodesic $[x, y]$. Let $c : I \to X, c' : I' \to X$ be geodesics. c' is called a reparametrization of c if there exists an increasing bijective function $r : I \to I'$ so that $c'(t) = c(r(t))$ for all $t \in A$.
Let X be a geodesic metric space. X is uniquely geodesic if for any two geodesics c, c' with the same endpoints, c is a reparametrization of c'.
A subset S of a geodesic metric space X is convex if any geodesic connecting two points in S is entirely contained in S.
Convention: All considered metric spaces (X, d) are proper, geodesic and complete.

2.2 Cat(0)-structure

Let (X, d) be a metric space. We introduce the notion of comparison triangles, compare [BH99, I Definition 1.10].
For three distinct points $x, y, z \in X$ a geodesic triangle $\triangle(x, y, z)$ is a choice of three geodesics $[x, y], [y, z], [z, x]$. x, y, z are the vertices of $\triangle(x, y, z)$. The vertices do not entirely determine the triangle. However, various properties of $\triangle(x, y, z)$ depend on the vertices but not on the choice of the connecting geodesics.
Let $\triangle(x, y, z) \subset X$ be a geodesic triangle. A triangle $\triangle_c(x_c, y_c, z_c), x_c, y_c, z_c \in \mathbb{R}^2$ in the euclidean plane is called a comparison triangle if $d(x_c, y_c) = d(x, y), d(y_c, z_c) = d(y, z), d(z_c, x_c) = d(z, x)$. For any triangle in X, the triangle inequality ensures the existence of a comparison triangle. Up to isometry, $\triangle_c(x_c, y_c, z_c)$ is uniquely defined by the distance between the vertices.

Definition 2.1. *A space* (X, d) *is a* $Cat(0)$*-space if any triangle* $\triangle(x, y, z)$ *is thinner than the comparison triangle* $\triangle_c(x_c, y_c, z_c)$*. To be precise, the following inequality is satisfied.*

$$d([x, y](s), [y, z](t)) \leq d([x_c, y_c](s), [y_c, z_c](t)), \forall s, t$$

Alexandrov introduced the following concept in geodesic metric spaces to measure angles for geodesics issuing from a common point.
Let $c, c' : [0, T] \to X$ be geodesics with $c(0) = c'(0)$. For $t, t' \in (0, T]$ consider the triangle $\triangle(c(0), c(t), c'(t'))$. In the comparison triangle one can compute the euclidean angle at $c(0)$ which we abbreviate $\angle_c(t, t')$. The limit

$$\angle_A(c, c') := \limsup_{t,t' \to 0} \angle_c(t, t')$$

is the so-called Alexandrov angle.
A $Cat(0)$-space has many of the properties which are well-known in euclidean space. As the results are classical, we refer to [BH99, I Proposition 1.4, I Proposition 2.4, II Proposition 8.2]:

Proposition 2.1. *Let* X *be a* $Cat(0)$*-space.*

- *X is uniquely geodesic.*
- *Due to the properness of X, $[x, y] : [0, T] \to X$ continuously depends on x, y in the following manner: Let x_i resp. y_i be a sequence in X which converges towards x resp. y. $[x_i, y_i][0, T_i] \to X$ uniformly converges to $[x, y]$.*
- *A closed metric ball is a convex set.*
- *For every closed convex set $S \subset X$ and for every x, there is a unique point $s_x \in S$ such that $d(s_x, x) = d(x, S)$. The mapping $\pi_S : x \mapsto s_x$ has the following properties:*
  - *The map π_S does not increase distances.*
  - *Let y be a point and $x \in [y, \pi_S(y)]$ some point on the geodesic connecting y with $\pi_S(y)$. It follows that $\pi_S(x) = \pi_S(y)$*
  - *Let x be a point and denote $d := d(x, \pi_S(x))$. Let $s \in S$ be some point and let $c := d(x, s) - d(x, \pi_S(x)) \geq 0$. It follows that*

    $$d(s, \pi_S(x)) \leq 2\sqrt{2dc + c^2}$$

  - *Denote by $\angle_A$ the Alexandrov angle. The angle at the point of projection satisfies.*

    $$\angle_A([x, \pi_S(x)], [s, \pi_S(x)]) \geq \pi/2, \forall x \notin S, s \in S - \pi_S(x)$$

- *Let $r : [0, \infty) \to X$ be a geodesic ray and $x \in X$ be a point. There exists a unique geodesic ray r' so that $r'(0) = x$ and the distance of r' to r is bounded. More precisely, there is a constant C so that for all t, $d(r(t), r'(t)) < C$. The uniqueness implies that two different geodesic rays, having one point in common, drift apart.*

2.2.1 Euclidean polyhedral complexes

One family of $Cat(0)$-spaces are uniquely geodesic euclidean polyhedral complexes.
A euclidean polyhedral cell $C \subset \mathbb{R}^n$ is the convex hull of a finite number of points $\{p_1 \dots p_k\}$. The dimension of C is the dimension of the smallest m-plane containing C. The interior of C is the interior of C as a subset of this plane.
Let H be a hyperplane in $\mathbb{R}^n$ so that the intersection $F := C \cap H \neq \emptyset$ is not empty. If C lies in a closed half-space, bounded by H, then F is called a face of C. F is a proper face if $F \neq C$. The dimension of F is the dimension of the smallest m'-plane containing F. The interior of F is the interior of F in this plane.
Let $x \in C$ be a point. The support $supp(x)$ is the unique face containing x in its interior.
A shape is an isometry equivalence class of faces. We define a polyhedral complex and follow [BH99, I Definition 7.37].

Definition 2.2. *Let $C_i, i \in I$ be a family of euclidean cells which correspond to a finite number of shapes. Let $X := \bigcup (C_i, i), i \in I$ be the disjoint union of cells. Let $K := X/\sim$ with respect to some equivalence relation. For each i, denote by p_i the canonical mapping $p_i : C_i \hookrightarrow X \to K$. K is called a euclidean polyhedral complex with a finite number of shapes if and only if the following conditions are satisfied:*

- *The map p_i, restricted to the interior of a face, is injective for each i.*

- *Assume that $p_i(x) = p_j(x')$ for some i, j. There exists an isometry $h : supp(x) \to supp(x')$ such that for each $y \in supp(x), y' \in supp(x')$ it follows that*

$$h(y) = y' \Leftrightarrow p_i(y) = p_j(y')$$

A euclidean polyhedral complex can be naturally endowed with the euclidean metric in the interior of each cell which can be extended to a metric on K.

Proposition 2.2. *A euclidean polyhedral complex K with a finite number shapes is a $Cat(0)$-space if and only if K is uniquely geodesic.*

Proof. [BH99, II Theorem 5.4] ☐

2.2.2 Boundary

Definition and topology Let (X,d) be a $Cat(0)$-space. We define the boundary ∂X as equivalence classes of geodesic rays, together with an appropriate topology.
For details we refer to [BH99, II Section 8]. Denote

$$\partial X := \{r : [0,\infty) \to X, r \text{ geodesic ray}\}/\sim$$

Here $r_1 \sim r_2 \Leftrightarrow \exists C : d(r_1(t), r_2) < C, \forall t$. Let x be a point and r a geodesic ray i.e a boundary point. Since X is proper, due to Proposition 2.1, there exists a unique $r' \sim r$ such that $r'(0) = x$. For $\eta \in \partial X$ and $x \in X$, we define $[x,\eta]$ as the ray in the equivalence class η issuing from x.
Let $\overline{X} := X \cup \partial X$. We endow $\overline{X}$ with the following topology: For any set $U \subset X$ and for each point $x \in X$ let $sh_x(U) \subset \overline{X}$, the U-shadow, be the set of points on the geodesic rays r, issuing from x, which intersect U first:

$$sh_x(U) := \{r(t) \cup r, r \in \partial X : r(0) = x, \exists 0 \leq t_0 \leq t : r(t_0) \in U\}$$

We define the basis of topology on $\overline{X}$ as all finite diameter open balls in X together with all shadows of open sets $U \subset X$.
With respect to this topology, $\overline{X}$ is compact and ∂X is a closed subset of $\overline{X}$.
We will often work on the boundary. Therefore, we define the boundary shadow:

$$\partial sh_x(U) := \partial X \cap sh_x(U)$$

The set of all boundary shadows of open sets U forms a basis of the topology on ∂X.
To compute neighborhoods on the boundary, the following Proposition is helpful.

Proposition 2.3. *Let X be a $Cat(0)$-space and $B := B_x(r)$ be a closed ball in X. B is a closed convex set and therefore, there exists the natural projection $\pi_B : X \to B$ which is the closest point projection on X.*
Let $\eta \in \partial X$ be a boundary point and let $[x,\eta]$ be the connecting geodesic.
The projection $\pi_B([x,\eta](t)) = [x,\eta](r)$ is constant for $t > r$.
We define the extended projection $\overline{\pi}_B : \overline{X} \to B$:

$$\overline{\pi}_B(y) := \begin{cases} \lim_{t\to\infty} \pi_B([x,y](t)) & y \in \partial X \\ \pi_B(y) & y \in X \end{cases}$$

$\overline{\pi}_B$ is a continuous map.

Proof. [BH99, II Proposition 8.8] □

2.3 Gromov hyperbolic spaces

$Cat(0)$-spaces are, locally and globally, non-positively curved. Moreover, there is the notion of coarsely negative curvature, the so-called Gromov δ-hyperbolicity. We refer to [BH99, III Definition 1.1].

Definition 2.3. *Let (X, d) be a metric space. A geodesic triangle $\triangle(x, y, z) \subset X$ is called δ-slim if each point $p \in [x, y]$ has distance at most δ to the set $[y, z] \cup [z, x]$. A metric space (X, d) is Gromov δ-hyperbolic if and only if each geodesic triangle is δ-slim.*

Let X, Y be metric spaces. A mapping $f : X \to Y$ is a (K, L)-quasi-isometric embedding for some $K \geq 0, L \geq 1$ if

$$L^{-1}d(f(x_1), f(x_2)) - K \leq d(x_1, x_2) \leq Ld(f(x_1), f(x_2)) + K, \forall x_i \in X$$

A quasi-isometric embedding is a quasi-isometry if there exists some constant C such that $d(y, f(X)) \leq C, \forall y \in Y$.
Let $f : X \to Y$ be a (K, L)-quasi-isometry and Y be a δ-hyperbolic space. X is δ'-hyperbolic for some constant δ' which only depends on K, L and δ.
If K tends to 0 and L tends to 1, δ' tends to δ.
In most cases the choice of K and L is not important, so we will skip them and denote f as a quasi-isometry.
Let $f : X \to Y$ and $g : Y \to Z$ be quasi-isometries. The concatenation $g \circ f : X \to Z$ is a quasi-isometry as well.
Additionally in [BH99, I Section 8] the following Lemma is shown:

Lemma 2.1. *Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be a (K, L)-quasi-isometry. There exists a (K', L')-quasi-isometry $g : Y \to X$ and a constant C such that*

$$d_X(x, g \circ f(x)) \leq C, d_Y(y, f \circ g(y)) \leq C, \forall x \in X, y \in Y$$

Therefore, the property to be quasi-isometric is an equivalence relation on the set of all metric spaces.
A (K, L)-quasi-isometric embedding $c : I \to X$ of a compact segment $I \subset \mathbb{R}$ is called a (K, L)-quasi-geodesic. If $I = [a, \infty)$, c is a (K, L)-quasi-geodesic ray and if $I = \mathbb{R}$, c is a quasi-geodesic line.
It is shown in [BH99, III Lemma 1.11] that quasi-geodesics with the same endpoints are uniformly close.

Lemma 2.2. *For any K, L, δ there is a constant $\lambda(K, L, \delta)$ such that any two (K, L)-quasi-geodesics in a δ-hyperbolic space, connecting the same endpoints, have Hausdorff-distance at most λ. Assume that δ is fixed. If K tends to 0 and L tends to 1, λ tends to 0.*

One of the main tools in Gromov hyperbolic spaces is the Gromov product.
Let X be a metric space. For a fixed base point $p \in X$ one defines:

$$(x \cdot y)_p := \frac{1}{2}(d(x,p) + d(y,p) - d(x,y))$$

Proposition 2.4. *Let X be a δ-hyperbolic metric space. The Gromov product has the following properties:*

i) The Gromov product is continuous in each factor.

ii) For x', y' let $x \in [p, x'], y \in [p, y']$. Then $(x' \cdot y')_p \geq (x \cdot y)_p$

iii) For $x \in [p, y]$ it follows that $(x \cdot y)_p = d(x, p)$

iv) Let $c : [0, t] \to X$ be a geodesic. It follows

$$d(p, c) - 4\delta \leq (c(0) \cdot c(t))_p \leq d(p, c)$$

v) $(x \cdot y)_p \geq \min\{(x \cdot z)_p, (y \cdot z)_p\} - \delta, \forall x, y, z$

Proof. The statements $i) - iii)$ follow from the triangle inequality.
$iv)$ is shown in [CP93, I Proposition 1.5] and $v)$ is proved in [BH99, III Remark 1.21]. ☐

2.3.1 Boundary metric

On a Gromov hyperbolic space X one can define a boundary, together with a topology, similarly to the definition of boundary of $Cat(0)$-spaces, compare [BH99, III Section 3]. In case X is a δ-hyperbolic, $Cat(0)$-space both definitions of boundary are equivalent, [BH99, III Proposition 3.7].
For simplicity we assume that X is a δ-hyperbolic $Cat(0)$-space.
We will use the fact that X is a δ-hyperbolic space to construct a family of metrics on the boundary. The topology, induced by any of the metrics, equals the original topology. Denote by $\overline{X}$ the extension of X as described in section 2.2.2. Various properties of the space X also hold in the extended space.

Proposition 2.5. *Let X be a δ-hyperbolic $Cat(0)$-space.*
Recall that the boundary at infinity is defined as equivalence classes of geodesic rays. For any two points $\eta \neq \zeta \in \partial X$ there exists a geodesic c such that the rays $r_+ := c(t)|_{[0,\infty)}, r_- := c(-t)|_{(-\infty,0]}$ satisfy $r_+ \in \eta, r_- \in \zeta$
Thus we have a notion of geodesics in $\overline{X}$. Any triangle, with vertices in $\overline{X}$, is 4δ slim.

Proof. The proof of the first part, which mainly uses an Arcelà Ascoli argument, can be found in [BH99, III Lemma 3.2].
The second part is straight forward. We refer to [CP93, I Proposition 3.2]. □

The notion of Gromov product extends to the space $\overline{X}$.

Definition 2.4. *Let X be a δ-hyperbolic $Cat(0)$-space. For any points $\eta, \zeta \in \partial X$ let $s_i, t_i \in \mathbb{R}$ be sequences tending to infinity.*
The Gromov product on the boundary is defined as

$$(\eta \cdot \zeta)_p := \lim_i ([p,\eta](s_i) \cdot [p,\zeta](t_i))_p$$

The existence of the limit and the independence of the sequences s_i, t_i follows from Proposition 2.4.

Proposition 2.6. *Let X be a δ-hyperbolic, $Cat(0)$-space.*

i) Let $\eta, \zeta \in \partial X$. For all sequences of points $x_i \in X$ resp. $y_i \in X$, which converge towards η resp. ζ, it follows:

$$(\eta \cdot \zeta)_p - 2\delta \leq \lim_i \inf (x_i \cdot y_i)_p \leq (\eta \cdot \zeta)_p$$

ii) Let Y be a proper geodesic $Cat(0)$-space. Assume that there exists a quasi-isometry $f : X \to Y$.
f extends to a homeomorphism $f_* : \partial X \to \partial Y$.

iii) The Gromov product is a priori not continuous on the boundary.

Proof. $iii)$ and the left inequality of $i)$ are due to [BH99, III Remark 3.17].
$ii)$ follows from [BH99, III Theorem 3.9].
It remains to show the right inequality of $i)$. Let $x_i \in X$ resp. $y_i \in X$ be a sequence of points which tends towards $\eta \in \partial X$ resp. $\zeta \in \partial X$. Let $[p, x_i]$ resp. $[p, y_i]$ be the connecting geodesic. Due to the definition of the shadow and the uniqueness of geodesics for all $\epsilon > 0$ and all t, there exists a i_0 such that for all $i \geq i_0$

$$d([p, x_i](t), [p,\eta](t)) \leq \epsilon$$

$$d([p, y_i](t), [p, \zeta](t)) \leq \epsilon$$

Therefore

$$([p, x_i](t) \cdot [p, y_i](t))_p \geq ([p, \eta](t) \cdot [p, \zeta](t))_p - 2\epsilon$$

It follows that

$$(x_i \cdot y_i)_p \geq (\eta \cdot \zeta)_p - 2\epsilon$$

and consequently

$$\liminf (x_i \cdot y_i)_p \geq (\eta \cdot \zeta)_p$$

□

For any Gromov hyperbolic space the Gromov product on the boundary is the main tool for defining an appropriate boundary metric which is compatible with the topology of the boundary.

Let X be a δ-hyperbolic space. X is also $(\delta + \epsilon)$-hyperbolic for any $\epsilon > 0$. Denote

$$\delta_{inf}(X) := \inf_{\delta'} X \text{ is } \delta'\text{-hyperbolic}$$

We define the function

$$\xi : \mathbb{R}_+ \to \mathbb{R}_+, \xi(\delta) := 2^{\frac{1}{2\delta}}$$

The following Proposition is crucial.

Proposition 2.7. *Let X be a δ-hyperbolic space. For any point $x \in X$ and for each $c \leq \xi(\delta)$ there is a metric $d_{c,x}$ on ∂X and a constant $\epsilon(c) < 1$ which satisfies:*

$$c^{-(\eta \cdot \zeta)_x} \geq d_{c,x}(\eta, \zeta) \geq (1 - \epsilon(c))(c^{-(\eta \cdot \zeta)_x})$$

The metric $d_{c,x}$ is called a Gromov metric and $(\partial X, d_{c,x})$ the Gromov boundary of X.

Proof. [BH99, III Proposition 3.21] □

Definition 2.5. *Let X be a δ-hyperbolic space. Let $x \in X$ be a point and $\delta_{inf}(X)$ be defined as above. Denote*

$$c := \frac{1}{2}\xi(\delta_{inf}(X))$$

By Proposition 2.7 the boundary metric

$$d_{\infty,x} := d_{c,x}$$

exists.

It is a consequence of the triangle inequality that any two metrics $d_{c,x}, d_{c,x'}$, with respect to the same constant c but different base points x, x', are bilipschitz.

In various situations the choice of metrics, up to bilipschitz equivalence, does not affect the result. Therefore, we will often skip the index x and abbreviate $d_c := d_{c,x}$.

2.4 Patterson-Sullivan theory and volume entropy

As we have a family of metrics $d_{c,x}$ on the boundary, we can define Hausdorff measure and Hausdorff dimension on the boundary. Let $x \neq x'$ be points in X. The Gromov metrics $d_{c,x}$ and $d_{c,x'}$ are bilipschitz equivalent. Therefore, the Hausdorff dimensions of the boundary, with respect to the metrics $d_{c,x}$ and $d_{c,x'}$, are equal.

We connect these quantities with the theory of Patterson-Sullivan measures. Patterson-Sullivan measures have been constructed by [Pat76] for fuchsian groups and generalized by Sullivan [Sul79] for groups of isometries acting properly discontinuously and freely on a finite-dimensional hyperbolic space. Sullivan's work led to the generalization by Coornaert [Coo93] for groups of isometries Γ acting properly discontinuously and freely on a δ-hyperbolic space X which is complete, proper and geodesic.

We will restrict ourselves to the case that the action of Γ is cocompact.

Let $x_0 \in X$ be a point. Denote by

$$N_{x_0}(R) := |x_0\Gamma \cap \overline{B_{x_0}(R)}|$$

the number of orbit points in the closed metric ball $\overline{B_{x_0}(R)}$ of radius R about x_0. The volume entropy of Γ is defined as

$$e(X,\Gamma) := \limsup_{R\to\infty} \frac{\log(N_{x_0}(R))}{R}$$

Convention: By entropy we always mean the volume entropy.

Due to the triangle inequality, $e(X,\Gamma)$ is independent of x_0. We will only make use of the counting function to compute the entropy. Therefore, we skip the base point and abbreviate $N(R)$. Assume that $e(X,\Gamma)$ is positive and finite. Since the quotient is compact, one observes the following connection between volume growth and orbit growth:

Lemma 2.3. *Let Γ be a group of isometries acting properly discontinuously cocompactly and freely on a metric space X so that the entropy $e(X,\Gamma)$ is finite. Let ℓ be a Γ-invariant non-zero Radon measure on X and let $x_0 \in X$ be a point. There exists some $C > 0$ such that*

$$C^{-1}\ell(B_{x_0}(R)) - C \leq N_{x_0}(R) \leq C\ell(B_{x_0}(R)) + C$$

Proof. Since X is proper and since the group Γ acts discretely and freely, $N_{x_0}(R)$ is finite for each R.

X/Γ is compact, hence for any point $y, y' \in X$ the distance $d(y, \Gamma y')$ has a universal upper bound $\overline{D}$ independent of y, y'.

Choose some $y_0 \in supp(\ell)$. Since the support is Γ-invariant, we can assume that $d(x_0, y_0) \leq \overline{D}$.
Each point in $B_{x_0}(R)$ has distance at most $\overline{D}$ to an orbit point $y \in \Gamma y_0$. So $B_{x_0}(R)$ can be covered by $N_{x_0}(R + 2\overline{D})$ balls of radius $\overline{D}$. The centers of these balls are orbit points of y_0. Since the group Γ acts by isometries, all the balls are translates of the ball $B_{y_0}(\overline{D})$. So the balls have the same measure $C := \ell(B_{y_0}(\overline{D}) > 0$ with respect to ℓ.

$$\ell(B_{x_0}(R)) \leq C N_{x_0}(R + 2\overline{D})$$

On the other hand, Γ acts discretely and freely. Therefore, there is some radius $\underline{D} > 0$ so that the projection $B_{y_0}(\underline{D}) \to X/\Gamma$ is an embedding.
Around each orbit point $y \in y_0\Gamma \cap B_{x_0}(R - \overline{D} - \underline{D})$ we can embed a disc of radius $\underline{D}$. By definition, all such balls are disjoint and contained in $B_{x_0}(R)$. They are translates of the ball $B_{y_0}(\underline{D})$ under Γ. Therefore, we can estimate the measure

$$\ell(B_{x_0}(R)) \geq \ell(B_{y_0}(\underline{D})) N_{x_0}(R - \overline{D} - \underline{D})$$

Since y_0 is in the support of ℓ, the set $B_{y_0}(\underline{D})$ has positive measure. We can enlarge C so that $\ell(B_{y_0}(\underline{D})) \geq C^{-1}$.
We showed the that there exists some constant C such that

$$C^{-1}\ell(B_{x_0}(R - 2\overline{D})) \leq N_{x_0}(R) \leq C\ell(B_{x_0}(R + \overline{D} + \underline{D}))$$

Let $r > 0$ be some constant. It remains to show that there exists some constant $C(r) > 0$ such that the quantities $N_{x_0}(R)$ and $N_{x_0}(R + r)$ at most differ by $C(r)$, independently of R.

$$N_{x_0}(R + r) \leq C(r)\ell(N_{x_0}(R)) + C(r)$$

We choose $C(r) := N_{x_0}(2r + 2\overline{D})$. The claim holds for $R < r + 2\overline{D}$.
Assume that $R \geq r + 2\overline{D}$.
Let x be a point in Γx_0 with $R < d(x, x_0) < R + r$. Let y be the point on the geodesic $[x, x_0]$ of distance $r + \overline{D}$ to x. There exists a point $x' \in \Gamma x_0$ of distance at most $\overline{D}$ to y. Therefore

$$d(x', x) \leq d(x', y) + d(y, x) \leq R$$

The distance between x_0 and x' can be estimated by

$$d(x_0, x') \leq r + 2\overline{D}$$

It follows that for each point x in Γx_0 of distance at most $R + r$ to x_0, there exists some point $x' \in \Gamma x_0, \cap B_{x_0}(r + 2\overline{D})$ of distance at most R to x. The number of such points x'

is at most $N_{x_0}(r+2\overline{D})$. Since $N_{x_0}(R) = N_{x'}(R)$, it follows that

$$N_{x_0}(R+r) \leq N_{x_0}(r+2\overline{D})N_{x_0}(R) \leq C(r)N_{x_0}(R)$$

□

We recall the construction of Patterson-Sullivan measures. Rigorous computations can be found in [Sul79, Section 1-3], [Coo93, Section 4-8].
We define the Poincaré series

$$g_s(z_0) := \sum_{z \in \Gamma z_0} \exp(-sd(z_0, z))$$

Proposition 2.8. *Let X be a δ-hyperbolic space. Let Γ be a discrete group of isometries acting freely on X and $x_0 \in X$ be some point.*
For $s > e(X,\Gamma)$, $g_s(x_0)$ is finite, whereas for $s < e(X,\Gamma)$, $g_s(z_0)$ diverges. If Γ acts cocompactly, $g_s(z_0)$ also diverges for $s = e(X,\Gamma)$.

Proof. [Coo93, Proposition 5.3, Corollary 7.3] □

For $s > e(X,\Gamma)$ one defines the following Radon measure $\nu_{x,s}$ on $\overline{X}$

$$\nu_{s,x} := \frac{1}{g_s(z_0)} \sum_{z \in \Gamma z_0} \exp(-sd(x,z))\delta_z$$

where δ_z is the Dirac measure. For any $x \in \Gamma z_0$, $\nu_{x,s}$ is a probability measure. One shows that for $s_i \searrow e(X,\Gamma)$, $\nu_{s_i,x}$ converges towards a Radon measure ν_x which is again finite. As Γ acts cocompactly, g_s diverges. Therefore, ν_x is supported on the boundary. For more details compare [Coo93, Theorem 5.4]
As Γ is a group of isometries, $\nu_{s,x}$ satisfies the following invariance:

$$\gamma * \nu_{s,\gamma(x)} = \nu_{s,x}$$

The limit measure ν_x meets the same invariance.

$$\gamma * \nu_{\gamma(x)} = \nu_x$$

Theorem 2.1. *Let X be a δ-hyperbolic $Cat(0)$-space and let Γ be a discrete group of isometries acting cocompactly on X. Let $d_{c,x}$ be the Gromov metric on the boundary with respect to some base c and some base point $x \in X$.*
The Hausdorff dimension of the boundary coincides with $\frac{e(X,\Gamma)}{log(c)}$.
Furthermore, the Hausdorff measure exists and is absolutely continuous with respect to ν_x. One can use the ergodicity of the action of Γ with respect to the measure class of ν_x to show that both measures have to coincide up to a multiplicative constant.

Proof. We refer to [Coo93, Theorem 7.7]. □

Recall that in Definition 2.5 we chose the base c for the Gromov metric $d_{c,x}$ as $c := \frac{1}{2}\xi(\delta_{inf}(X))$ and called the resulting Gromov metric $d_{\infty,x}$.

Remark 2.1. *With respect to the normalized Gromov metric $d_{\infty,x}$, the Hausdorff dimension on the boundary remains unchanged under scaling the metric on X.*

The measure ν_x is not complete. For simplicity we can use a standard construction, see i.e. [Rud87, I Theorem 1.36] to extend ν_x to a complete measure. From now on we will, without stating explicitly, assume that we always take the completion of any measure instead of the measure itself.

Radon-Nikodym derivative For distinct base points x, y, the resulting measures ν_x, ν_y are absolutely continuous with respect to each other. In the later context we need to estimate the difference of the two measures.
The diameter of X/Γ is defined as

$$diam := \sup_{x,y \in X} d(x, \Gamma y)$$

Since Γ acts cocompactly, $diam$ is bounded.

Lemma 2.4. *Let ν_x, ν_y be Patterson-Sullivan measures. The difference between ν_x and ν_y is bounded by the following inequalities. Let A be a measurable set.*

$$\exp(-e(X,\Gamma)d(x,y))\nu_y(A) \le \nu_x(A) \le \exp(e(X,\Gamma)d(x,y))\nu_y(A)$$

Furthermore, the measure of the whole space is bounded by the following inequality.

$$\exp(-e(X,\Gamma)diam) \le \nu_x(\partial\overline{X}) \le \exp(e(X,\Gamma)diam)$$

Proof. The first inequality follows from the triangle inequality.
The second is a consequence of the fact that $\nu_{\gamma(x)}(\gamma(A)) = \nu_x(A)$ and there exists some x_0 so that $\nu_{x_0}(\overline{X}) = 1$. □

For more accurate estimations we have to compute the Radon-Nikodym derivative. Let X be a $Cat(0)$- space. By [BH99][II Lemma 8.18] the limit

$$b(x, y, \eta) := \lim_{t \to \infty} t - d([x, \eta](t), y)$$

exists. $b(x, y, \eta)$ is the horospherical distance or Busemann distance with respect to η.

Lemma 2.5. *Let $x_0, x_1 \in X$ be points in a δ-hyperbolic $Cat(0)$-space X. Let $\eta \in \partial X$ be a boundary point. There exists a constant $C(\delta)$, which only depends on δ, and a neighborhood $U \subset \overline{X}$ of η so that for any $y \in U \cap X$ it follows*

$$|d(x_0, y) - d(x_1, y) - b(x_0, x_1, \eta)| < C(\delta)$$

Assume that x_1 is contained in the geodesic ray $[x_0, \eta]$. For all $\epsilon > 0$ there exists a neighborhood $U_\epsilon \subset \overline{X}$ of η such that for all $y \in U_\epsilon$ it follows:

$$|d(x_0, y) - d(x_1, y) - b(x_0, x_1, \eta)| < \epsilon$$

Proof. The first claim is shown in [Coo93, Lemma 2.2]. It remains to show the second claim. The set

$$U_{2\epsilon} := sh_{x_0}(B_{x_1}(\epsilon))$$

is an open neighborhood for η. Let y be a point in U_ϵ. The geodesic $[x_0, y]$ has distance at most ϵ to x_1. Therefore

$$|d(x_0, y) - d(x_1, y) - d(x_0, x_1)| \leq 2\epsilon$$

Since in this special case the Busemann distance satisfies

$$d(x_0, x_1) = b(x_0, x_1, \eta)$$

the claim is proved. □

Corollary 2.1. *The Radon-Nikodym derivative can be estimated by*

$$\exp(-e(X,\Gamma)(b(x_0, x_1, \eta) + C(\delta))) \leq \frac{d\nu_{x_0}}{d\nu_{x_1}}(\eta) \leq \exp(-e(X,\Gamma)(b(x_0, x_1, \eta) - C(\delta)))$$

If $x_1 \in [x_0, \eta]$ it follows that:

$$\frac{d\nu_{x_0}}{d\nu_{x_1}}(\eta) = \exp(-e(X,\Gamma)b(x_0, x_1, \eta)) = \exp(-e(X,\Gamma)d(x_0, x_1))$$

Remark 2.2. *If X is a $Cat(\kappa)$-space, $\kappa < 0$, it is well-known, see [BM96, section 1.1] for instance, that*

$$\frac{d\nu_{x_0}}{d\nu_{x_1}}(\eta) = \exp(-e(X,\Gamma)b(x_0, x_1, \eta))$$

3 Geometry of flat metrics

We introduce the geometry of flat surfaces. For rigorous computations we refer to [Str84], [MT02] and [Min92].

Let X be a closed Riemann surface of genus $g \geq 2$ which admits a natural holomorphic cotangent bundle $T'X$. A holomorphic quadratic differential q on X is a holomorphic section of the bundle $T'X \otimes_X T'X$. Let q be a non-zero quadratic differential and let Σ be the set of zeros of q. It is a consequence of the Riemann-Roch Theorem that the sum of the zeros, counted with multiplicity, equals $4g - 4$. For any point $x \in X - \Sigma$ one can choose a simply connected neighborhood U and $\sqrt{q}$, one branch of the root of q in U, which is a holomorphic 1-form. For any point $y \in U$ there is a path c in U connecting x with y. We define $\phi : U \to \mathbb{C}, \phi(y) := \int_c \sqrt{q}$. Since $\sqrt{q}$ is holomorphic and U simply connected, the integral is independent of the chosen path. ϕ is a locally biholomorphic map which we take as a chart.

Such charts form an atlas on $X - \Sigma$ with the property that the transition functions are of the form $z \mapsto \pm z + c$ due to the choice of the root. After identifying $\mathbb{C}$ with $\mathbb{R}^2$ we can pull back the flat metric in each chart and obtain a flat metric on $X - \Sigma$. About a zero of q we take a small disk D where q is of the local form $\frac{k+2}{2} z^k dz^2$ and subdivide D into sectors $i2\pi/k \leq arg(z) \leq (i+1)2\pi/k$.

The integral of $\sqrt{q}$, along the points in each sector, defines a biholomorphic mapping onto a half circle in $\mathbb{C}$. At each point $\varsigma \in \Sigma$ the metric is locally isometric to $k+2$ half circles which are glued at the boundary in clockwise order.

For any point $x \in X$ we call a neighborhood standard if it is isometric to a finite union of euclidean half discs glued along the boundary. If x is a regular point, each standard neighborhood is isometric to a euclidean disc.

Consequently, the flat metric on $X - \Sigma$ can be extended to a metric d_q on X which is a

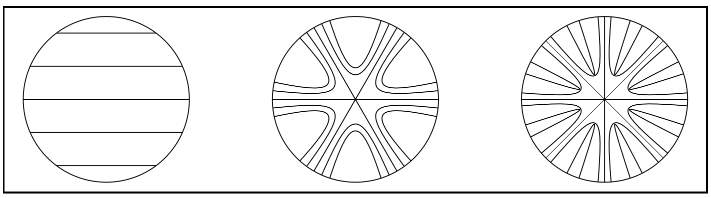

Figure 1: The horizontal line segments at a regular point and at a singularity for k=4 resp. 8.

singular cone metric with cone angle $(k+2)\pi$ at each zero of q. The metric d_q is called a flat. We denote $S = (X, d_q)$ as a flat surface.

By construction, the metric d_q defines the topology of the underlying Riemann surface X.

The length of a curve c, with respect to the flat metric, equals

$$l_q(c) = \int_c |\sqrt{q}|$$

The area ℓ_q of a quadratic differential is the area of the flat metric it defines. When we scale q with a positive number λ, the metric scales with $\sqrt{\lambda}$. We normalize the quadratic differential such that it has area one.
A straight line segment on $S - \Sigma$ is defined as the pull-back of a straight line segment on $\mathbb{R}^2 = \mathbb{C}$ in each chart. A straight half-line emanating from one singularity is called a seperatrix. A straight line segment which emanates from one singularity and ends at another is called a saddle connection. With respect to the flat metric one can define the angle in the following manner:
Let $x \in S$ be a point. Recall that a standard neighborhood of U of x is isometric to a finite number $n \geq 2$ of half discs, isometrically glued along the boundary.
The boundary of the standard neighborhood is a topological circle. We choose an orientation of the boundary.

Definition 3.1. *Let c_1, c_2 be straight line segments, issuing from x. Let U be a standard neighborhood of x. The metric at x is cone with cone angle $n\pi \geq 2\pi$. The complement $U - c_1 \cap c_2$ consists of two connected components U_1, U_2 which are isometric to euclidean circle sectors with angle $\vartheta_i, i = 1, 2$ possibly greater than 2π. We measure the flat angle $\angle_x(c_1, c_2)$ which is the sector angle at U_1 or U_2 due to the choice of the orientation. ϑ takes values in $[0, n\pi)$.*

It is shown in [Str84, Theorem 8.1] that local geodesics on S are concatenations of straight line segments. The points of transition are singularities, and there is some constraint on the euclidean angle of the in- and outgoing straight line segments.

Lemma 3.1. *A path $c : [0, T] \to S$ is a local geodesic if and only if it is continuous and a sequence of straight lines segments outside Σ. In the singularities $\varsigma = c(t)$ the consecutive line segments make angle, measured in the flat metric, at least π with respect to both boundary orientations.*

$$\angle_\varsigma(c|_{[t,t+\epsilon]}, c|_{[t-\epsilon,t]}) \geq \pi$$

Due to the fact that local geodesics are characterized by local properties, each local geodesic can be infinitely extended in both directions.

Proposition 3.1. *Let S be a flat surface and $c : [0, T] \to S$ a local geodesic. There exists a locally geodesic line $c' : \mathbb{R} \to S$ which, restricted to the interval $[0, T]$, equals c.*

Local geodesics are uniquely defined by their free homotopy class with fixed endpoints.

Proposition 3.2. *In any homotopy class of arcs with fixed endpoints there exists a unique local geodesic which is also length-minimizing. Furthermore, for any closed curve there exists a length-minimizing local geodesic in its free homotopy class.*

Proof. [Str84, Corollary 18.2] □

Let S be a flat surface and c, c' be local geodesics issuing from a common point x. Since both of them are locally of the form of two straight line segments, one can compute the euclidean angles $\angle_x(c, c')$ and $\angle_x(c', c)$ with respect to some boundary orientation. The Alexandrov angle equals $\min\{\pi, \angle_x(c, c'), \angle_x(c', c)\}$.

Definition 3.2. *A flat cylinder of height h and circumference c in S is an isometric embedding of $[0, c] \times (0, h)/\sim, (0, t) \sim (c, t)$ into S. We call a cylinder maximal if it cannot be extended.*

The boundary of a maximal flat cylinder is a union of saddle connections, see [MT02, Lemma 1.6].

Remark 3.1. *By the Gauss-Bonnet Theorem, for any smooth metric on a closed surface of genus ≥ 2, the integral over the curvature is negative. On the other hand, outside the zeros the metric is flat. So intuitively the curvature is concentrated in the singularities.*

3.1 The universal cover

Let X be a closed Riemann surface of genus $g \geq 2$. We consider the universal cover $\tilde{X}$, a topological disc, together with the Deck-transformation group Γ.
We will make use of the Jordan curve Theorem.

Proposition 3.3. *Let $\alpha : S^1 \hookrightarrow \mathbb{R}^2$ be a simple closed curve. The complement $\mathbb{R}^2 - \alpha$ consists of two connected components which are both bounded by α.*

Assume that X is endowed with a flat metric $S = (X, d_q)$. d_q can be lifted to a flat metric d_q on the universal cover $\tilde{X}$. We call $\tilde{S} = (\tilde{X}, d_q)$ the flat universal cover of S. S is a complete and proper space. The flat universal cover $\tilde{S} = (\tilde{X}, d_q)$ is a complete proper metric space as well. An arc on $\tilde{S}$ is geodesic if and only if its projection to S is a local geodesic. Since in any homotopy class of arcs with fixed endpoints on S there exists a unique length-minimizing local geodesic, $\tilde{S}$ is a uniquely geodesic metric space. Finally, the projection $\pi : \tilde{S} \to S$ is a local isometry. Consequently, each element $\gamma \in \Gamma$ of the Deck transformation group is an isometry as well.

3.2 Comparison with the natural hyperbolic metric

It is a result of the uniformization Theorem that for any flat surface there exists a natural hyperbolic metric σ on X which is in the same conformal class as d_q.
Both the flat universal cover as well as the hyperbolic universal cover are geodesic metric spaces. The Deck transformation group acts properly cocompactly by isometries with respect to both metrics. Therefore, we can apply the Švarc-Milnor Lemma.

Lemma 3.2. *Let X be a geodesic metric space. Suppose that a group Γ acts properly, cocompactly and isometrically on X. Then Γ is finitely generated.*
Choose a base point $x \in X$. The map

$$\Gamma \to X, \gamma \mapsto \gamma x$$

is a quasi-isometry with respect to a word metric on Γ.

Proof. In [BH99, I Proposition 8.19] one finds an accessible proof as well as references to the original work. □

Consequently, the flat and the hyperbolic metric on the universal cover are quasi-isometric.
Furthermore, both metrics define the same topology of the underlying Riemann surface.
We will show that $\tilde{S}$ is a Gromov hyperbolic $Cat(0)$-space.
The Poincaré disc is a Gromov hyperbolic space. That is why the universal cover, endowed with the flat metric, is Gromov hyperbolic as well.

3.3 Non-positive curvature of the universal cover

Let $S = (X, d_q)$ be a closed flat surface of genus $g \geq 2$. Since the metric is locally flat, we cannot expect local properties of spaces with negative curvature. Therefore, from the viewpoint of hyperbolic geometry, being $Cat(0)$ is the best we can achieve.
Recall that a polyhedral complex with a finite number of shapes is a disjoint union of cells which are isometrically glued along some faces. Up to isometry, the number of cells is finite. For a precise characterization we refer to Definition 2.2.
We show that S as well as the universal cover $\tilde{S}$ are polyhedral complexes with a finite number of shapes.
We make use of triangulations of the surface S which are explicitly described in [Vor96, Section 2], [KMS86].
Let Σ be the set of singularities on S. A triangulation $\mathcal{T}$ of S is an isometry from an euclidean polyhedral complex to S so that each cell is a euclidean triangle. Each

0-dimensional face is a point $\varsigma \in \Sigma$.
The number of cells has an upper bound which only depends on the topology of S and on the set of vertices V.

Proposition 3.4. *For each flat surface S there exists such a triangulation.*

Proof. We refer to [BS07, Proposition 12] and [MS91, section 4]. ☐

Proposition 3.5. *Let S be a flat surface. The flat universal cover $(\tilde{S}, d_q)$ is a uniquely geodesic polyhedral complex with a finite number of shapes and therefore, by Proposition 2.2, it is a $Cat(0)$-space.*

Proof. We showed that S is a polyhedral complex with a finite number of shapes. The polyhedral structure of S lifts to a polyhedral decomposition of the universal cover with an infinite number of cells. Each cell on $\tilde{S}$ is a lift of a cell of S. Since the covering projection is a local isometry, the number of shapes is finite. ☐

We showed that the flat universal cover $\tilde{S}$ is a proper δ-hyperbolic $Cat(0)$-space which is quasi-isometric to the Poincaré disc.
We compute δ explicitly in geometric terms of $\tilde{S}$.
We need the Gauss Bonnet formula for quadratic differentials which is extensively described in [Hub06, Proposition 5.3.3]:
Let S be a flat surface. Let $P \subset S$ be a compact topological subsurface whose boundary consists of a finite union of locally geodesic arcs. For each boundary point $x \in \partial P$, we define ϑ as follows:
Choose a small standard neighborhood of x in S. Since the boundary of P is piecewise locally geodesic, the two outgoing boundary segments at x are straight line-segments s_1, s_2. We define $\vartheta(x)$ as the flat angle of s_1, s_2 at x, measured at the circle sector inside of P.
At each point x in $\mathring{P}$ let $n(x)$ be the vanishing order of the quadratic differential. The curvature term at x is defined as

$$\kappa(x) := -\pi n(x)$$

Therefore, for each point $x \in X$ which is not a zero of the quadratic differential q, the curvature $\kappa(x)$ is 0.

Proposition 3.6. *Let S be a flat surface and let P be a compact subsurface of S with piecewise locally geodesic boundary. Denote by $\chi(P)$ the Euler characteristic of P. Then*

$$2\pi\chi(P) = \sum_{x \in \mathring{P}} \kappa(x) + \sum_{x \in \partial P} (\pi - \vartheta(x))$$

We mainly deal with the special case that P is a polygon with piecewise geodesic boundary.

Corollary 3.1. *Let S be a flat surface and $P \subset S$ be a simply connected compact polygon with piecewise geodesic boundary. Let x_i be the points on the boundary such that the angle $\vartheta(x_i) \neq \pi$ and let ς_j be the zeros in the interior of P of order n_j. Then*

$$2\pi = -\sum_j n_j \pi + \sum_i (\pi - \vartheta(x_i))$$

Let S be a singular flat surface which is not necessarily closed and let Σ be the set of singularities on S. We define the packing density ρ.

$$\rho(S) := \sup_{x \in S} d(x, \Sigma)$$

If S is compact, $\rho(S)$ is finite. The flat universal cover $\pi : \tilde{S} \to S$ is a flat surface, and consequently the packing density is also defined on $\tilde{S}$. Any point $\tilde{\varsigma} \in \tilde{S}$ is a singularity if and only if $\pi(\tilde{\varsigma}) \in S$ is a singularity as well. Furthermore, geodesics in the universal cover project to local geodesics in the base space. Since lifts of local geodesics are again geodesics, it follows:

$$\rho(S) = \rho(\tilde{S})$$

Proposition 3.7. *$\tilde{S}$ is δ-hyperbolic with $\delta = 2\rho(S)$. There exist triangles in $\tilde{S}$ which are not $\rho(S)/2$-slim. If S has area 1, there is a lower bound on $\rho(S)$ which only depends on the topology of S.*

Proof. Let $\triangle(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ be a triangle in $\tilde{S}$. $\tilde{S}$ is uniquely geodesic and therefore, the two geodesics emanating from a point $\tilde{x}_i$ might coincide for some time but after spreading, they remain disjoint. Denote by $\tilde{y}_i$ the point at which $[\tilde{x}_i, \tilde{x}_j]$ and $[\tilde{x}_i, \tilde{x}_k]$ start spreading apart. The interior P of the triangle is either empty or a topological disc bounded by $\triangle(\tilde{y}_1, \tilde{y}_2, \tilde{y}_2)$, see Figure 2.
If $P = \emptyset$, the triangle is a tripod, consequently it is 0-slim.
It suffices to show that $\triangle(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ is $2\rho(S)$-slim. At $\tilde{y}_i$ the two emanating geodesics are locally straight line segments which spread apart. The flat angle, measured inside $\tilde{P}$, satisfies $\vartheta(\tilde{y}_i) > 0$.
Assume there is a singularity $\tilde{\varsigma}$ in P. By Corollary 3.1, $\kappa(\tilde{\varsigma}) \leq -\pi$. Furthermore, each point in the interior has at most curvature 0 and so

$$\sum_{\tilde{x} \in \mathring{P}} \kappa(\tilde{x}) \leq -\pi.$$

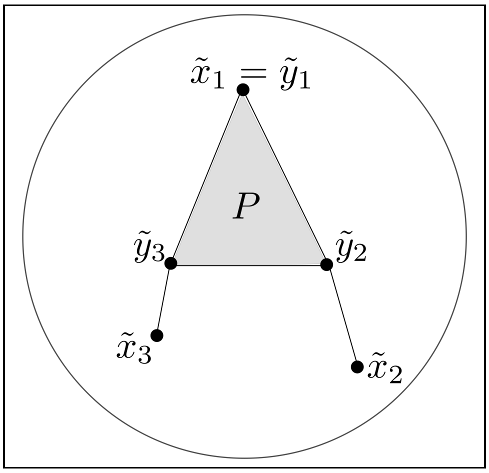

Figure 2: The geodesics which correspond to the triangle of the $\tilde{x}_i$ might share some arc until they spread apart. We call the spread point $\tilde{y}_i$.

The boundary consists of three geodesics. At each point $\tilde{x}$ in the interior of each geodesic, $\vartheta(\tilde{x}) \geq \pi$ hence $\pi - \vartheta(\tilde{x}) \leq 0$.
So the positive curvature terms can appear at most in the corners of the triangle. But in each corner $\tilde{y}_i$ one can have a positive amount of at most π which is attained if and only if $\vartheta(\tilde{y}_i) = 0$. This contradicts the Gauss Bonnet formula.
We showed that P cannot contain a singularity.
By definition, each ball of radius $\rho(S)$ contains a singularity.
Recall that $[\tilde{y}_1, \tilde{y}_2] \subset \tilde{S}$ is a closed convex set and $\tilde{S}$ is $Cat(0)$- space. Denote by

$$\pi_{\tilde{y}_1,\tilde{y}_2} : \tilde{S} \to [\tilde{y}_1, \tilde{y}_2]$$

the closest point projection, see Proposition 2.1. Let $\tilde{x} \in [\tilde{y}_1, \tilde{y}_2]$ be a regular point which is not an endpoint. Choose a standard neighborhood U of $\tilde{x}$ which is isometric to a euclidean disc. By the fact that the projection is locally orthogonal, $\pi^{-1}_{\tilde{y}_1,\tilde{y}_2}(\tilde{x}) \cap U$ is a straight line segment which is perpendicular to $[\tilde{y}_1, x]$. We can extend this line segment to a geodesic half-line $\tilde{c}$ which starts at $\tilde{x}$ and points inside P. We parametrize $\tilde{c}$ such that $\tilde{c}(0) = \tilde{x}$.
Consider the disc $B_t := B_{\tilde{c}(t)}(t)$. One observes that $B_{t'} \subset B_t, t' < t$. As $\tilde{c}$ points inside P, the intersection of P and B_t is not empty.
Let be $\epsilon > 0$ be so small that $2\epsilon + 4\sqrt{\epsilon\rho(S) + 3\epsilon^2} < d(\tilde{x}, \Sigma)$ and let t_0 be chosen in a way that the distance of B_{t_0} to the set of singularities Σ satisfies

$$\epsilon < d(B_{t_0}, \Sigma) < 2\epsilon$$

By definition $t_0 < \rho(S) - \epsilon$.
The closure $\overline{B}_{t_0}$ is a closed disc and therefore convex. It can be slightly thickened to an open euclidean disc B'_t of radius $t_0 + \epsilon$. The point $\tilde{x}$ is contained in ∂B_{t_0}. In a small neighborhood of $\tilde{x}$, $[\tilde{y_1}, \tilde{y_2}]$ is a straight line segment which is tangent to $\overline{B}_{t_0} \subset B'_t$. By convexity of $\overline{B}_{t_0}$, the line segment $[\tilde{y}_1, \tilde{y}_2]$ intersects $\overline{B}_{t_0}$ only at $\tilde{x}$.
Especially

$$\pi_{\tilde{y}_1,\tilde{y}_2}(\tilde{c}(t_0)) = \tilde{x}$$

There exists a singularity $\tilde{\varsigma}$ with $t_0 \leq d(\tilde{c}(t_0), \tilde{\varsigma}) \leq t_0 + 2\epsilon$. ς cannot be contained in the interior of P. Let $g := [\tilde{x}, \tilde{c}(t_0)] * [\tilde{c}(t_0), \tilde{\varsigma}]$ be the piecewise geodesic arc which has to intersect the triangle $\triangle(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ at some point $\tilde{y}$.
We want to show that $\tilde{y} \notin [\tilde{y}_1, \tilde{y}_2]$.
If $\tilde{y} \in [\tilde{y}_1, \tilde{y}_2]$ observe that

$$t_0 = d(\tilde{c}(t_0), \tilde{x}) = d(\tilde{c}(t_0), [\tilde{y_1}, \tilde{y_2}]) \leq d(\tilde{c}(t_0), \tilde{y}) \leq t_0 + 2\epsilon$$

and so,

$$d(\tilde{\varsigma}, \tilde{y}) \leq 2\epsilon$$

Moreover, by Proposition 2.1

$$d(\tilde{x}, \tilde{y}) \leq 2\sqrt{4\epsilon(t_0 + 2\epsilon) + 4\epsilon^2} \leq 4\sqrt{\epsilon\rho(S) + 3\epsilon^2}$$

We deduce

$$d(\tilde{\varsigma}, \tilde{x}) \leq 2\epsilon + 4\sqrt{\epsilon\rho(S) + 3\epsilon^2}$$

But we chose ϵ so small that such a singularity does not exist.
Consequently, $\tilde{y}$ is contained in the geodesics $[\tilde{y_2}, \tilde{y_3}] \cap [\tilde{y_1}, \tilde{y_3}]$ and therefore

$$d(\tilde{x}, [\tilde{y_2}, \tilde{y_3}] \cap [\tilde{y_1}, \tilde{y_3}]) \leq d(\tilde{x}, \tilde{y}) \leq 2t_0 + 2\epsilon \leq 2\rho(S)$$

We showed that each regular point on $[\tilde{y_1}, \tilde{y_2}]$ has distance at most $2\rho(S)$ to the other two geodesics. Since regular points are dense, this holds for each point on $[\tilde{y_1}, \tilde{y_2}]$.
Therefore, each triangle is $2\rho(S)$ slim.
We have to show that there exists a triangle which is not $\rho(\tilde{S})/2$ slim. Recall that by definition of the packing density there is a ball of radius $\rho(\tilde{S})$ in $\tilde{S}$ which does not contain a singularity. This ball is isometric to a euclidean disc. One can inscribe a maximal euclidean equilateral triangle and compute that it is not $\rho(S)/2$ slim.
Finally, we show that for a flat surface S of area 1 there is a lower bound on $\rho(S)$ which only depends on the genus of S.

Let $\varsigma_i, 1 \dots n$ be the set of singularities in the flat surface S. Let $n_i\pi$ be the cone angle at ς_i. Recall that by Riemann Roch Theorem

$$\sum (n_i - 2) = 4g - 4$$

Let $\epsilon > 0$. Around each singularity ς_i one can choose a disc of radius

$$r = \sqrt{\frac{1}{\sum_i n_i \pi}} - \epsilon$$

The volume of the union of the discs is at most $\sum_i r^2 n_i \pi < 1$. Therefore, there exists a point in the complement of the discs. This point has distance at least r to each singularity. Therefore $\rho(S) \geq r$. Since $n_i \geq 3$ for each i, the constant r has a lower bound which only depends on the genus. □

3.4 Asymptotic rays

The following fact concerning geodesic rays in the Poincaré disc is classical: Two geodesic rays with the same endpoint on the boundary converge towards each other exponentially fast.

We investigate the behavior of geodesic rays in the flat universal cover $\pi : \tilde{S} \to S$ of a closed flat surface S. $\tilde{S}$ is quasi-isometric to the Poincaré disc and the map extends to a homeomorphism between the topological boundaries. Consequently, the boundary of the flat universal cover is a topological circle.

The following Proposition is well-known:

Proposition 3.8. *Let S be a closed flat surface and $\tilde{S}$ be the flat universal cover of S. Let $\tilde{\alpha}_1, \tilde{\alpha}_2$ be geodesic lines in $\tilde{S}$. Assume that $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ have finite Hausdorff distance, but there is some constant $c > 0$ such that $d(\tilde{\alpha}_1(t), \alpha_2) > c, \forall t$.*
It follows that the projections of $\tilde{\alpha}_i$ to the flat surface S are closed curves which are freely homotopic to core curves of the same maximal flat cylinder.

Proof. [MS85, Theorem 1] □

The converse also holds. Let α_1, α_2 be locally geodesic core curves of the same maximal flat cylinder on a flat surface S so that α_1 is not a reparametrization of α_2. One can choose complete lifts $\tilde{\alpha}_i$ of α_i in the universal cover $\tilde{S}$. The geodesic lines $\tilde{\alpha}_i$ have finite Hausdorff distance but do not converge towards each other.

It was shown by Masur [Mas86, Theorem 2] that flat cylinders always exist.

Proposition 3.9. *Each closed flat surface S of genus $g \geq 2$ contains a countable set of maximal flat cylinders.*

Therefore, on each flat universal cover $\tilde{S}$ of a closed flat surface S of genus $g \geq 2$ one finds infinitely many pairs of geodesic lines which tend towards the same boundary points but do not approach.
However, on the boundary of the flat cover $\tilde{S}$ we can choose a Gromov metric and define the corresponding positive finite Hausdorff measure. With respect to the Hausdorff measure let $\eta \in \partial S$ be a typical point and let $\tilde{\alpha}, \tilde{\beta}$ be two geodesic rays with the same positive endpoint η. We will show that the lines $\tilde{\alpha}, \tilde{\beta}$ eventually coincide in positive direction.
We need some criterion for the non-typical boundary points which we call quasi-straight. In this section we define the quasi-straight points and describe some properties. The methods are highly motivated by [DLR09]. The fact that quasi-straight points form a set of measure 0 will be shown in section 5.3.
Let $\tilde{S}$ be the universal cover of a closed flat surface S. Let $\tilde{\alpha}$ be a parametrized geodesic line in $\tilde{S}$. The complement $\tilde{S} - \tilde{\alpha}$ consists of two connected components $\tilde{S}^{\pm}$. For each point $\tilde{\alpha}(t)$ we choose a standard neighborhood U. $U - \tilde{\alpha}$ consists of two connected components $U^{\pm}$ so that $U^i \subset \tilde{S}^i, i \in \pm$.
We choose the orientation in a way that ϑ^+ resp. ϑ^- measures the flat angle

$$\angle_{\tilde{\alpha}(t)}(\tilde{\alpha}|_{[t-\epsilon,t]}, \tilde{\alpha}|_{[t,t+\epsilon]})$$

inside of U^+ resp. U^-.

Definition 3.3. *Let $\tilde{\alpha}$ be a geodesic line.*

- *$\tilde{\alpha}$ is called quasi-straight if there is some $i \in \pm$ so that*

$$\sum_{t>0} (\vartheta^i(\tilde{\alpha}(t)) - \pi) < \infty$$

- *We call a boundary point $\eta \in \partial\tilde{S}$ a quasi-straight point if there is a quasi-straight geodesic line which tends to η in positive direction.*
- *The set of quasi-straight points is denoted as str_∂.*

Let $\tilde{\alpha}$ be a geodesic line. At each regular point $\tilde{\alpha}(t)$ the angle satisfies $\vartheta^i(\tilde{\alpha}(t)) - \pi = 0$. At each singularity the angle is bounded, $\vartheta^i(\tilde{\alpha}(t)) < \infty$. Therefore, the sum, restricted to a compact set $A \subset \mathbb{R}$, is finite.
Consequently, a geodesic line $\tilde{\alpha}$ is quasi-straight if and only if each reparametrization of $\tilde{\alpha}$ is quasi-straight.

Since the choice of $\tilde{S}^+$ is arbitrary, we assume that for any quasi-straight geodesic line the angle ϑ^+ is bounded

$$\sum_{t>0}(\vartheta^+(\tilde{\alpha}(t)) - \pi) < \infty$$

Lemma 3.3. *Let $\tilde{\alpha}$ be a parametrized geodesic line which is not quasi-straight and let $\tilde{S}^+$ be a component of $\tilde{S} - \tilde{\alpha}$. Let $\eta \in \partial\tilde{S}$ be the endpoint in positive direction of $\tilde{\alpha}$. There exists a sequence of times t_i which tends to infinity and a sequence of geodesic lines $\tilde{\beta}_i \in \tilde{S}^+ \cup \tilde{\alpha}$ which remain on one side of $\tilde{\alpha}$ and share exactly one point $\tilde{\alpha}(t_i)$ with $\tilde{\alpha}$. Precisely,*

$$\tilde{\beta}_i \cap \tilde{\alpha} = \tilde{\alpha}(t_i)$$

Furthermore, let $\eta_{i,+}$ resp. $\eta_{i,-}$ be the endpoints of $\tilde{\beta}_i$ in positive resp. negative direction. Both endpoints $\eta_{i,j}, j \in \pm$ tend towards η.

Proof. The geodesic line $\tilde{\alpha}$ is not quasi-straight. Therefore, there is an increasing sequence of times $t_i > 0$ such that the angle $\vartheta^+(\tilde{\alpha}(t_i))$ is strictly greater than π. This is only possible if $\tilde{\alpha}(t_i)$ is a singularity. Since the set of singularities is discrete, the sequence t_i tends to infinity.
Let U be a standard neighborhood of the point $\tilde{\alpha}(t_i)$. Denote by $U^+ := U \cap \tilde{S}^+$ the circle sector in $\tilde{S}^+$. By definition, the angle at $\alpha(t_i)$ in U^+ is strictly greater than π. We can locally choose two line segments $\tilde{c}_1, \tilde{c}_2$ which issue from $\tilde{\alpha}(t_i)$ in the interior of U^+ and form an internal angle at least π. The concatenation $\tilde{c_1} * \tilde{c_2}$ is a geodesic which only shares $\tilde{\alpha}(t_i)$ with $\tilde{\alpha}$. It can be extended to a geodesic line $\tilde{\beta}_i$. Due to the uniqueness of geodesics, $\tilde{\beta}_i$ cannot hit $\tilde{\alpha}$ outside $\tilde{\alpha}(t_i)$, consequently $\tilde{\beta}_i \subset \tilde{S}^+ \cup \tilde{\alpha}$.
Recall that the visual boundary $\partial\tilde{S}$ consists of equivalence classes of geodesic rays.
Let

$$\eta_{i,+} := \tilde{\beta}_i(t)\Big|_{[0,\infty])}, \eta_{i,-} := \tilde{\beta}_i(-t)\Big|_{[0,\infty)}$$

be the endpoints of $\tilde{\beta}_i$ in $\partial\tilde{S}$. It remains to show that the sequence $\eta_{i,j}, j \in \pm$ converge towards η, the positive endpoint of $\tilde{\alpha}$. The topology on the boundary is the topology of each Gromov metric. We choose a Gromov metric $d_{\tilde{p},c}$ with base point $\tilde{p} := \tilde{\alpha}(0)$ and some constant $c > 0$.
It suffices to show that

$$lim_i(\eta_{i,j} \cdot \eta)_{\tilde{p}} = \infty, j \in \pm$$

We make use of the following observation:
Let $s_0 < t_0 \in \mathbb{R}$ so that

$$\sum_{t\in(s_0,t_0)}(\vartheta^+(\tilde{\alpha}(t)) - \pi) \geq \pi$$

Let $\tilde{x} \in \tilde{S}^+$ be a point in $\tilde{S}^+$. We claim that, up to reparametrization, one of the two geodesics $[\tilde{\alpha}(s_0), \tilde{x}], [\tilde{x}, \tilde{\alpha}(t_0)]$ coincides with $\tilde{\alpha}$ on a subsegment of $\tilde{\alpha}|_{[s_0,t_0]}$.
Assume that the geodesics $[\tilde{x}, \tilde{\alpha}(t_0)]$ and $[\tilde{x}, \tilde{\alpha}(s_0)]$ both do not share a subsegment with $\tilde{\alpha}|_{[s_0,t_0]}$. Then the interior of the triangle $\triangle(\tilde{\alpha}(t_0), \tilde{\alpha}(s_0), \tilde{x})$ violates the Gauss Bonnet formula as, by definition of s_0, t_0, the interior angle at the line segment $\tilde{\alpha}|_{[s_0,t_0]}$ is too large.
Therefore, $[\tilde{\alpha}(s_0), \tilde{x}]$ coincides with $\tilde{\alpha}$ on the interval $[s_0, s']$, for some $s' > s_0$ or $[\tilde{\alpha}(t_0), \tilde{x}]$ coincides with $\tilde{\alpha}$ on the interval for some $[t', t_0], t' < t_0$, up to reparametrization.
Recall that

$$\sum_{t>0} (\vartheta^+(\tilde{\alpha}(t)) - \pi) = \infty$$

Up to removing the first elements of the sequence t_i, we can assume that

$$\sum_{t \in (0,t_i)} (\vartheta^+(\tilde{\alpha}(t)) - \pi) \geq \pi, \forall t_i$$

We can choose a sequence s_i, so that $0 \leq s_i \leq t_i$, $\lim_i s_i = \infty$ and

$$\sum_{t \in (s_i,t_i)} (\vartheta^+(\tilde{\alpha}(t)) - \pi) \geq \pi$$

Let $\tilde{x} \in \tilde{\beta}_i$ be a point on $\tilde{\beta}_i$.
The geodesic $[\tilde{\alpha}(t_i), \tilde{x}]$ is a segment of $\tilde{\beta}_i$ and therefore no subsegment of $[\tilde{\alpha}(t_i), \tilde{x}]$ of positive length can coincide with a subsegment of $\tilde{\alpha}$. So, the geodesic which connects $\tilde{x}$ with $\tilde{\alpha}(s_i)$ necessarily coincides with $\tilde{\alpha}$ along some interval $\tilde{\alpha}|_{[s_i,s']}$.
Consequently, the concatenation of $[\tilde{p}, \tilde{\alpha}(s_i)] = \tilde{\alpha}|_{[0,s_i]}$ with $[\tilde{\alpha}(s_i), \tilde{x}]$ is a geodesic. We deduce

$$(\tilde{x} \cdot \tilde{\alpha}(t_i))_{\tilde{p}} = (\tilde{x} \cdot \tilde{\alpha}(t_i))_{\tilde{\alpha}(s_i)} + s_i \geq s_i$$

Therefore, the Gromov product of $\eta_{i,j}, j \in \pm$ and η is at least s_i

$$(\eta_{i,j} \cdot \eta)_{\tilde{p}} \geq s_i, j \in \pm$$

Since the sequence s_i tends to infinity, we can estimate the Gromov product

$$\lim_i (\eta_{i,j} \cdot \eta)_{\tilde{p}} = \infty, j \in \pm$$

□

Since the Group of Deck transformations Γ acts by isometries and therefore preserves angles, the set of parametrized quasi-straight geodesic lines is Γ-invariant. Consequently, the set of quasi-straight boundary points is Γ-invariant as well.

Proposition 3.10. *Let $\tilde{S}$ be the flat universal cover of a closed flat surface S.*

i) Let $\tilde{\alpha}_1, \tilde{\alpha}_2$ be parametrized geodesic lines with the same positive endpoint η. Assume that $\tilde{\alpha}_1$ is not quasi-straight. Then there are times $r, s \in \mathbb{R}$ so that $\tilde{\alpha}_1(s+t) = \tilde{\alpha}_2(r+t)$ for all $t > 0$.

ii) Let η be a quasi-straight point. Any geodesic line with positive endpoint η is quasi-straight.

Proof. We show *i*) first:
Since geodesic rays are uniquely defined by their endpoints either $\tilde{\alpha}_1, \tilde{\alpha}_2$ are disjoint or there are times $r, s \in \mathbb{R}$ so that $\tilde{\alpha}_1(s+t) = \tilde{\alpha}_2(r+t)$ for all $t > 0$.
Assume that $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are disjoint. Denote by $\tilde{S}^+$ the connected component of $\tilde{S} - \tilde{\alpha}_1$ containing $\tilde{\alpha}_2$. Let C be the connected component of $\tilde{S}^+ - \tilde{\alpha}_2$ which is bounded by $\tilde{\alpha}_1, \tilde{\alpha}_2$, see Figure 3. Let ζ be the negative endpoint of $\tilde{\alpha}_2$ which might coincide with

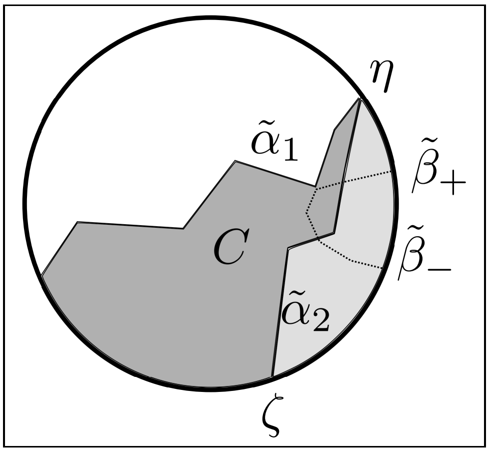

Figure 3: $\tilde{\beta}$ is in $\tilde{S}^+$ which is marked gray and $C \subset \tilde{S}^+$ the dark gray part. The distance of ζ to η is larger than the distance of η to both endpoints of $\tilde{\beta}$. Consequently, both rays $\tilde{\beta}_\pm$ have to leave C. But they intersect $\tilde{\alpha}_2$ at most once.

the negative endpoint of $\tilde{\alpha}_1$. Since the boundary of $\tilde{S}$ is a topological circle, we can parametrize the boundary at infinity of $\tilde{S}^+$ by an arc starting at the negative endpoint of $\tilde{\alpha}_1$ and ending at the positive endpoint η. By Lemma 3.3 there is some geodesic line $\tilde{\beta} \subset \tilde{S}^+$ which shares one point $\tilde{\alpha}(t_0)$ with $\tilde{\alpha}_1$ such that both endpoints of $\tilde{\beta}$ are closer to η than to ζ with respect to the parametrization of the boundary at infinity of $\tilde{S}^+$. Therefore, both endpoints of $\tilde{\beta}$ are disjoint from the boundary at infinity of C. We parametrize $\tilde{\beta}$ such that $\tilde{\beta}(0) = \tilde{\alpha}_1(t_0)$. Therefore, $\tilde{\beta}(0)$ does not lie on $\tilde{\alpha}_2$.
The geodesic rays $\tilde{\beta}_+(t) := \tilde{\beta}(t)\Big|_{[0,\infty)}$ and $\tilde{\beta}_-(t) := \tilde{\beta}(-t)\Big|_{[0,\infty)}$ emanate from $\tilde{\alpha}_1(t_0)$

and therefore start in C. Since both endpoints are outside the boundary at infinity of C, both rays have to leave C. Therefore they intersect $\tilde{\alpha}_2$ what violates the fact that geodesics are unique.

We deduce that $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are not disjoint.

To show *ii*) observe that a reparametrization of the geodesic line $\tilde{\alpha}_1$ is not a quasi-straight line if and only if $\tilde{\alpha}_1$ is not quasi straight either. Consequently, for any not quasi-straight boundary point η each geodesic line with positive endpoint η is not quasi-straight either. □

Remark 3.2. *Recall that the boundary of $\tilde{S}$ is defined as equivalence classes of geodesic rays. We call a geodesic ray $\tilde{r}$ quasi-straight if the corresponding boundary point is quasi-straight. Let $\tilde{r}, \tilde{r}'$ be geodesic rays in the same equivalence class η. If the η is not quasi-straight, the rays r, r' eventually coincide.*

3.5 Concatenation of compact geodesics and intersections of closed curves

Let c be a compact locally geodesic arc on a closed flat S and assume that ς, the positive endpoint of c, is a singularity. There exists a one-parameter family of possible local extensions of c.

We make use of this feature to show that for any local geodesics c, c', so that the positive endpoint of c and the negative endpoint of c' are a singularities, we find some local geodesic g which first coincides with c and eventually coincides with c'.

Proposition 3.11. *Let S be a closed flat surface. Let c, c' be parametrized locally geodesic arcs on S so that c ends at a singularity ς and c' issues from some singularity ς'. There is a local geodesic g so that g first passes through c and eventually passes through c'. The length of g is bounded from above by $l(g) < C_l(S) + l(c) + l(c')$. The constant $C_l(S)$ only depends on the flat surface S.*

We need the following technical facts:

Lemma 3.4. *Let S be closed a flat surface of genus $g \geq 2$ and $x \in S$ be a point. Let θ be an outgoing direction at ς. Let $\angle_\varsigma$ be the flat angle at ς with respect to a choice of orientation. For any ϵ there exists a singularity ς which emanates from x so that $\angle_x([x, \varsigma], \theta) \leq \epsilon$.*

Proof. We refer to [Vor96, Proposition 3.1]. □

Lemma 3.5. *Let S be a flat surface and let $\varsigma \in S$ be a singularity. There exist 4 saddle connections $s_1, \ldots, s_4$ emanating from ς with the following property:*

Let c be a local geodesic with endpoint ς. The concatenation of c with at least one s_i is again a local geodesic.

Proof. Let $k\pi \geq 3\pi$ be the total angle at ς. Let s_1 be some saddle connection emanating from ς. With respect to some orientation we define an angle $\angle_\varsigma$ at ς. Let $s_i, i = 2 \ldots 4$ be consecutive saddle connections such that $\angle_\varsigma(s_i, s_{i+1}) \in \left[\frac{2\pi}{3}, \frac{3\pi}{4}\right], i = 1 \ldots 3$. By Lemma 3.4 such saddle connections exist.
Since the total angle is $k\pi \geq 3\pi$ we observe that

$$k\pi - 2\pi > \angle_\varsigma(s_4, s_1) \geq \frac{3}{4}\pi$$

Let c be some incoming locally geodesic line segment. To show that the concatenation of c and some saddle connection s_i is a geodesic, it suffices to show that $\angle_\varsigma(s_i, c) \geq \pi, \angle_\varsigma(c, s_i) \geq \pi$. Since

$$\angle_\varsigma(c, s_i) = k\pi - \angle_\varsigma(s_i, c)$$

it suffices to show that there exists some s_i so that $\pi \leq \angle_\varsigma(c, s_i) \leq (k-1)\pi$.
But we chose the sequence s_i in a way that we can always ensure the existence of such a saddle connection. □

A further tool in the universal cover are shadows of singularities. Shadows have already been defined in section 2.2.2.

Lemma 3.6. *Let $\tilde{S}$ be the universal cover of a flat surface and let $\tilde{x} \neq \tilde{\varsigma}$ be points in $\tilde{S}$. We require that $\tilde{\varsigma}$ is a singularity. The shadow $\partial sh_{\tilde{x}}(\tilde{\varsigma})$ contains an open subset of the boundary. It is a closed set.*

Proof. Let $[\tilde{x}, \tilde{\varsigma}]$ be the geodesic connecting $\tilde{x}$ with $\tilde{\varsigma}$. Let $\angle_{\tilde{\varsigma}}$ measure the flat angle at $\tilde{\varsigma}$ with respect to some orientation.
Let I be the subinterval of directions at $\tilde{\varsigma}$ such for each $\theta \in I$ the angle satisfies

$$\pi \leq \angle_{\tilde{\varsigma}}([\tilde{x}, \tilde{\varsigma}], \theta) \leq 2\pi$$

As the cone angle at ς is at least 3π, it follows that $\pi \leq \angle_{\tilde{\varsigma}}(\theta, [\tilde{x}, \tilde{\varsigma}])$.
Let $\tilde{c}$ be a ray issuing from $\tilde{\varsigma}$ in direction I. The concatenation $[\tilde{x}, \tilde{\varsigma}] * \tilde{c}$ is a geodesic.
We can choose a standard neighborhood U of $\tilde{\varsigma}$. The rays which issue from $\tilde{\varsigma}$ in direction of I sweep out a circle sector of U of circular angle π. We choose an open subset V of this sector.
By construction, any geodesic from $\tilde{x}$ to a point $\tilde{y} \in V$ passes through $\tilde{\varsigma}$. By the construction of the topology, $\partial sh_{\tilde{x}}(V) \subset \partial sh_{\tilde{x}}(\tilde{\varsigma})$ is an open subset of the boundary $\partial\tilde{S}$.
It remains to show, that $\partial sh_{\tilde{x}}(\varsigma)$ is a closed subset of $\partial\tilde{S}$. Denote by $\overline{D} := \overline{B_{\tilde{x}}(d(\tilde{x}, \tilde{\varsigma}))}$

the closed disc with center $\tilde{x}$ which contains $\tilde{\varsigma}$ on the boundary.
Recall the closest point projection $\pi_{\overline{D}} : \partial\tilde{S} \to \overline{D}$, see Proposition 2.5. A boundary point $\eta \in \partial\tilde{S}$ is mapped to $[\tilde{x}, \eta](d(\tilde{x}, \tilde{\varsigma}))$, the intersection of the boundary of $\overline{D}$ with the geodesic connecting the center of $\bar{D}$ and η.
Therefore

$$\partial sh_{\tilde{x}}(\tilde{\varsigma}) = \partial\tilde{S} \cap \pi_{\bar{D}}^{-1}(\tilde{\varsigma})$$

Since the closest point projection is a continuous map, the shadow is necessarily a closed set. □

Finally, we need some criterion for closed local geodesics α on S, to ensure that each complete lift of α in the flat universal cover is not quasi-straight.
By definition of a flat cylinder C, there exists a core curve c which is locally geodesic and does not pass through any singularity. Therefore, at each point $c(t)$ the angle of the ingoing and outgoing ray is exactly π, independent of the choice of orientation.
The converse also holds:

Lemma 3.7. *Let S be a closed flat surface and let α be a, not necessarily simple, closed local geodesic. Let $\tilde{\alpha}$ be a complete lift of α in the flat universal cover. α is freely homotopic to a core curve of a flat cylinder if and only if at one side of $\tilde{\alpha}$ the flat angle is π at each point.*

Proof. We refer to [DLR09, Lemma 17]. □

Proof of the Proposition. Let c, c' be geodesic arcs on S so that c ends at a singularity ς and c' issues from ς'. We have to show that there exists a local geodesic g of uniformly bounded length which first passes through c and eventually through c'.
We first show the existence of such a connecting local geodesic g. Let α be a closed local geodesic in S which is not freely homotopic to a simple closed curve.
We claim that there exists a local geodesic g which first passes through c and then through α.
Take a complete lift $\tilde{\alpha}$ of α in the universal cover. We claim that $\tilde{\alpha}$ is not quasi straight.
Let $\tilde{S}^{\pm}$ be the components of $\tilde{S} - \tilde{\alpha}$.
By Lemma 3.7, there is a point $\tilde{p}_+ = \tilde{\alpha}(t_+)$ so that the angle $\vartheta^+(\tilde{p})$ at $\tilde{p}$, measured inside $\tilde{S}^+$, satisfies $\vartheta^+(\tilde{p}) > \pi$. Moreover, there is a point $\tilde{p}_- = \tilde{\alpha}(t_-)$ so that $\vartheta^-(\tilde{p}_-) > \pi$.
$\tilde{\alpha}$ is preserved by a cyclic subgroup $< \gamma_\alpha >< \Gamma$ of the Deck transformation group. Up to replacing γ_α by γ_α^2, we can assume that γ_α preserves $\tilde{S}^+$.
It follows that

$$\vartheta^+(\gamma_\alpha^k(\tilde{p}_+)) = \vartheta^+(\tilde{p}_+), \forall k$$

$$\vartheta^-(\gamma_\alpha^k(\tilde{p}_-)) = \vartheta^-(\tilde{p}_-), \forall k$$

Consequently $\tilde{\alpha}$ is not quasi-straight. Moreover, the geodesic line $\tilde{\alpha}'(t) := \tilde{\alpha}(-t)$ is not quasi-straight either.

We also choose a lift $\tilde{c} : [0, T] \to \tilde{S}$ of c. The shadow $sh_{\tilde{c}(0)}(\tilde{c}(T))$ contains an open subset of the boundary. The visual metric boundary of the hyperbolic plane and the Gromov boundary of the flat metric are homeomorphic and the Deck transformation group Γ acts on both boundaries in same topological way. It is well-known that each element in a cocompact Fuchsian group acts with north-south dynamics on the boundary of the Poincaré disc and that the attracting fixed points are dense, see i.e. [Kat92].

Therefore, up to translating $\tilde{\alpha}$ with an element $\gamma \in \Gamma$, we can assume that the endpoints of $\tilde{\alpha}$ are in the interior of $\partial sh_{\tilde{c}(0)}(\tilde{c}(T))$. Let η be the positive endpoint of $\tilde{\alpha}$. The geodesic ray $[\tilde{c}(0), \eta]$ has to pass through $\tilde{c}$ first. Since $\tilde{\alpha}$ is not quasi-straight, the geodesic ray $[\tilde{c}(0), \eta]$ and the line $\tilde{\alpha}$ eventually coincide along a subray of $\tilde{\alpha}$.

Analogously let ζ be the positive endpoint of the geodesic line $\tilde{\alpha}'$ which is the negative endpoint of $\tilde{\alpha}$. The geodesic ray $[\tilde{c}'(0), \zeta]$ passes through $\tilde{c}'$. Moreover, $\tilde{\alpha}'$ and $[\tilde{c}(0), \zeta]$ eventually coincide.

We constructed a geodesic $\tilde{g}_1$ in $\tilde{S}$ which first passes through $\tilde{c}$ and eventually coincides with a subray of $\tilde{\alpha}$. We truncate $\tilde{g}_1$ to a compact segment so that $\tilde{g}_1$ eventually coincides with subsegment of $\tilde{\alpha}$ of positive length and ends at a preimage of $\alpha(0)$. The projection of $g_1 := \pi(\tilde{g}_1)$ is a local geodesic in S which first passes through c and eventually through the closed curve α. The endpoint of g_1 is $\alpha(0)$. In the same manner one can construct a local geodesic g_2 which begins in c' and which passes through $\alpha'(t) := \alpha(-t)$. The endpoint of g_2 is also $\alpha(0)$. The concatenation of the local geodesics $g := g_1 * g_2^{-1}$ is an arc which first passes through c and eventually through c'. Outside the point of transition $g_1 * g_2^{-1}$, is a local geodesic. At $\alpha(0)$ the arc $g_1 * g_2^{-1}$ locally coincides with α and is therefore locally geodesic as well.

It remains to show that we can choose such a local geodesic g with uniformly bounded length. We choose a set of saddle connections s_i on the flat surface S with following property:

Any local geodesic whose endpoint is a singularity can be concatenated with some saddle connection s_i to an extended local geodesic. The number of singularities on S is finite and by Lemma 3.5 at each singularity it suffices to choose 4 saddle connections. Therefore, the total number of such chosen saddle connections is finite. For each pair of saddle connections s_i, s_j, we construct the local geodesic $g_{i,j}$ which first passes through s_i and eventually through s_j. Since there is only a finite number of such pairs, the length of $g_{i,j}$ is bounded from above by a constant $C_l(S)$.

For any two geodesic arcs c, c' there are two saddle connections s_i, s_j such that the concatenation $c * s_i$ and $s_j * c'$ is a local geodesic. So $g := c * g_{ij} * c'$ is also a local geodesic of length

$$l(c * g_{ij} * c) \leq C_l(S) + l(c) + l(c')$$

□

Corollary 3.2. *Let S be a closed flat surface. Let $x \in S$ be a point and s be a saddle connection. There exists a local geodesic c starting at x which passes through s. The length of c is bounded by $l(c) \leq diam(S) + C_l(S) + l(s)$.*

Proof. There is some local geodesic connecting x with a singularity ς of length at most $diam(S)$. As explained above, one finds a local geodesic connecting $[x, \varsigma]$ with s. □

We will use $C_l(S)$ as a universal constant for the flat metric.
Finally, we discuss the intersections of closed local geodesics on flat surfaces. Let (X, d) be a closed surface endowed with a metric d which is either a hyperbolic metric or a flat metric.
Let $[\alpha]$ be a free homotopy class of closed curves in X. We define:

$$l_d([\alpha]) := \min_{\alpha' \in [\alpha]} l_d(\alpha')$$

Since in each free homotopy class of closed curves, there exist length-minimizing geodesic representatives α_d the minimum is attained:

$$l_d([\alpha]) = l_d(\alpha_d)$$

If d is a flat metric the representative α_d is not necessarily unique.
Analogous let $[c]$ be a homotopy class of arcs with fixed endpoints in X.

$$l_d([c]) := \min_{c' \in [c]} l_d(c)$$

In any such homotopy class there exists a unique length-minimizing geodesic representatives c_d.

$$l_d([c]) = l_d(c_d)$$

A useful property of hyperbolic metrics is the following: Let α, β be closed curves on a orientable closed topological surface X of negative Euler characteristics and let $[\alpha], [\beta]$ be their free homotopy classes. The geometric intersection number $i([\alpha], [\beta])$ is the minimal number of intersection points of α', β' where $\alpha' \in [\alpha], \beta' \in [\beta]$. With respect to any hyperbolic metric σ on X, in $[\alpha]$,$[\beta]$ there are unique geodesic representatives $\alpha_\sigma, \beta_\sigma$.

The number of intersection points of $\alpha_\sigma, \beta_\sigma$ equals the geometric intersection number of the corresponding free homotopy classes.
On flat surfaces the following analogon holds:

Lemma 3.8. *Let α, β be closed curves on a flat surface S and α_q, β_q be a choice of locally geodesic representatives in the free homotopy class. If the number of intersection points of α_q, β_q is bigger than $i([\alpha],[\beta])$, the local geodesics α_q and β_q share some arcs which start and end at singularities. We also allow degenerated arcs which are singularities.*

Proof. Assume that α_q and β_q have more points in common than $i(\alpha, \beta)$. Either α_q and β_q share some arc or there is a bigon bounded by an arc in α_q and β_q. This bigon lifts to a geodesically bounded bigon in the universal cover. Since $Cat(0)$-spaces are uniquely geodesic, this is impossible. So α_q,β_q share some arc. As local geodesics in flat surfaces are straight line segments outside the singularities, two local geodesics having some arc in common can drift apart at most at singularities. So they can share at most arcs with singularities as start- and endpoints. □

The Lemma shows that it is possible to homotope α_q and β_q slightly away from their common saddle connection such that the length only changes by an arbitrary small amount and the number of intersections of the homotoped curves is minimal.

3.6 Decomposition of flat surfaces

3.6.1 Removing long cylinders

In this section we show same standard facts about decompositions of flat surfaces. Most of the concepts can be found in [MS91, Section 5].
The following Lemma is due to [MS91, Lemma 5.1, Lemma 5.2].

Lemma 3.9. *Let S be a flat surface and $\pi : \tilde{S} \to S$ the flat universal cover. Let $\tilde{x} \in \tilde{S}$ be a regular point. Take $\tilde{D}$ a disc of radius r and center $\tilde{x}$ which does not contain a singularity. Let $\tilde{D}'$ be the subdisc of radius $r/2$ and center $\tilde{x}$. Assume that $\tilde{D}'$ does not embed into S.*
Then $\pi(\tilde{x})$ lies in a flat cylinder, which has circumference at most r and height at least $\sqrt{\frac{3}{4}}r$.

Proof. We only sketch the proof.
Since $\tilde{D}'$ does not contain a singularity and does not embed, there are two points in $\tilde{D}'$ which lie in the same Γ-orbit. Take the two closest points and let $\tilde{c}$ be the connecting geodesic in $\tilde{D}'$. $\tilde{c}$ does not pass through a singularity and therefore does not change

direction. So it projects to a simple closed local geodesic. As $\tilde{D}$ is a euclidean disc, one can transport $\tilde{c}$ in direction of the normal bundle. One can proceed at least until one reaches the boundary of $\tilde{D}$. The projection of this variation is a flat cylinder. Using euclidean geometry one computes that the height of this cylinder is at least $\sqrt{\frac{3}{4}}r$. □

From Lemma 3.9 one easily deduces:

Lemma 3.10. *Let S be a closed flat surface of area 1 and genus $g \geq 2$. Let Σ_S be the set of singularities on S. There exists some uniform constant c_{height} such that the following holds:*

i) Each point $x \in S$ with $d(x, \Sigma_S) > \sqrt{\frac{4}{3}}c_{height}$ is contained in a maximal flat cylinder of height at least c_{height}.

ii) Any two maximal flat cylinders of height at least c_{height} are either disjoint or equal.

Proof. We refer to [MS91, Theorem 5.3] □

Let S be a closed flat surface of area 1 and genus $g \geq 2$. Let $\bigcup C_i$ be the disjoint union of all maximal flat cylinders C_i each of height at least c_{height}. By Lemma 3.10, $\bigcup C_i$ contains all points which are of distance at least $\sqrt{\frac{4}{3}}c_{height}$ to each singularity. Since the core curves of different maximal flat cylinders are simple, disjoint and pairwise not freely homotopic, the number of such cylinders is bounded from above by a constant which only depends on the topology of S.
For each such cylinder C_i we choose $C_i^* \subset C_i$ the closed central subcylinder of C_i so that $C_i - C_i^*$ consists of two flat cylinders both of height $\frac{c_{height}}{3}$. Let $\bigcup S_j := S - \bigcup_i C_i^*$ be the complement of the cylinders.

Definition 3.4. *We call $\bigcup S_j$ the thin-cylinder decomposition of S.*

Proposition 3.12. *Let S be a closed flat surface of area 1 and let $\bigcup S_j$ be the thin-cylinder decomposition of S.*
Each connected component S_j is an open subsurface with locally geodesic boundary. The boundary is disjoint from the singularities.
The diameter of S_j is bounded from above by some constant c_{diam} which only depends on the topology of S.

Proof. Let C_i be the maximal cylinder in the homotopy class of C_i^*. By construction of C_i^*, each point in $S - \bigcup_i C_i$ is of distance at most $\sqrt{\frac{4}{3}}c_{height}$ to a singularity.
Each connected component of $C_i - C_i^*$ is a flat cylinder of height $\frac{c_{height}}{3}$ and of circumference at most c_{height}^{-1} as the area of C_i is at most 1 and the height of C_i is at least

c_{height}. Therefore, each component of $C_i - C_i^*$ is of diameter at most $\frac{c_{height}}{3} + c_{height}^{-1}$. Moreover, the boundary of each component of $C_i - C_i^*$ contains a singularity.
Consequently, the distance of each point in $S - \bigcup C_i^*$ to a singularity is uniformly bounded by a constant which only depends on c_{height}. The number of singularities on S is bounded from above by a constant which only depends on the topology of S. So, the diameter of each connected component $S_j \subset S - \bigcup C_i^*$ is bounded from above by some uniform constant.
To show that S_j is geodesically bounded, one observes that each boundary component is the geodesic core curve of some cylinder C_i. □

$\bigcup_j S_j$ is obtained from S by removing disjoint annuli. No core curves of different removed annuli are freely homotopic in S. Therefore, each subsurface S_j is of negative Euler characteristic.

Lemma 3.11. *Let $S_j \subset S$ be a component of the thin-cylinder decomposition of a closed surface S of genus $g \geq 2$ and area 1.*
Let α be some essential closed curve in S_j which might be freely homotopic to a boundary component. There exists some locally geodesic representative $\alpha_q \subset S$ in the free homotopy class of α which is contained in S_j.

Proof. Assume first that α is freely homotopic to the multiple of some boundary component. Each boundary component of S_j is the core curve of some cylinder C_i and there exists a geodesic core curve of C_i in S_j.
Assume next that α is not freely homotopic to the boundary. α cannot not be realized disjointly from S_j as otherwise α were in the free homotopy class of the multiple of some boundary component. Therefore, the geodesic representative α_q of α intersects S_j.
It remains to show that each geodesic representative does not intersect the boundary of S_j. Let β_q be a boundary component of S_j which is the locally geodesic core curve of some cylinder C_i. By definition $i(\alpha_q, \beta_q) = 0$. By Lemma 3.8 two intersecting locally geodesic closed curves, which are freely homotopic to disjoint curves, intersect each other at least at some singularities. Since β_q does not contain any singularity α_q is disjoint from each boundary component of S_j. □

Lemma 3.12. *Let S be flat surface of genus $g \geq 2$ and area 1. Denote by $\bigcup S_j$ the thin-cylinder decomposition of S and let c_{height} be the constant as in Lemma 3.10.*
Then, for each closed curve $\alpha \subset S$ of length less than c_{height} there exists a length minimizing representative α_q of the free homotopy class $[\alpha]$ which is a contained in some subsurface S_j.

Proof. We can assume that α is length-minimizing in its free homotopy class of closed curves and therefore locally geodesic. Let C_j be a maximal cylinder which contains a component of the complement $S - \bigcup S_j$. Let β be a geodesic core curve of C_j. C_j is isometric to a flat cylinder and α a local geodesic. If α intersects β then it has to cross through C_j and has length at least c_{height}, which is impossible. Therefore, α is freely homotopic to some curve which is disjoint from the core curves of all cylinders and therefore can be homotoped in some subsurface S_j.
By Lemma 3.11, there exists a geodesic representative α_q in the free homotopy class of α which is contained in S_j. □

3.6.2 Length of simple closed curves in the hyperbolic metric and the Rafi thick-thin decomposition

Finally, we compare the length of simple closed curves with respect to the flat and hyperbolic metric in the same conformal class.
Let $S = (X, d_q)$ be a closed flat surface of genus $g \geq 2$. Let (X, σ) be the hyperbolic metric in the same conformal class as d_q.
Let $[\alpha]$ be a free homotopy class of closed curves in X. We recall the notation

$$l_*([\alpha]) := \min_{\alpha' \in [\alpha]} l_i(\alpha'), * = q, \sigma$$

Since in each free homotopy class of closed curves, there exist a length-minimizing geodesic representative $\alpha_*, * = q, \sigma$ the minimum is attained:

$$l_*([\alpha]) = l_*(\alpha_*), * = q, \sigma$$

The length $l_q([\alpha])$ can be bounded by the length $l_\sigma([\alpha])$.

Proposition 3.13. *Denote by f the function*

$$f : \mathbb{R}_+ \to \mathbb{R}_+, x \mapsto \frac{1}{2} x \exp\left(\frac{x}{2}\right)$$

Let $S = (X, d_q)$ be a closed flat surface of genus $g \geq 2$ and area 1. Let (X, σ) be the hyperbolic metric in the same conformal class.
Let $[\alpha]$ be a free homotopy class of a simple closed curve. The length $l_q([\alpha])$ can be estimated by $l_\sigma([\alpha])$.

$$l_q([\alpha]) \leq f(l_\sigma([\alpha]))$$

Proof. The proof uses extremal length. We refer to [Mas85, Corollary 3]. □

Remark 3.3. *The converse does not hold: There exists a family of flat surfaces* $S_i = (X_i, d_{q,i})$ *of genus* g *and a sequence of simple closed local curves* $\alpha_i \in S_i$ *so that* $\lim_i l_{q,i}([\alpha_i]) = 0$*. The length of the corresponding hyperbolic geodesic representatives* $l_{\sigma,i}([\alpha_i])$ *in the hyperbolic metric is bounded from below by a positive constant.*

For details we refer to the family of examples in [Raf07, Section 5].
In his work [Raf07] compared the length of simple closed local geodesics in terms of the hyperbolic resp. the flat metric.
The following facts concerning the geometry of hyperbolic surfaces are standard:

Lemma 3.13. *Let* (X, σ) *be a finite-volume hyperbolic surface with cusps.*

i) Around any simple closed local geodesic α_σ *there exists a neighborhood which is an equidistant convex collar. The distance from the boundary to the locally geodesic core curve is bounded from below by a function* $r(l_\sigma(\alpha_\sigma))$ *which is independent from the surface* (X, σ)*.* r *is decreasing and unbounded.*

ii) There exists some small $\epsilon > 0$ *which only depends on the topology of* X *with the following property:*
Let $X_< \subset X$ *be the set of points with injectivity radius of at most* ϵ*. By the Margulis Lemma,* $X_<$ *is a disjoint union of annuli. Each connected component is convex. We call* $X_> := X - X_<$ *the thick part of the surface. The diameter of each connected component of* $X_>$ *is bounded from above by a constant which again only depends on the topology of* X*.*

iii) There exists a maximal collection of pairwise disjoint non-peripheral not free-homotopic simple closed curves in the hyperbolic surface (X, σ) *with following properties:*
The length of each curve is bounded from above by a constant which only depends on the Euler characteristic $\chi(X)$*. Each connected component in the complement is homeomorphic to a 3-punctured sphere, a so-called pair of pants. Such a decomposition by curves of uniformly bounded length is called a Bers-short pants-decomposition.*

iv) Let ϵ *be the constant from ii) and let* $(X_>, X_<)$ *be the* ϵ *thick-thin decomposition of* X*. There exists a Bers-short pants decomposition* α_i *of* X *so that the core curve of each* ϵ*-thin component is part of the pants decomposition. Let* $Y \subset X_>$ *be a component of the thick part. Each pants curve* α_i *is either disjoint from* Y *or contained in* Y*. Let* $\alpha_i \subset Y$ *be such a pants curve. There exists a second curve*

$\beta_i \subset Y$ that intersects α_i essentially. The length of β_i is bounded from above by some uniform constant.

Proof. Since the statements are classic, we only refer to the references

i) The so-called Collar Lemma, was shown in [Kee74].

ii) We choose ϵ smaller than the constant of the Margulis Lemma which only depends on the topology of X. The fact that $X_<$ is a disjoint union of annuli can be found in [Thu80], [BP92].

iii) The existence of a short pants decomposition can be found in [Ber85].

iv) Let $(X_>, X_<)$ be the ϵ thick-thin decomposition. The core curves α_i of each component of the thin part are of length at most 2ϵ. The core curves can be extended to a Bers short pants decomposition.
Let $\alpha_i \subset X_>$ be a pants curve in the thick part. The existence of a short intersecting curve $\beta_i \subset X_>$ was shown in [MM00], see also [Raf07, Section 3.1].

□

Let ϵ be the Margulis constant as in Proposition 3.13. Let $S = (X, d_q)$ be a flat surface and σ the corresponding hyperbolic metric on the Riemann surface X which is in same conformal class as d_q.
With respect to the hyperbolic metric σ and the Margulis constant ϵ, let $(X_>, X_<)$be the thick thin decomposition and let Y be a connected component of the thick part $X_>$. There is a unique subsurface Y_q in the homotopy class of Y which has locally geodesic boundary with respect to the flat metric d_q so that for each boundary curve of Y there is a unique q-geodesic representative in Y_q.
Y_q might be degenerated, so it might be a graph. Nevertheless, let α be a simple closed curve which can be homotoped into Y. At least one locally geodesic representative in the flat metric α_q is entirely contained in Y_q.
Assume that Y is not a topological pair of pants. We define $\lambda(Y)$, the q-size of Y, as the flat length of the shortest essential non-peripheral simple closed local geodesic on Y_q.
If Y is a topological pair of pants, $\lambda(Y)$ is defined as the maximal flat q-length of boundary components.

Proposition 3.14. *Let $S = (X, d_q)$ be a closed flat surface and σ the hyperbolic metric on X in the same conformal class as d_q. There exists some constant $c > 0$, which only* *depends on the topology of X, such that for any non-peripheral simple closed curves α, β in Y it follows that:*

$$c^{-1}\lambda(Y_q)l_\sigma([\alpha]) < l_q([\alpha]) < c\lambda(Y_q)l_\sigma([\alpha])$$

Moreover, the diameter of each connected component is comparable to λ.

$$c^{-1}\lambda(Y) \leq diam(Y_q) \leq c\lambda(Y)$$

Proof. [Raf07] □

4 Hausdorff dimension and entropy

Let S be a closed flat surface and $\pi : \tilde{S} \to S$ the flat universal cover. We showed that $\tilde{S}$ is a δ-hyperbolic $Cat(0)$-space. We recall the notation of Section 2.3.1.
Let $\delta_{inf}(\tilde{S})$ be the infimum of all δ' so that $\tilde{S}$ is δ'-hyperbolic. We defined the continuous function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$. Denote $\xi := \frac{1}{2}\xi(\delta_{inf})$.
There exists a family of Gromov metrics $d_{\tilde{x},\xi}, \tilde{x} \in \tilde{S}$ with respect to the base point $\tilde{x}$. Any two of such metrics are bilipschitz equivalent.
We investigate the Hausdorff dimension of the boundary with respect to such a metric. Since Hausdorff dimension is invariant under bilipshitz equivalence, we skip the base point and abbreviate

$$d_\infty := d_{\xi,\tilde{x}}$$

4.1 Hausdorff dimension in moduli space and asymptotic behavior

Lemma 4.1. *Let S be a closed flat surface and $\pi : \tilde{S} \to S$ the flat universal cover. Let d_∞ be a Gromov metric on the boundary. The Hausdorff dimension of the boundary $(\partial\tilde{S}, d_\infty)$ is at least* 1.

Proof. The Gromov boundary is a homeomorphic to the boundary of the Poincaré disc and therefore a topological circle. Consequently, the topological dimension of $(\partial\tilde{S}, d_\infty)$ is 1. The topological dimension is a lower bound for the Hausdorff dimension, compare [Fal03] for details. Therefore, the Hausdorff dimension is at least 1. □

It is natural to ask how the Hausdorff dimension and the entropy vary under slight changes of the flat metric.
Let $\mathcal{Q}_g$ be the moduli space of flat structures which is the set of all isometry equivalence classes of flat surfaces of genus $g \geq 2$ and area 1. For rigorous statements and computations we refer to [Vee90].

Let $S \in \mathcal{Q}_g$ be a flat surface and $\epsilon > 0$.
We define

$$B_S(\epsilon) := \{S' \in \mathcal{Q}_g : \exists f : S' \to S, f \text{ is a } (1+\epsilon)-\text{bilipschitz homeomorphism}\}$$

The sets $B_S(\epsilon), S \in \mathcal{Q}_g, \epsilon > 0$ form a basis of the topology on Q_g.
We consider the following mappings

i) $\delta_{inf} : \mathcal{Q}_g \to \mathbb{R}$, the minimal Gromov hyperbolic constant of the flat universal cover $\tilde{S}$.

ii) $e(\tilde{S}, \Gamma_S) : \mathcal{Q}_g \to \mathbb{R}$ the entropy of the Deck transformation group Γ_S acting on the flat universal cover.

iii) $hdim : \mathcal{Q}_g \to \mathbb{R}$ the Hausdorff dimension of the Gromov boundary of the universal cover with respect to the Gromov metric d_∞.

iv) $l_0 : \mathcal{Q}_g \to \mathbb{R}$ the length of the shortest essential simple closed curve on the flat surface S.

Proposition 4.1. *The functions $\delta_{inf}, e(\tilde{S}, \Gamma_S), hdim$ and l_0 are continuous in moduli space.*

Recall that in section 2.3 we mentioned the existence of a function $r(K, L, \delta)$ with

$$\lim_{K \to 0, L \to 1} r(K, L, \delta) = \delta$$

so that the following property is satisfied :
Let $\phi : X \to Y$ be a (K, L)-quasi-isometry and Y be a δ-hyperbolic space. Then X is a $r(K, L, \delta)$-hyperbolic space.

Proof of the Proposition. Let S be a point in $\mathcal{Q}_g$ and let $S_i \in \mathcal{Q}_g$ be a sequence which converges towards S. Therefore, there exists a sequence of homeomorphisms $f_i : S \to S_i, g_i : S_i \to S$ which are L_i-bilipschitz, so that the constant L_i tends to 1. f_i, g_i lift to L_i-bilipschitz homeomorphism between the flat universal covers $\tilde{f}_i : \tilde{S} \to \tilde{S}_i, \tilde{g}_i : \tilde{S}_i \to \tilde{S}$.
We first show that δ_{inf} is continuous.
For each $\epsilon > 0$, $\tilde{S}$ is a $(\delta_{inf}(S)+\epsilon)$-hyperbolic space. Each flat covering $\tilde{S}_i$ is also Gromov hyperbolic with Gromov hyperbolic constant $r(0, L_i, \delta_{inf}(\tilde{S}) + \epsilon)$. Therefore

$$r(0, L_i, \delta_{inf}(S) + \epsilon) \geq \delta_{inf}(S_i)$$

$$r(0, L_i, \delta_{inf}(S_i) + \epsilon) \geq \delta_{inf}(S)$$

r tends to $\delta_{inf} + \epsilon$ if L_i tends to 1. Consequently

$$\lim_i \delta_{inf}(\tilde{S}_i) = \delta_{inf}(\tilde{S})$$

The entropy $e(\tilde{S}, \Gamma_S)$ and $e(\tilde{S}_i, \Gamma_{S_i})$ satisfies the relation

$$L_i e(\tilde{S}, \Gamma_S) \leq e(\tilde{S}_i, \Gamma_{S_i}) \leq L_i e(\tilde{S}, \Gamma_S)$$

and therefore

$$\lim_i e(\tilde{S}_i, \Gamma_{S_i}) = e(\tilde{S}, \Gamma_S)$$

By Proposition 2.1 the Hausdorff dimension of the boundary equals

$$hdim(S) = \frac{e(\tilde{S}, \Gamma_S)}{log\left(\frac{1}{2}\xi(\delta_{inf}(S))\right)}$$

Since ξ depends continuously on $\delta_{inf}(S)$, the Hausdorff dimension of the Gromov boundary is a continuous function in the moduli space $\mathcal{Q}_g$.
It remains to show that l_0 is continuous which follows from:

$$L_i^{-1} l_0(S) \leq l_0(S_i) \leq L_i l_0(S)$$

□

Consequently, in compact sets of moduli space the Hausdorff dimension of the Gromov boundary with respect to the metric d_∞ is bounded from above. It remains to investigate the quantities under the degeneration of the flat surface.
A sequence $S_i \in \mathcal{Q}_g$ is called divergent if it eventually leaves every compact subset of $\mathcal{Q}_g$.
The function $l_0 : \mathcal{Q}_g \to \mathbb{R}_+$, which measures the length of the shortest essential simple closed curve on the flat surface S, is continuous. Consequently, in compact subsets of $\mathcal{Q}_g$, l_0 has a positive lower bound.
It is a classical fact, see i.e. [Mas92, Proposition 1.2], that the converse also holds.

Proposition 4.2. *Let $\epsilon > 0$. The set $l_0^{-1}((\epsilon, \infty)) \subset \mathcal{Q}_g$ is contained in a compact subset of $\mathcal{Q}_g$.*

Therefore, a sequence $S_i \in \mathcal{Q}_g$ diverges if and only if $\lim_{i\to\infty} l_0(S_i) = 0$.
We will show that the entropy $e(\Gamma_i, \tilde{S}_i)$ tends to infinity if and only if the sequence S_i diverges.
We need the following Lemmas:

Lemma 4.2. *Let S be a closed flat surface of genus $g \geq 2$. Let $\alpha, \beta : [0,1] \to S$ be closed curves sharing at least one point $p := \alpha(0) = \beta(0) = \alpha(1) = \beta(1)$.*
Assume that the group $< \alpha, \beta >$, considered as a subgroup of $\pi_1(S,p)$, is neither cyclic nor trivial.
Let $a = l_q(\alpha)$ resp. $b = l_q(\beta)$ be the length of α resp. β. We assume that $a \leq b$.
The entropy of S can be estimated in terms of a, b:

$$e(\tilde{S}, \Gamma) \geq \max \left\{ \frac{1}{2b} \log \left(\frac{b}{a} \right), \frac{\log(2)}{b} \right\}$$

We emphasize that α, β are not necessarily local geodesics. The term length in this context actually means the length of α, β and not the length of geodesic representatives. Moreover, we do not require that α and β intersect transversely.

Proof. Since S is a closed surface of genus $g \geq 2$, it is a classical fact that the group $< \alpha, \beta >< \pi_1(S,p)$ is free , see i.e. [Jac70, Corollary 2].
As $< \alpha, \beta >$ is neither cyclic nor trivial, the positive semi-group $< \alpha, \beta >_s$ is free with respect to the generating system $\{\alpha, \beta\}$.
Let $\pi : \tilde{S} \to S$ be the universal cover of S and let $\tilde{p} \in \pi^{-1}(p)$ be a preimage of p. We can choose connected arcs $\tilde{\alpha}, \tilde{\beta} : [0,1] \to \tilde{S}$, emanating from $\tilde{p}$, so that $\pi \circ \tilde{\alpha} = \alpha, \pi \circ \tilde{\beta} = \beta$.
Let $\tilde{p}_\alpha := \tilde{\alpha}(1)$, resp. $\tilde{p}_\beta := \tilde{\beta}(1)$ be the endpoints. By definition of a, b,

$$d(\tilde{p}_\alpha, \tilde{p}) \leq a, d(\tilde{p}_\beta, \tilde{p}) \leq b$$

Let γ_α resp. γ_β be the element of the Deck transformation group which maps $\tilde{p}$ to $\tilde{p}_\alpha$ resp. $\tilde{p}_\beta$.
Let Φ be the canonical isomorphism of the positive semi-group of words with letters α, β to the semi-group $< \gamma_\alpha, \gamma_\beta >_s$ with distinguished generating system $\{\gamma_\alpha, \gamma_\beta\}$ which is defined as $\Phi(\alpha) := \gamma_\alpha, \Phi(\beta) := \gamma_\beta$.
Let $w = a_1 \dots a_n$ be a word which contains k times the letter α and $l = n - k$ times the letter β. Let w_i be the sub-word of w truncated after the i-th letter.
Due to the triangle inequality

$$d(\tilde{p}, \Phi(w)(\tilde{p})) \leq \sum_{i=1}^{n} d(\Phi(w_{i-1})(\tilde{p}), \Phi(w_i)(\tilde{p}))$$

Since Γ is a group of isometries,

$$d(\Phi(w_{i-1})(\tilde{p}), \Phi(w_i)(\tilde{p})) = d(\Phi(w_{i-1})(\tilde{p}), \Phi(w_{i-1}, a_i)(\tilde{p})) = d(\tilde{p}, \Phi(a_i)(\tilde{p}))$$

If follows that the distance between $\tilde{p}$ and its image under $\Phi(w)$ can be estimated by the following formula:

$$d(\tilde{p}, \Phi(w)(\tilde{p})) \leq ka + lb$$

Let $U(R)$ be the set of words so that the number of α-letters is $\lfloor R/a \rfloor$ and the number of β-letters is $\lfloor R/b \rfloor$. The function $\lfloor * \rfloor : \mathbb{R} \to \mathbb{Z}$ rounds down each number. The cardinality of $U(R)$ is

$$\begin{aligned} |U(R)| &= \binom{\lfloor R/a \rfloor + \lfloor R/b \rfloor}{\lfloor R/b \rfloor} \geq \frac{\prod_{i=1}^{\lfloor R/b \rfloor} (i + R/a - 1)}{\lfloor R/b \rfloor !} \geq \left(\frac{R/a - 1}{R/b} \right)^{R/b-1} \\ &= \left(\frac{b}{a} - \frac{b}{R} \right)^{R/b-1} \end{aligned}$$

Let $w \in U(R)$ be such a word. The distance $d(\tilde{p}, \Phi(w)\tilde{p})$ is at most $2R$. The Deck transformation group acts freely. Therefore, the counting function

$$N_{\tilde{p}}(R) := |\Gamma \tilde{p} \cap \overline{B_{\tilde{p}}(R)}|$$

is bounded below by

$$N(R) \geq |U(R/2)| \geq \left(\frac{b}{a} - \frac{2b}{R} \right)^{R/2b-1}$$

Consequently

$$e(\tilde{S}, \Gamma_S) \geq \frac{1}{2b} \log \left(\frac{b}{a} \right)$$

The other inequality is analogous. Since $a \leq b$, for any word w of length n the distance can be estimated by

$$d(\tilde{p}, \Phi(w)(\tilde{p})) \leq bn$$

Let $V(R)$ be the set of words of length $\lfloor R/b \rfloor$. The cardinality of $V(R)$ is $2^{\lfloor R/b \rfloor}$ and therefore

$$N(R) \geq 2^{\lfloor R/b \rfloor} \geq \frac{1}{2} 2^{R/b}$$

□

The following Corollary is an immediate consequence.

Corollary 4.1. *Let S_i be a sequence of flat surfaces and let α_i, β_i be two closed curves on S_i satisfying the conditions as in Lemma 4.2. If the length of α_i tends to zero and the length of β_i is bounded from above, then the entropy of S_i tends to infinity.*

Let S be a point in $\mathcal{Q}_g$. Assume that S contains a short simple closed curve α. It is the goal to show that there exists a curve β whose length is uniformly bounded from above so that α, β satisfy the conditions of Lemma 4.2.
We recall the thin-cylinder decomposition, which was defined in section 3.6.1.

There exists a finite union of open subsurfaces $\bigcup S_j \subset S$ of negative Euler characteristic. The closures of any two different subsurfaces S_{j_1}, S_{j_2} are disjoint. The diameter of each connected component S_j is bounded from above by some constant c_{diam} which only depends on the topology of S.
The complement $S - \bigcup S_j$ is contained in a disjoint union of maximal flat cylinders $\bigcup C_i$. The height of each cylinder C_i is bounded from below by a uniform constant c_{height} which also only depends on the topology of S.
None of the cylinders C_i is entirely contained in the complement. Assume that C_i intersect some subsurface S_j. Then S_j contains a geodesic core curve of C_i.
For each closed curve α which is contained in a subsurface S_j, there exists a geodesic representative α_q in the free homotopy class of α which is also contained in S_j.
Moreover, let α be a closed curve in S whose length is less than c_{height}. There exists a geodesic representative α_q in the free homotopy class of α, which is contained in some subsurface S_j.

Proposition 4.3. *Let $S = (X, d_q) \in Q_g$ be a closed flat surface of genus $g \geq 2$ and area 1. Let $\bigcup S_j$ be the thin-cylinder decomposition of S. Let α be a simple closed curve in S_1 which might be freely homotopic to a boundary curve. There exists some uniform constant $c > 0$ which only depends on the genus of X and a simple closed local geodesic $\beta_q \subset S$ which intersects S_1. The length of β_q is bounded from above by c. No multiple of β_q is in the free homotopy class of any multiple of α in S.*

Proof. Denote by X the underlying Riemann surface of $S = (X, d_q)$. Let σ be hyperbolic metric on X which is in the same conformal class as the flat metric d_q.
Recall that on a hyperbolic surface (X, σ) there exists a short pants decomposition with pants curves $\beta_{\sigma,i}$, see section 3.6.2. The hyperbolic length of each pants curve $\beta_{\sigma,i}$ is bounded from above by the Bers constant which only depends on the genus of X. Let $\beta_{q,i}$ be a flat geodesic representative in the free homotopy class of $\beta_{\sigma,i}$. We showed in Proposition 3.13 that the flat length of $\beta_{q,i}$ is bounded from above by a uniform constant c.
Assume first that there is a pants curve $\beta_{\sigma,i}$ which intersects S_1 essentially, but cannot be homotoped inside of S_1. No multiple of $\beta_{\sigma,i}$ is in the free homotopy class of a multiple of α and the flat geodesic representative $\beta_{q,i}$ also intersects S_1.
If such a pants curve does not exist, recall that the Euler characteristic of S_1 is negative. There are at least two pants curves $\beta_{\sigma,i_1}, \beta_{\sigma,i_2}$ which can be homotoped inside of S_1, possibly as boundary components.
By Lemma 3.11 the flat geodesic representatives $\beta_{q,i_j}, j = 1, 2$ are contained in S_1. One observes that for at least one i no multiple of $\beta_{q,i}$ is freely homotopic to a multiple of

α. ☐

Theorem 4.1. *The entropy is bounded from below by a positive constant.*
A sequence of flat surfaces diverges in $\mathcal{Q}_g$ if and only if the entropy tends to infinity.

Proof. Recall that the Hausdorff dimension of the boundary is at least 1 with respect to the Gromov metric d_∞. It was shown in Proposition 3.7 that the minimal Gromov hyperbolic constant δ_{inf} is bounded from below by a positive constant which only depends on the topology of S. By Theorem 2.1 the entropy is related to these quantities by

$$log\left(\frac{1}{2}\xi(\delta_{inf})\right) hdim(\partial\tilde{S}, d_\infty) = e(\tilde{S}, \Gamma_S)$$

Therefore, the entropy is bounded from below by some uniform positive constant.
Let S_i be a sequence of flat surfaces. The entropy depends continuously on the point in $\mathcal{Q}_g$. If the entropy $e(\tilde{S}_i, \Gamma_{S_i})$ tends to infinity, the sequence eventually leaves every compact set.
On the other hand, assume that the sequence S_i leaves every compact set. Equivalently, there exists a sequence of essential simple closed curves α_i in S_i so that the length of α_i tends to zero. Let

$$\bigcup_j S_{i,j} \subset S_i$$

be the thin-cylinder decomposition of S_i.
Assume that α_i is shorter than c_{height} the minimal height of the removed cylinders. By Lemma 3.12, α_i is contained in some subsurface $S_{i,j}$. By Proposition 4.3, there exists an essential curve β_i' which intersects $S_{i,j}$ and no multiple of β_i' is freely homotopic to a multiple of α_i. The length of β_i' is smaller than some uniform constant c.
The diameter of $S_{i,j}$ is bounded from above by some uniform constant c_{diam}. We choose a short excursion from β_i' to α_i of length at most c_{diam}. We concatenate β_i' with the excursion and call the resulting closed curve β_i.
The length of β_i is uniformly bounded and the two curves α_i, β_i satisfy the conditions of Lemma 4.2.
By Corollary 4.1 the entropy of $e(\tilde{S}_i, \Gamma_{S_i})$ tends to infinity. ☐

Corollary 4.2. *Let $S_i \in \mathcal{Q}_g$ be a sequence of flat surfaces. The Hausdorff dimension of the boundary $hdim(S_i)$, with respect to the Gromov metric d_∞, tends to infinity if and only if the sequence S_i diverges.*

Proof. The function *hdim* is bounded from below by 1.
Let $S_i \in \mathcal{Q}_g$ be a sequence of surfaces.

$hdim$ is a continuous function and therefore S_i diverges if $hdim(S_i)$ tends to infinity. On the other hand, assume that S_i diverges. The entropy $e(\tilde{S}_i, \Gamma_{S_i})$ tends to infinity. By Proposition 2.1 the Hausdorff dimension of the boundary equals

$$hdim(S_i) = \frac{e(\tilde{S}_i, \Gamma_{S_i})}{log\left(\frac{1}{2}\xi(\delta_{inf}(\tilde{S}_i))\right)}$$

By Proposition 3.7 the function δ_{inf} is uniformly bounded from below and therefore

$$\frac{1}{2}\xi(\delta_{inf}(S)) = 2^{\frac{1}{2\delta_{\inf}}}$$

is bounded from above.
Consequently, $hdim(S_i)$ tends to infinity. □

4.2 Hausdorff dimension under branched coverings

Let $\pi : Y \to X$ be a holomorphic branched covering of Riemann surfaces. The number of sheets and the number of branch points together with the branching index is called the combinatorics of the covering.
Holomorphic branched coverings satisfy the Riemann Hurwitz formula.

Proposition 4.4. *Let $\pi : Y \to X$ be an n-sheeted holomorphic branched covering between compact Riemann surfaces. Let $ind(y)$ be the branching index at the point $y \in Y$ and $\chi(X)$ be the Euler characteristics of X. The Euler characteristic of X and Y are related by*

$$\chi(Y) = n\chi(X) - \sum_{y \in Y} ind(y)$$

The Euler characteristics of a closed Riemann surface is $2-2g$ for some integer $g \in \mathbb{N}$. Consequently, for some kind of combinatorics there cannot exist a branched covering which realizes the combinatorics. For instance, let X be a Riemann surface. There is no even-sheeted branched covering $\pi : Y \to X$ with only one branch point.
On the other hand, it was shown by [Hus62], [EKS84] that for closed surfaces of positive genus the Riemann-Hurwitz formula is the only constraint on the existence of a holomorphic branched covering.

Proposition 4.5. *Let X be a compact Riemann surface of genus $g \geq 1$ and let $x_1 \cdots x_k$ be distinct points of X.*
Let $m_{i,j}, i = 1 \dots k, j = 1, \dots, l_i$ be positive numbers so that $\sum_{i,j} m_{i,j}$ is even. Let $n \geq \max_i \sum_j (m_{i,j} + 1)$. There exists an n-sheeted holomorphic branched covering $\pi : Y \to X$ with marked points $y_{i,j} \in Y$ so that $\pi(y_{i,j}) = x_i$ and $ind(y_{i,j}) = m_{i,j}$.

Let $\pi : Y \to X$ be a holomorphic branched covering.
Assume that X is endowed with a flat metric $S := (X, d_{q_X})$. The metric d_{q_X} can be pulled back to a flat metric d_{q_Y} on Y. Denote by $T := (Y, d_{q_Y})$ the flat surface.
The induced covering map $\pi : T \to S$ is called a flat branched covering which is locally isometric outside the branch points. The preimages of the singularities on S and the set of branch points form the singularities on T. Since the flat covering is locally isometric, the volume of T is the product of the number of sheets and the volume of S.
We investigate how the entropy and the minimal Gromov hyperbolic constant of the flat universal covers of S and T are related.

Proposition 4.6. *Let $\pi : T \to S$ be an n-sheeted branched flat covering with k branch points.*
Let $\tilde{S}$ resp. $\tilde{T}$ be the flat universal cover of S resp. T. Denote by $\delta_{inf}(\tilde{S})$ resp. $\delta_{inf}(\tilde{T})$ the minimal Gromov hyperbolic constant of $\tilde{S}$ resp. $\tilde{T}$.
Then $\delta_{inf}(\tilde{T})$ can be estimated in terms of $\delta_{inf}(\tilde{S})$ and k.

$$\frac{\delta_{inf}(\tilde{S})}{24k} \le \delta_{inf}(T) \le 4\delta_{inf}(\tilde{S})$$

Proof. Denote by Σ_S resp. Σ_T the set of singularities on S resp. T.
Recall that by Proposition 3.7 the quantity $\rho(S) := \sup_{x \in S} d(x, \Sigma_S)$ and the quantity $\delta_{inf}(\tilde{S})$ are nearly equal.

$$\frac{\rho(S)}{2} \le \delta_{inf}(\tilde{S}) \le 2\rho(S)$$

We showed that $\rho(\tilde{S}) = \rho(S)$.
Let $y \in T$ be a point and let $x = \pi(y) \in S$ be the image of y. Let $\varsigma \in S$ be a closest singularity to x and let $[x, \varsigma]$ be a shortest connecting geodesic. One can choose a lift $[y, \varsigma']$ of $[x, \varsigma]$ which is maximal with respect to the property that it does not contain a branch point. Since branch points are singularities and since the flat covering is locally isometric, $d(y, \Sigma_T) \le d(\pi(y), \Sigma_S)$. Consequently,

$$\rho(T) \le \rho(S)$$

On the other hand, let $B_T \subset T$ be the set of all branch points on T. Let $B_S := \pi(B_T) \subset S$ be the image of B_T. The cardinality of B_S is at most k. Let $B_{\tilde{S}} := \pi^{-1}(B_S) \subset \tilde{S}$ be the preimage of B_S in the flat universal cover.
There is a disc $\tilde{D}_1$ of radius $\rho(\tilde{S}) = \rho(S)$ in $\tilde{S}$ which does not contain a singularity.

$$\tilde{D}_1 \cap \Sigma_{\tilde{S}} = \emptyset$$

Let $\tilde{x}$ be the center of $\tilde{D}_1$. Let $\tilde{D}_2 \subset \tilde{D}_1$ be the sub-disc of radius $\frac{\rho(S)}{2}$ and center $\tilde{x}$. We have to distinguish two cases.

- The projection of $\tilde{D}_2$ embeds into S.
Since $\tilde{D}_2$ is an euclidean disc, the volume of any subdisc of $\tilde{D}_2$ of radius r equals πr^2. As $\tilde{D}_2$ embeds into S, there are at most k points of $B_{\tilde{S}}$ in $\tilde{D}_2$. Therefore, the discs of radius $r = \frac{\rho(S)}{3\sqrt{k}}$ around each point of $B_{\tilde{S}} \cap \tilde{D}_2$ do not cover $\tilde{D}_3 \subset \tilde{D}_2$, the subdisc of center $\tilde{x}$ and radius $\frac{\rho(S)}{3}$.
Consequently, there is a point $\tilde{x}' \in \tilde{D}_3$ which is of distance at least $\frac{\rho(S)}{3\sqrt{k}}$ to any point in $B_{\tilde{S}} \cap \tilde{D}_2$. Since $\tilde{x}'$ has also distance $\frac{1}{6}\rho(S)$ to the boundary of $\tilde{D}_2$, it has distance at least $\frac{\rho(S)}{6\sqrt{k}}$ to any point in $B_{\tilde{S}}$ and any singularity.
Let $x' := \pi(\tilde{x}') \in S$. By construction, the distance between x' and the set $B_S \cup \Sigma_S$ is again at least $\frac{\rho(S)}{6\sqrt{k}}$.
Let $y' \in \pi^{-1}(x') \subset T$ be a preimage of x'. Let Σ_T be the set of singularities on T. Σ_T consists of the branch points and preimages of singularities on S and therefore

$$\Sigma_T \subset \pi^{-1}(\Sigma_S \cup B_S) \subset T$$

Consequently,

$$\rho(T) \geq d(y', \Sigma_T) \geq d(x', \Sigma_S \cup B_S) \geq \frac{\rho(S)}{6\sqrt{k}} \geq \frac{\rho(S)}{6k}$$

- $\tilde{D}_2$ is immersed but not embedded. By Lemma 3.9, the projection of $\tilde{D}_2$ in S is contained in a cylinder C' of height at least $\frac{\sqrt{3}\rho(S)}{2}$. As the cardinality of B_S is at most k, there is a sub-cylinder $C \subset C'$ of height $\frac{\sqrt{3}\rho(S)}{2k}$ which does not contain an element of B_S. Therefore, one finds a point $x' \in C$ of distance $\frac{\sqrt{3}\rho(S)}{4k}$ to a singularity and to B_S.

In both cases

$$\rho(T) \geq \frac{\rho(S)}{6k}$$

Using the relationship between ρ and δ_{inf} one can estimate:

$$\frac{\delta_{inf}(\tilde{S})}{24k} \leq \frac{\rho(S)}{12k} \leq \frac{\rho(T)}{2} \leq \delta_{inf}(T) \leq 2\rho(T) \leq 2\rho(S) \leq 4\delta_{inf}(\tilde{S})$$

□

We construct a family of examples which show that the bounds are asymptotically sharp:
A Strebel differential is a flat surface which, after removing a finite union of saddle connections, is isometric to disjoint union of flat cylinders. We consider the subset of Strebel differentials which admit only one cylinder. The set of Strebel differentials with

one cylinder is dense in the moduli space of flat structures [Mas79].
Let S' be a Strebel differential with one cylinder of area 1. Assume that the core curve of the cylinder is horizontal. One can stretch the vertical component by a large factor λ and shrink the horizontal component by λ^{-1}. The resulting flat surface S is again a Strebel differential with one cylinder of area 1. The horizontal cylinder is of short circumference and large height h. The Gromov hyperbolic constant of the flat universal cover $\delta_{inf}(\tilde{S})$ is nearly $\frac{h}{2}$. We can distribute k points x_i on S so that each point in S has distance at most $\frac{2h}{k}$ to some point x_i. We can construct a flat branched covering $\pi : T \to S$, so that each preimage of each point x_i is a branch point and therefore a singularity.
In this special example one observes that the quantity $\delta_{inf}(\tilde{T})$ is nearly $\frac{\delta_{inf}(\tilde{S})}{k}$. The precise statement is summarized in the following remark:

Remark 4.1. *There exists a family of branched covering $\pi_i : T_i \to S_i, S_i \in Q_g, i \in \mathbb{N}$. The genus of S_i is fixed and the number of branch points on T_i equals i. The quantity $i \cdot \rho(T_i)$ is bounded from above independent of i.*

To compute the entropy we first consider the following example:
Let S be a flat surface and $D \subset S$ be a small euclidean disc. On D one can make a small horizontal slit of length l and take $n \geq 3$ copies $S_1, \dots S_n$ of S endowed with the same slit. The copies can be isometrically glued along the slit. The right-hand side of the slit in S_i is glued to the left-hand side of the slit in S_{i+1}. Finally, the left-hand side of the slit in S_1 is glued on the right-hand side of the one in S_n. A detailed construction for $n = 2$ can be found in [Str84, Section 12.3].
Denote by T the resulting flat surface. There is a canonical projection $\pi : T \to S$ which is a flat n-sheeted covering. The branch points are the endpoints of the slits.
Let $\tilde{T}$ be the flat universal cover of T. The lifts of all slits in T to the flat universal cover $\tilde{T}$ is a countable disjoint union of isometrically embedded n-valent trees of edge length l. Each vertex is the preimage of one of the branch points. Let $B(R)$ be a ball of radius R in $\tilde{T}$ whose center is a vertex. $B(R)$ contains at least $n^{R/l-1}$ vertices.
Let $\tilde{y}_0 \in \tilde{T}$ be one such vertex. Recall that the counting function $N_T(R)$ is defined as

$$N_T(R) := |\Gamma \tilde{y}_0 \cap \overline{B_{\tilde{y}_0}(R)}|$$

One deduces that

$$e(\tilde{T}, \Gamma_T) = \limsup_{R \to \infty} \frac{log(N_T(R))}{R} \geq \frac{log(n)}{l} + log(1/2)$$

Therefore, one cannot expect a growth rate which is smaller than an expression inverse proportional to distance between the branch points and logarithmic in the combinatorics of the cover. We will show that this inequality is almost sharp.

Theorem 4.2. *Let $\pi : T \to S$ be a branched flat finite-sheeted covering. The entropy $e(\tilde{T}, \Gamma_T)$ of $\tilde{T}$ is bounded by the inequality*

$$e(\tilde{T}, \Gamma_T) \leq (a(S) + b(T))(e(\tilde{S}, \Gamma_S) + 1)$$

where $b(T)$ is logarithmic in the combinatorics of the covering and inverse proportional to the distance between the two closest branch points in T.

The methods to prove the statement are mainly combinatorial. We first show the claim for the special case that the branch points in T project to singularities in S. Afterwards we show that the general case can be reduced to the first.
The following Lemma is well-known:

Lemma 4.3. *Let $\pi : T \to S$ be a flat branched covering. Denote by $B_T \subset T$ the branch points and $B_S := \pi(B_T)$ its image. Assume that the branching index is bounded from above by some constant*

$$n := \sup_{y \in B_T} ind(y) + 1$$

- *Let $c : [0, t] \to S$ be an arc in S which passes k times through points in B_S. Let $y \in \pi^{-1}(c(0))$ be a preimage of the starting point of $c(0)$ in T. There are at most k^n connected arcs c' in T which project to c and have the starting point y.*

- *Let $c'_0 : [0, t] \to T$ be an arc in T which passes k times through branch points. Denote by $y = c'_0(0)$ the starting point of c'_0. There are at most k^n connected arcs c' in T which project to $\pi(c'_0)$, have the starting point y and pass through k branch points.*

Proof. The proofs of both statements are similar and use the following fact:
Let $y' \in B_T$ be a branch point and let $x' := \pi(y')$. Let α be a short line segment issuing from x' which does not pass through any image of a branch point. There are at most n lifts $\alpha'_j \subset T$ so that $\pi(\alpha'_j) = \alpha, \alpha'_j(0) = y'$.

- Let $c : [0, t] \to S$ be an arc in S. One can decompose c into subarcs $c_i : [t_i, t_i + 1] = c|_{[t_i, t_i+1]} \to S, i = 1, \ldots k$ so that each interior point of c_i is disjoint from B_S for any i. We chose a preimage $y \in \pi^{-1}(c_1(t_1))$ of the starting point of c_1. There are at most n different connected arcs $c'_{1,j}$ so that $\pi \circ c'_{1,j} = c_1, c_{1,j}(t_1) = y$. To extend the lift $c'_{1,j}$ to a complete connected lift of c, one has to find a lift $c'_{2,k}$ of c_2

so that $c'_{2,k}(t_2) = c'_{1,j}(t_2)$. Again there are at most n of such lifts $c'_{2,k}$ for each j. One iterates the process and observes that there are at most k^n connected lifts of c with the starting point y.

- The proof is analogous to the one in the first case. Let $c := \pi \circ c'_0 \subset S$. One can choose at most n maximal lifts $c'_{i,1}$ of c which emanate from y with the property that they do not meet a branch point. At the branch point, there are again at most n possibilities to continue the lift.
  One iterates the process k times and eventually obtains all lifts c' which pass through k branch points so that $\pi \circ c' = c$. All in all there are at most k^n of such lifts which emanate from y.

□

Let S be a flat surface and s_1, s_2 be saddle connections on S. Recall that by Proposition 3.11 there exists a constant $C_l(S)$, which is independent of s_1, s_2, and a geodesic g of length $l(g) \leq l(s_1) + l(s_2) + C_l(S)$ which first coincides with s_1 and eventually with s_2.

Proposition 4.7. *Let $\pi : T \to S$ be an n-sheeted flat covering which branches at most over singularities in S. Let $l_0(S)$ be the length of the shortest saddle connection in S and $C_l(S)$ be the constant as mentioned. $N_T(R)$ denotes the counting function on the flat universal cover $\tilde{T}$ of T.*

$$N_T(R) := |\Gamma_T \tilde{y}_0 \cap \overline{B_{\tilde{y}_0}(R)}|$$

Denote by $l_0(*)$ the length of the shortest saddle connection on the space $* = S, T$.
One can estimate the counting functions of S and T.

$$N_T(R) \leq n^{\frac{R}{l_0(S)}+1} 2^{\frac{R}{l_0(S)}\left(1+\frac{C_l(S)}{l_0(S)}\right)} N_S\left(R\left(1+\frac{C_l(S)}{l_0(S)}\right)\right)$$

Proof. Let $\tilde{S}$ resp. $\tilde{T}$ be the flat universal cover of S resp. T. Denote by $\Sigma_*, * = S, T, \tilde{S}, \tilde{T}$ the set of singularities in the corresponding space $*$.
Since the singularities of T project to singularities of S, $l_0(\tilde{S}) = l_0(S) = l_0(T) = l_0(\tilde{T})$.
Let $\tilde{x}_0 \in \tilde{S}$ and $\tilde{y}_0 \in \tilde{T}$ be points and $x_0 \in S, y_0 \in T$ their projections. We choose $\tilde{x}_0, \tilde{y}_0$ such that $\pi(y_0) = x_0$. For simplicity of notation we assume that $\tilde{x}_0, \tilde{y}_0$ are singularities.
Let $\mathcal{P}(R)$ be the set of parametrized loops h in S which have the following properties:

i) h emanates from x_0 and ends at x_0.

ii) h is a parametrized by arc length. The length of h is at most R.

iii) h is locally geodesic outside Σ_S.

Let $\tilde{y} \in \Gamma\tilde{y}_0 \cap B_{\tilde{y}_0}(R) \subset \tilde{T}$ be a point in the flat universal cover of T. Let $[\tilde{y}_0, \tilde{y}]$ be the parametrized connecting geodesic. We define the mapping

$$\Phi : \Gamma\tilde{y}_0 \cap B_{\tilde{y}_0}(R) \to \mathcal{P}(R), \tilde{y} \mapsto h := \pi \circ [\tilde{y}_0, \tilde{y}] \subset S$$

The flat covering map $\pi : T \to S$ only branches over the singularities of S. Consequently, away from Σ_S, h is a local geodesic. One checks that h is contained in $\mathcal{P}(R)$.
Let $h \in P(R)$ be such a loop. We have to estimate the maximal number of possible preimages.
Recall that the projection $\pi : T \to S$ is an n-sheeted branched covering. Therefore, the branching index of each branch point is at most $n - 1$. Each branch point of the branched covering $\pi : \tilde{T} \to S$ satisfies the same property.
The loop h is, away from the singularities, a local geodesic and the starting and endpoint of h is a singularity. Therefore, h is a concatenation of saddle connections. Since each saddle connection has length at least $l_0(S)$, h passes through at most $\frac{R}{l_0(S)} + 1$ singularities.
By Lemma 4.3 there are at most $n^{\frac{R}{l_0(S)}+1}$ different arcs in $\tilde{T}$ which emanate from $\tilde{y}_0$ and project to the same loop h. Therefore, the number of singularities in $\Gamma_T\tilde{y}_0 \cap B_{\tilde{y}_0}(R)$ can be estimated in terms of the cardinality of $\mathcal{P}(R)$.

$$N_T(R) \leq n^{\frac{R}{l_0(S)}+1} |\mathcal{P}(R)|$$

It remains to compare the cardinality of the set $\mathcal{P}(R)$ with the counting function $N_S(R)$.
Denote by $\mathcal{L}(R)$ the set of all locally geodesic loops in S of length at most R with the same starting point x_0.
We define an injective mapping Ψ which maps a piecewise geodesic loop $h \in \mathcal{P}(R)$ to a geodesic loop $g \in \mathcal{L}\left(R\left(1 + \frac{C_l(S)}{l_0(S)}\right)\right)$ which we equip with a combinatorial datum which consists in a coloring of each saddle connection in g with the color red or green.
Let $h \in \mathcal{P}(R)$ be a piecewise locally geodesic loop in S of length at most R. As h is locally geodesic outside the singularities and the endpoints of h are singularities, h is a concatenation of saddle connections $s_1 * \ldots * s_m$. Let $s_{i-1} * s_i$ be the incoming and outgoing saddle connections at the singularity $h(t_i)$. By Proposition 3.11 there exists a local geodesic α_i whose length is at most $C_l(S) + l(s_{i-1}) + l(s_i)$ and which first leaves s_{i-1} and eventually passes through s_i. If h is not locally geodesic at $h(t_i)$, we replace $s_{i-1} * s_i$ by the local geodesic α_i.
We can do this construction successively at all points $h(t_i)$ where h is not locally geodesic. Let g be the resulting locally geodesic loop. The starting point and endpoint of g equals

the starting and endpoint of h which is x_0.
The length of g is bounded from above by:

$$l(g) \leq l(h) + \frac{RC_l(S)}{l_0(S)} \leq R\left(1 + \frac{C_l(S)}{l_0(S)}\right)$$

The map $h \to g$ is not necessarily injective. It is possible that the same loop g arises from different points $h, h' \in \mathcal{P}(R)$. That is why we add a combinatorial datum.
h is a concatenation of saddle connections $s_1 * \ldots * s_m$. g can also be written as such a concatenation $s'_1 * \ldots * s'_n$ which arises from $s_1 * \ldots * s_m$ by gluing in additional saddle connections. We color each saddle connection s'_i with one of the colors red or green, so that the subsequence of green saddle connections equals the original sequence $s_1 * \ldots * s_m$. Let g' be the colored loop. The mapping

$$\Psi(h) := g'$$

is injective, as we can reconstruct h from g' by removing the red saddle connections. Since each locally geodesic loop g consists of at most $\frac{l(g)}{l_0(S)}$ saddle connections, g can be colored in at most $2^{\frac{l(g)}{l_0(S)}}$ different ways.
Therefore

$$|\mathcal{P}(R)| \leq 2^{\frac{R}{l_0(S)}\left(1+\frac{C_l(S)}{l_0(S)}\right)} \left|\mathcal{L}\left(R\left(1 + \frac{C_l(S)}{l_0(S)}\right)\right)\right|$$

Since $|\mathcal{L}(R)| = N_S(R)$, the two formulae allow to compare the counting functions with respect to S and T.

$$N_T(R) \leq n^{\frac{R}{l_0(S)}+1} 2^{\frac{R}{l_0(S)}\left(1+\frac{C_l(S)}{l_0(S)}\right)} * N_S\left(R\left(1 + \frac{C_l(S)}{l_0(S)}\right)\right)$$

□

Corollary 4.3. *Let $\pi : T \to S$ be an n-sheeted flat branched covering which branches at most over the singularities. Let $C_l(S), l_0(S)$ be the constants defined above. The entropy of T can be estimated in terms of the following formula:*

$$e(\tilde{T}, \Gamma_T) \leq \frac{log(n)}{l_0(S)} + \left(1 + \frac{C_l(S)}{l_0(S)}\right)\left(e(\tilde{S}, \Gamma_S) + log(2)\right)$$

To compute the entropy of general branched coverings, one makes use of the following obvious observation

Lemma 4.4. *Let $\pi : T \to S$ be a flat n-sheeted branched covering with exactly one ramification point y_0. Assume that y_0 ramifies with maximal index $n - 1 \geq 2$ over $x_0 = \pi(y_0)$. Let $c : [0, t] \to S$ be a connected arc which is locally geodesic outside x_0. Then, for each $y \in \pi^{-1}(c(0))$ there exists some local geodesic g in T such that $\pi \circ g = c$ and $g(0) = y$.*

Proposition 4.8. *Let $\pi : T \to S$ be an n-sheeted flat branched covering and let $B_T \subset T$ be the set of all branch points in T. Denote by $l_b(T)$ the minimal distance between any two branch points on T and let $k := |B_T|$ be the cardinality of B_T.*
Finally, let ς_S be a distinguished singularity on S and let $\pi : T_0 \to S$ be a 3-sheeted flat covering which ramifies with maximal index over ς_S and has no other branch points. Then

$$N_T(R) \leq n^{R/l_b(T)} N_{T_0}\left(R\left(1 + \frac{2diam(S)}{l_b(T)}\right)\right) k^{\frac{R}{l_b(T)}}$$

Proof. We choose $\varsigma_T \in \pi^{-1}(\varsigma_S) \in T$ a singularity in T which is a preimage of ς_S. Denote by $\varsigma_{T_0} = \pi^{-1}(\varsigma_S) \in T_0$ the unique preimage of ς_S in T_0.
Let $B_T = \{b_1 \dots b_k\} \subset T$ be the set of all branch points in T. Let $c_i = [\pi(b_i), \varsigma_S], l_q(c_i) \leq diam(S)$ be a shortest geodesic in S connecting any point $\pi(b_i) \in S$ with ς_S.
Let $\mathcal{L}_*(R), * = T, T_0$ be the set of geodesic loops of length at most R in $*$, which connect ς_* with itself. We define a mapping

$$\Phi : \mathcal{L}_T(R) \to \mathcal{L}_{T_0}\left(R\left(1 + \frac{2diam(S)}{l_b(T)}\right)\right) \times B_T^{\frac{R}{l_b(T)}}$$

Let $g \in \mathcal{L}_T(R)$ be such a loop of length at most R. One can cut g at each branch point $a_i \in B_T$ and obtains local geodesics $g_i, i = 1 \dots m$. The length of each of the local geodesics is at least $l_b(T)$ and therefore

$$m \leq \frac{R}{l_b(T)}$$

Let $\pi \circ g_i \subset S$ be the projection of g_i to S. $\pi \circ g_i$ is a local geodesic. Since $\pi \circ g_i$ starts at the image of some branch point and ends at some other, there are some geodesics c_j, c_k so that $h_i := c_j * \pi \circ g_i * c_k^{-1}$ is a connected arc with the starting and endpoint ς_S. If $i = 1$ resp. $i = m$, one only has to concatenate c_j at the endpoint resp. at the starting point of $\pi \circ g_i$.
Let $h_{i,q} \in [h_i] \subset S$ be the length-minimizing local geodesic in the homotopy class of arcs with fixed endpoints, see Figure 4.
Denote by

$$h := h_{1,q} * h_{2,q} \dots * h_{m,q}$$

the concatenation of all such arcs. h is a loop which is locally geodesic outside ς_S. It follows from Lemma 4.4 that we can choose a lift α of h in T_0 which is a local geodesic and emanates from ς_{T_0}. Since α is a lift of h, α ends at a preimage of the endpoint of h which is a preimage of ς_S. By construction $\pi^{-1}(\varsigma_S) = \varsigma_{T_0}$ and therefore α is a locally

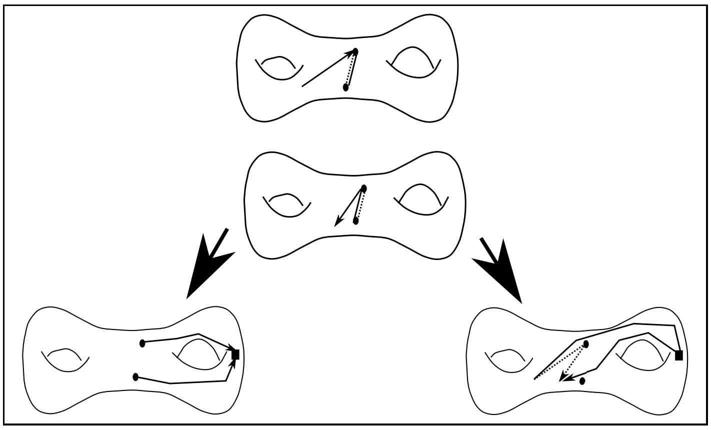

Figure 4: The two surfaces are identified along the dotted and the solid line and form a branched covering of the surface. We fix one point in the base, here the solid square, and for each image of a branch point, we fix a connecting geodesic, here marked on the lower left. A geodesic in the covering projects to a path here marked dotted which is locally geodesic outside the image of the branch points. We cut off each piece of the path at each branch point image and we glue in the excursion to the square. Finally, we straighten each such path so that the concatenation is locally geodesic outside the square.

geodesic loop. The length of α equals the length of h. Therefore

$$\begin{aligned} l(\alpha) &= \sum_i l(h_i) \leq \sum_i (l(g_i) + 2l(c_i)) \leq l(g) + 2mdiam(S) \\ &\leq R + \frac{2R}{l_b(T)} diam(S) \end{aligned}$$

We define the mapping

$$\Phi : g \mapsto (\alpha, (a_i)_i)$$

where $a_i \in B_T$ is the endpoint of g_i.

$$\begin{array}{ccc} (g = g_1 * \ldots * g_m \subset T, (a_i)_i) & & (\alpha \subset T_0, (a_i)_i) \\ \downarrow \pi & & \uparrow \\ (\pi(g) = \pi(g_1) * \ldots * \pi(g_m) \subset S, (a_i)_i) & \rightsquigarrow & (h = h_1 * \ldots * h_m \subset S, (a_i)) \end{array}$$

Again we have to count the number of possible preimages.
Let $g, g' \subset T$ be different geodesics loops in T.
If the associated data $(a_i)_i \in B_T^m$ resp. $(a'_j)_j \in B_T^{m'}$ are different, $\Phi(g) \neq \Phi(g')$.
Assume that the data $(a_i)_i$ are equal, but the projections $\pi(g), \pi(g') \subset S$ are different.
In each homotopy class of arcs there exists exactly one local geodesic. Therefore, at least one of the homotopy classes $[g_i]$ and $[g'_i]$ differ.

Consequently, the concatenations h and h' are different. Let $\Phi(g) = (\alpha, a_i), \Phi(g') = (\alpha', a_i)$. Since the loop α resp. α' project to h resp. h', $\Phi(g) \neq \Phi(g')$.
Therefore, $\Phi(g) = \Phi(g')$ only if $\pi(g) = \pi(g')$ and only if the branch point data $a_i = a_i'$ equal as well.
By Lemma 4.3 for each m there are at most n^m different loops $g \subset T$ with the same starting point which pass through m branch points and which project to the same arc $\pi(g) \subset S$.
Since a geodesic in $g \subset T$ of length at most R can pass through at most $\frac{R}{l_b(T)}$ branch points, it follows that

$$m \leq \frac{R}{l_b(T)}$$

Therefore, the map Φ, restricted to loops of length at most R, is at most $n^{R/l_b(T)}$-to-1.
Consequently,

$$N_T(R) \leq n^{\frac{R}{l_b(T)}} N_{T_0}\left(R\left(1 + \frac{2diam(S)}{l_b(T)}\right)\right)|B_T|^{\frac{R}{l_b(T)}}$$

□

Corollary 4.4. *Let $\pi : T \to S$ be a flat branched n-sheeted covering. Let $l_b(T)$ be the minimal distance of branch points in T. Let $\pi : T_0 \to S$ be defined as above.*

$$e(T, \Gamma_T) \leq \left(1 + \frac{2diam(S)}{l_b(T)}\right) e(T_0, \Gamma_{T_0}) + \frac{log(n) + log(k)}{l_b(T)}$$

Theorem 4.2 follows from Corollary 4.4 and Corollary 4.3.

5 Asymptotic behavior

In this section we investigate the asymptotic behavior of bi-infinite geodesics on the flat surface. In the first two sections, we compare the length of geodesic arcs with respect to the flat and the hyperbolic metric in the same conformal class.
Finally, we define a geodesic flow and construct an appropriate measure on the space of geodesic on the flat surface. With respect to this measure we show how the typical geodesic winds through the flat surface.

5.1 Asymptotic comparison of hyperbolic and flat geodesics

It is the goal of this part to investigate the relationship between the singular euclidean metric and the hyperbolic metric in the same conformal class. We therefore compare the geodesic flows. We make use of standard results in ergodic theory following [Kre85, section 1.5].

Definition 5.1. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measurable space together with the measure preserving flow* g_t.
A family of integrable functions $F_{i,j}, i, j \in \mathbb{R}$ *is called a subadditive process with respect to the flow* g_t *if and only if:*

- $F_{i,j} \circ g_t = F_{i+t,j+t}$
- $F_{0,\tilde{t}} + F_{\tilde{t},t} \geq F_{0,t}$
- $\inf_t t^{-1} \int F_{0,t} > -\infty$

In ergodic theory the Theorem of Kingman is classical.

Theorem 5.1. *Let* $F_{i,j}$ *be a subadditive process on* $(\Omega, \mathcal{A}, \mu)$ *with respect to* g_t.
Then the sequence $T^{-1}F_{0,T}$ *converges to a* g_t*-invariant measurable limit* F *a. e. Moreover, when* μ *is a finite measure then* $T^{-1}F_{0,T}$ *converges with respect to the* L_1*-norm.*

We apply this concept to our setting:
Let $S = (X, d_q)$ be a closed flat surface and σ the hyperbolic metric on the Riemann surface X which is in the same conformal class as d_q. Let $\tau : T^1X \to X$ be the unit tangent bundle of X .
Let $\ell_*, * = d_q, \sigma$ be the Lebesgue measure on X induced by the hyperbolic resp. flat metric. Denote by $\mathfrak{m}$ the corresponding Liouville measure for the hyperbolic metric on the unit tangent bundle. Let $\mathfrak{m}_x, x \in X$ be the induced family of measures of the fiber $T^1_xX \cong S^1$ which is absolutely to continuous to the Lebesgue measure of S^1 and which is invariant under parallel transport along unit speed geodesics.
Let $\pi : \tilde{X} \to X$ be the universal cover. The measures are defined in the same way on the universal cover $\tilde{X}$ endowed with the lifted structure i.e. the unit tangent bundle $\tau : T^1\tilde{X} \to \tilde{X}$, the hyperbolic or flat metric. The covering map π naturally extends to $\pi : T^1\tilde{X} \to T^1X$.
Let c be an arc on X. We called $c_*, * = d_q, \sigma$ the shortest representative in the homotopy class of c with fixed endpoints. Such a length-minimizing arc always exists and is unique. We introduced the notion

$$l_*([c]) := \min_{c' \in [c]} l_*(c') = l_*(c_*), * = d_q, \sigma$$

as the length of the length-minimizing representative c_i in the homotopy class.
Let $g_t : T^1X \to T^1X$ be the geodesic flow with respect to the hyperbolic metric σ on T^1X. It is a result of Hopf, [Hop71] that the geodesic flow g_t acts ergodically with

respect to $\mathfrak{m}$ on the unit tangent bundle T^1X .
To each point $v \in T^1X$ and to each $i, j \in \mathbb{R}$, one can naturally associate an arc on X.

$$c_{i,j} : [0, j-i] \to X, c_{i,j}(t) := \tau(g_{i+t}(v))$$

$c_{i,j}$ is a local geodesic on (X, σ).
We define

$$F_{i,j} : T^1X \to \mathbb{R}_+ : v \mapsto l_q([c_{i,j}])$$

By definition of $c_{i,j}$,

$$F_{i,j} \circ g_t = F_{i+t,j+t}$$

We will first show some properties of $F_{i,j}$:
Let $\pi : \tilde{X} \to X$ be the universal cover and $(\tilde{X}, *), * = \sigma, d_q$ the lifted metrics. $(\tilde{X}, \sigma)$ is isometric to the Poincaré disc and therefore a δ-hyperbolic space where δ is independent of X.
On the universal cover the flat and the hyperbolic metric are (L, L)-quasi-isometric for some L.

Lemma 5.1. *Let $S = (X, d_q)$ be a closed flat surface of genus $g \geq 2$. Let σ be the hyperbolic metric on X in the same conformal class as d_q. Denote by L the quasi-isometric constant between the two metrics on the universal cover.*
Then, the function $F_{i,j} : T^1X \to \mathbb{R}_{\geq 0}$ meets the following properties:

i)

$$L^{-1}|i-j| - L \leq F_{i,j} \leq L|i-j| + L$$

ii) There exists a constant $\lambda(L)$ which only depends on L such that for each $s, t \geq 0$,

$$F_{0,s+t} \geq F_{0,s}(x) + F_{s,s+t} - \lambda$$

iii) Let λ be the constant from ii)

$$F_{0,s} \leq F_{0,s+t}, \forall t \geq \lambda' := L(\lambda + 1)$$

iv) $F_{i,j}$ is a subadditive process with respect to $\mathfrak{m}$

Proof. Statement $i)$ is obvious.
We show statement $ii)$. Let $v \in T^1X$ be a point in the unit tangent bundle. $x := \tau(v)$ resp. $y := \tau(g_s(v)), z := \tau(g_{s+t}(v))$. Furthermore, let $[x, z]_*$ be the connecting geodesics with respect to the metrics $* = d_q, \sigma$.

We can choose a lift $[\tilde{x}, \tilde{z}]_*$ in the universal cover and $\tilde{y} = [\tilde{x}, \tilde{z}]_\sigma(s)$. As geodesics in the hyperbolic metric are (L, L)-quasi-geodesics in the flat metric, the Hausdorff distance of $[\tilde{x}, \tilde{z}]_\sigma$ and $[\tilde{x}, \tilde{z}]_q$ is uniformly bounded by some constant $\frac{\lambda}{2}$ with respect to the flat metric. λ only depends on L. Consequently, there is a point $[\tilde{x}, \tilde{z}]_q(r)$ such that $d_q([\tilde{x}, \tilde{z}]_q(r), \tilde{y}) \leq \frac{\lambda}{2}$. It follows that

$$\begin{aligned} F_{0,t}(v) + F_{t,t+s}(v) &= d_q(\tilde{x}, \tilde{y}) + d_q(\tilde{y}, \tilde{z}) \\ &\leq d_q(\tilde{x}, [\tilde{x}, \tilde{z}]_q(r)) + \frac{\lambda}{2} + d_q([\tilde{x}, \tilde{z}]_q(r), \tilde{z}) + \frac{\lambda}{2} \\ &= d_q(\tilde{x}, \tilde{z}) + \lambda \\ &= F_{0,s+t}(v) + \lambda \end{aligned}$$

iii) follows from *i*) and *ii*).
To prove *iv*) it remains to show that $F_{i,j}$ is integrable. The geodesic flow g_t acts continuously on T^1X. The flat as well as the hyperbolic metric are length metrics on $\tilde{X}$. Since both metrics induce the same topology on $\tilde{X}$, $F_{i,j}$ is continuous and therefore measurable. Since $F_{i,j}$ is bounded and the unit tangent bundle T^1X is of finite volume, the integrability of $F_{i,j}$ holds. □

We can apply Kingman's Theorem and conclude the existence of a g_t-invariant measurable limit function

$$F := \lim_r r^{-1} F_{0,r}$$

As g_t acts ergodically for $\mathfrak{m}$, F is constant a. e.

Theorem 5.2. *The entropy and the constant F are related.*

$$e(\tilde{S}, \Gamma_S) \geq F^{-1}$$

Proof. Assume on the contrary that $e(\tilde{S}, \Gamma_S) < F^{-1} - 3\epsilon$ for some $0 < \epsilon < F^{-1}$.
Since $r^{-1}F_{0,r}$ converges towards F with respect to the L_1-norm,

$$\lim_{r\to\infty} \int_{v \in T^1X} |r^{-1}F_{0,r} - F| d\mathfrak{m} = 0$$

We notice that $\frac{1}{F^{-1}-\epsilon} > F$.
Let x be a typical point for l_σ in X. Let $c_r(x)$ be the measure of directions in the T^1_xX fiber over x so that $F_{0,r}$ is exceptionally large.

$$c_r(x) := \mathfrak{m}_x \left(v \in T^1_xX : F_{0,r(F^{-1}-\epsilon)}(v) > r \right)$$

Since x is l_σ-typical, $c_r(x)$ tends to zero if r tends to infinity.
Let $\tilde{x} \in \pi^{-1}(x) \subset \tilde{X}$ be a preimage of x in the universal cover. Denote by $\overline{B^*_{\tilde{x}}(r)}$ the closed ball with center $\tilde{x}$ and radius r with respect to the metric $* = \sigma, d_q$.
It is a standard fact from hyperbolic geometry, see i.e. [BP92], that

$$\lim_{r\to\infty} \frac{\log\Big(\ell_\sigma\Big(\overline{B^\sigma_{\tilde{x}}(r)}\Big)\Big)}{r} = 1$$

l_σ is a Γ-invariant Radon measure on $\tilde{X}$. By Lemma 2.3 the entropy satisfies the following formula:

$$e(\tilde{S},\Gamma_S) = \limsup_{r\to\infty} \frac{log\left(\ell_\sigma\left(\overline{B^q_{\tilde{x}}(r)}\right)\right)}{r}$$

One can estimate F by comparing the volume of metric balls:

$$\begin{aligned} & \ell_\sigma\Big(\overline{B^\sigma_{\tilde{x}}(r(e(\tilde{S},\Gamma_S)+\epsilon))}\Big) - \ell_\sigma\Big(\overline{B^q_{\tilde{x}}(r)}\Big) \\ \leq\ & \ell_\sigma\Big(\overline{B^\sigma_{\tilde{x}}(r(e(\tilde{S},\Gamma_S)+\epsilon))} - \overline{B^q_{\tilde{x}}(r)}\Big) \\ =\ & \ell_\sigma\Big(\Big\{\tau(g_t(v))|(v,t)\in T^1_{\tilde{x}}\tilde{X}\times[0,r(e(\tilde{S},\Gamma_S)+\epsilon)] : F_{0,t}(\pi(v)) > r\Big\}\Big) \end{aligned}$$

Let λ' be the constant of Lemma 5.1 and let r be so large that

$$r(F^{-1}-\epsilon) - r(e(\tilde{S},\Gamma_S)+\epsilon) \geq r\epsilon \geq 2\lambda'$$

By Lemma 5.1, we conclude

$$F_{0,r(F^{-1}-\epsilon)}(\pi(v)) \geq F_{0,t}(\pi(v)), \forall v \in T^1\tilde{X}, t \in [0, r(e(\tilde{S},\Gamma_S)+\epsilon)]$$

Therefore, we can estimate

$$\begin{aligned} & \ell_\sigma\Big(\overline{B^\sigma_{\tilde{x}}(r(e(\tilde{S},\Gamma_S)+\epsilon))} - \overline{B^q_{\tilde{x}}(r)}\Big) \\ \leq\ & \ell_\sigma\left(\Big\{\tau(g_t(v))|(v,t)\in T^1_{\tilde{x}}\tilde{X}\times[0,r(e(\tilde{S},\Gamma_S)+\epsilon)] : F_{0,r(F^{-1}-\epsilon)}(\pi(v)) > r\Big\}\right) \end{aligned}$$

By definition of $c_r(x)$ it follows

$$\begin{aligned} & \ell_\sigma\left(\Big\{\tau(g_t(v))|(v,t)\in T^1_{\tilde{x}}\tilde{X}\times[0,r(e(\tilde{S},\Gamma_S)+\epsilon)] : F_{0,r(F^{-1}-\epsilon)}(\pi(v)) > r\Big\}\right) \\ =\ & \ell_\sigma\Big(\overline{B^\sigma_{\tilde{x}}(r(e(\tilde{S},\Gamma_S)+\epsilon))}\Big)c_r(x) \end{aligned}$$

We summarize

$$\begin{aligned} & \ell_\sigma\Big(\overline{B^\sigma_{\tilde{x}}(r(e(\tilde{S},\Gamma_S)+\epsilon))}\Big) - \ell_\sigma\Big(\overline{B^q_{\tilde{x}}(r)}\Big) \leq \ell_\sigma\Big(\overline{B^\sigma_{\tilde{x}}(r(e(\tilde{S},\Gamma_S)+\epsilon))}\Big)c_r(x) \\ \Leftrightarrow\ & \frac{log(1-c_r(x)) + \log\Big(\ell_\sigma\Big(\overline{B^\sigma_{\tilde{x}}(r(e(\tilde{S},\Gamma_S)+\epsilon)}\Big)\Big)}{r} \leq \frac{log\Big(\ell_\sigma\Big(\overline{B^q_{\tilde{x}}(r)}\Big)\Big)}{r} \end{aligned}$$

If r tends to infinity, the left term tends to $e(\tilde{S}, \Gamma_S)+\epsilon$ whereas the right term is bounded from above by $e(\tilde{S}, \Gamma_S)$, what is a contradiction. $\square$

5.2 Flow limit and size of subsurfaces

We showed that the entropy bounds the asymptotic difference between hyperbolic and quadratic differential length. A similar quantity was introduced in section 3.6.2. We recall the facts:
Let $S = (X, d_q)$ be a closed flat surface and σ the hyperbolic metric in the same conformal class as d_q. Denote by $(X_>, X_<)$ the thick-thin decomposition of (X, σ) with respect to the Margulis constant $\epsilon > 0$.
Let Y be a connected component of $X_>$. In the same homotopy class as Y there exists a subsurface Y_q so that for each boundary component of Y the length minimizing representative in Y_q is unique. Y_q might be degenerated to a graph.
If Y is not a topological pair of pants, we define $\lambda(Y)$ as the flat length of the shortest essential non-peripheral simple closed curve in Y. If Y is a topological pair of pants, $\lambda(Y)$ is defined as the flat length of the longest boundary component.
Let $[\alpha]$ be a free homotopy class of a non-peripheral simple closed curve α which can be realized in Y. It was shown that the hyperbolic length $l_\sigma([\alpha])$, multiplied with $\lambda(Y)$, and the flat length $l_q([\alpha])$ equal up to a multiplicative constant c which only depends on the topology of X:

$$cl_q([\alpha]) \geq \lambda(Y) l_\sigma([\alpha]) \geq c^{-1} l_q([\alpha])$$

Moreover, the diameter of Y_q is comparable to $\lambda(Y)$.

$$c \cdot diam_q(Y_q) \geq \lambda(Y) \geq c^{-1} diam_q(Y_q)$$

Denote by $F_{i,j}$ again the subadditive process as in the previous section and $F := \lim_T T^{-1} F_{0,T}$ the measurable limit.

Theorem 5.3. *Let $S = (X, d_q)$ be a closed flat surface of genus $g \geq 2$. Let σ be the hyperbolic metric on X which is in same conformal class as d_q.*
Denote by $(X_>, X_<)$ the thick-thin decomposition of (X, σ) with respect to the Margulis constant ϵ. Let Y be a connected component of $X_>$.
Then there exists a constant $A := A(\chi(X)) > 0$ which only depends on the topology of X such that

$$F \geq A\lambda(Y)$$

Proof. We recall that $\tau : T^1X \to X$ is the projection of the unit tangent bundle to the Riemann surface X, whereas $\pi : \tilde{X} \to X$ is the projection of the universal cover.

Assume first that the connected component $Y \subset X_>$ is not a topological pair of pants. By Lemma 3.13 there exist non-peripheral intersecting simple closed local geodesics $\alpha_\sigma, \beta_\sigma \subset Y$ whose hyperbolic lengths are bounded from above and below by some uniform constants.
Let α_q resp. β_q be flat geodesic representatives in the free homotopy class of α_σ resp. β_σ. By Proposition 3.14 there is a constant c which only depends on the topology of X so that for any simple closed curve γ in Y

$$cl_q([\gamma]) \geq \lambda(Y) l_\sigma([\gamma]) \geq c^{-1} l_q([\gamma]),$$

We define a constant m

$$m := \frac{2c \cdot l_q(\beta_q)}{\lambda(Y)} + 4$$

m is bounded from above and below by constants which only depend on the topology of X.
The hyperbolic length of α_σ is bounded from above and below by some uniform constants. By the Collar Lemma, there exists a hyperbolic convex collar C_α around α_σ which satisfies the following properties.

- The hyperbolic length of the shortest arc in C_α which connects different boundary components has a universal positive upper and lower bounds which only depend on the length of α_σ.
- The hyperbolic area of C_α is bounded from below by some positive constant which only depends on the topology of X as well.
- Moreover, there exists some $s_0 > 0$, which depends on m and on the hyperbolic length of α_σ, so that each hyperbolic geodesic $g_\sigma : [0, s_0] \to C_\alpha$, of length at least s_0 intersects β_σ at least m times.

At each point x in C_α we choose a maximal set of directions $I_x \subset T^1_x X$ so that for every point $v \in I_x$ the hyperbolic geodesic

$$g_\sigma : [0, s_0] \to X, g_\sigma : t \mapsto \tau(g_t(v))$$

is entirely contained in C_α.
We call this set of directions the direction collar CD_α, see Figure 5. One can uniformize C_α in the hyperbolic plane and observe that the Lebesgue measure of each fiber I_x has a positive lower bound which depends on s_0 and the length of α_σ, consequently on the topology of X. Therefore, the volume of CD_α has a positive lower bound which only depends on the topology of X.

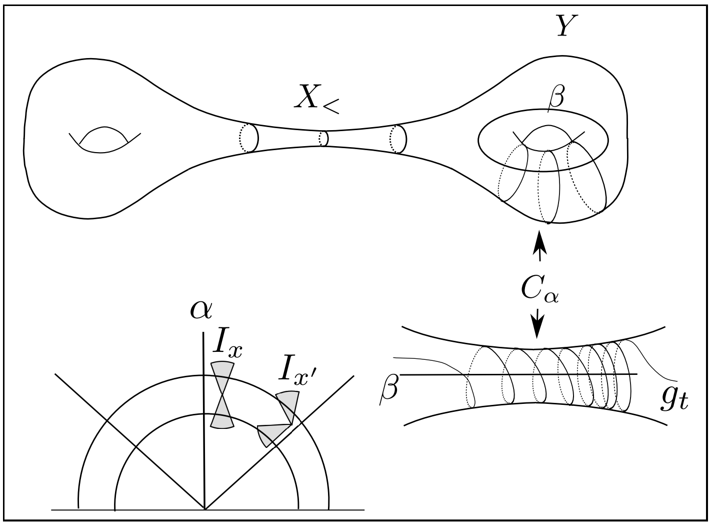

Figure 5: One sees the configuration of α and β on the surface. We uniform C_α such that the identification map is $z \mapsto az, a > 0$. At each point in C_α we choose two intervals of directions I_x, such that the geodesic flow in this direction twists a fixed amount of times in the collar C_α.

We use CD_α as a tool to estimate the quotient of hyperbolic and flat lengths of geodesic arcs, which follows from the following Lemma

Lemma 5.2. *Let $g_\sigma : [0, T] \to X$ be a geodesic arc for the hyperbolic metric and let g_q be the q-geodesic representative of the homotopy class with fixed endpoints $[g_\sigma]$. Let $t_j > 0, j = 1, \dots, n_{CD}$ which satisfy the following properties:*

i) $t_j < t_{j+1} - s_0 < T - 2s_0$

ii) $g'_\sigma(t_j) \in CD_\alpha$

Then there is a constant $c > 0$ which only depends on the topology of X and $k > 0$ which depends on the flat metric but not on g_σ so that

$$l_q(g_q) \geq c^{-1}\lambda(Y)(n_{CD} - 2k)$$

Proof. If the number of such n_{CD} is at most $2k$ the Lemma is obvious. So, assume that $n_{CD} > 2k$.

The main line of the argument is the following: For each t, so that $g'_\sigma(t) \in CD_\alpha$ the geodesic subarc $\tilde{g}_\sigma|_{[t,t+s_0]}$ twists in the collar C_α. We expect that there is a corresponding subarc of g_q which has the same behavior. The length of this flat subarc is easy to estimate. It is comparable to the length of α_q multiplied with the number of twists. The main difficulty is to synchronize the behavior of the flat and the hyperbolic geodesic

$g_*, * = q, \sigma$.
That is why we go over to the universal cover. Let $\tilde{g}_*$ be lifts of $g_*, * = q, \sigma$ to the universal cover with common endpoints $\tilde{x}, \tilde{y}$.
Let $\tilde{\beta}_{\sigma,i}, i = 1, \ldots, n_{int}$ be the complete lifts of β_σ which intersect $\tilde{g}_\sigma$. The lifts are ordered by their distance to $\tilde{x}$. Let $\tilde{\beta}_{q,i}$ be the complete lifts of β_q which share their endpoints at infinity with $\tilde{\beta}_{\sigma,i}$.
Each line $\tilde{\beta}_{\sigma,i}$ separates the endpoints $\tilde{x}$ and $\tilde{y}$. This is not necessarily true for $\tilde{\beta}_{q,i}$.
However, we claim that there is a bound $k > 0$, which only depends on the flat metric, so that $\tilde{\beta}_{q,i}$ separates $\tilde{x}, \tilde{y}$ for all $k < i < n_{int} - k$.
The line $\tilde{\beta}_{q,i}$ is a (L, L)-quasi-geodesic in the Poincaré disc where L only depends on the flat metric. (L, L)-quasi-geodesics with common endpoints have universally bounded Hausdorff distance $H(L)$. Therefore, for each geodesic line $\tilde{\beta}_{\sigma,i}$ which has distance greater than $H(L)$ to $\tilde{x}, \tilde{y}$, the corresponding quasi-geodesic $\tilde{\beta}_{q,i}$ also separates $\tilde{x}$ and $\tilde{y}$.
Since β is simple and of uniformly bounded length, we conclude from the Collar Lemma that any two geodesic lines $\tilde{\beta}_{\sigma,i}, \tilde{\beta}_{\sigma,i+1}$ are disjoint and their distance is bounded from below by some uniform constant which only depends on the hyperbolic length of β_σ. So, there is a bound $k > 0$ which depends on the quasi-isometric constant L and the length of β_σ such that $\tilde{\beta}_{\sigma,i}$ is of distance at least $H(L)$ to $\tilde{x}, \tilde{y}$ for all $k \leq i \leq n_{int} - k$. So, the corresponding flat geodesic line $\tilde{\beta}_{q,i}$ separates $\tilde{x}$ and $\tilde{y}$.
The geodesic lines $\tilde{\beta}_{q,i}, \beta_{\sigma,i}$ admit a coarse synchronization of the geodesics g_q, g_σ in the following way: To each subarc of g_σ that connects two lines $\beta_{\sigma,i}, \beta_{\sigma,j}, k < i, j < n_{int} - k$ we associate the shortest subarc of g_q that connects the corresponding lines $\beta_{q,i}, \beta_{q,j}$.
After projecting the subarcs to the base surface they can be closed up with a piece of β_σ, resp. β_q to closed curves which are in same free homotopy classes up to attaching multiples of β.
Let $t_j, k \leq j \leq n_{Cd} - k$ so that $g'_\sigma(t_j) \in CD_\alpha$. $\tilde{g}_\sigma|_{[t_j, t_j+s_0]}$ intersects at least m lifts $\tilde{\beta}_{\sigma,i_j}, \ldots, \tilde{\beta}_{\sigma,i_j+m}$.
Moreover, the lines $\tilde{\beta}_{q,i_j} \ldots \tilde{\beta}_{q,i_j+m}$ intersect $\tilde{g}_q$. Let $\tilde{b}_j \subset \tilde{g}_q$ be the shortest subarc of $\tilde{g}_q$ connecting $\tilde{\beta}_{q,i_j}$ with $\tilde{\beta}_{q,i_j+m}$.
For different times t_j, the subarcs $\tilde{b}_j$ are disjoint up to endpoints and therefore

$$l_q(\tilde{g}_q) \geq \sum_{j=k}^{n_{CD}-k} l_q(\tilde{b}_j)$$

It remains to show that each arc $\tilde{b}_j$ is of length at least $c^{-1}\lambda(Y)$. Let $\tilde{a}_j \subset \tilde{g}_\sigma$ be the corresponding subarc of $\tilde{g}_\sigma$ connecting $\tilde{\beta}_{\sigma,i_j}$ with $\tilde{\beta}_{\sigma,i_j+m}$.
One can close $\pi(\tilde{a}_j)$ up along β_σ to a closed curve $\overline{a_j}$ which is in the free homotopy class

$\left[\alpha^{m-2}\right]$. As β_q and β_σ are freely homotopic, we can close $\pi(\tilde{b}_j)$ up along a piece of β_q to a closed curve $\overline{b_j}$ which is in the free homotopy class $\left[\alpha^{m'}\beta^l\right]$, $|m'-(m-2)| \leq 1$ and therefore not necessarily simple. Let $\overline{b_{q,j}}$ be the flat geodesic representative of the free homotopy class $\left[\alpha^{m'}\beta^l\right]$.
We can remove a subarc of $\overline{b_{q,j}}$ so that the resulting arc is a loop in the free homotopy class $\left[\alpha^{m'}\right]$ or $\left[\alpha^{m'}\beta\right]$. Since $l_q([\alpha^{m'}\beta]) \geq l_q([\alpha^{m'}]) - l_q([\beta]) = m' l_q(\alpha_q) - l_q(\beta_q)$ we conclude

$$l_q\left(\overline{b_{q,j}}\right) \geq l_q\left(\left[\alpha^{m'}\right]\right) - l_q\left([\beta]\right) \geq (m-3) l_q(\alpha_q) - l_q(\beta_q)$$

It follows from Proposition 3.14 that there exists a uniform constant $c > 0$ so that

$$l_q(b_j) \geq l_q\left(\overline{b_j}\right) - l_q(\beta) \geq c^{-1}\lambda(Y)(m-3) - 2l_q(\beta_q) \geq c^{-1}\lambda(Y)$$

□

We return to proof of the theorem. Let $v \in T^1X$ be a $\mathfrak{m}$-typical point. Since g_t acts ergodically on T^1X, there is some bound $T_0 > 0$ so that the geodesic $g_t(v), 0 < t < T$ spends at least some proportional amount of time rT in CD_α for all $T > T_0$. r depends only on the area of CD_α and the area of T^1X and therefore only on the topology of X. So there are at least $\frac{rT}{s_0}$ times t_j so that $g_{t_j}(v) \in CD_\alpha, 0 < t_j < t_j - s_0 < T - 2s_0$.
By Lemma 5.2

$$T^{-1}F_{0,T}(v) \geq T^{-1}c^{-1}\lambda(Y)\left(\frac{rT}{s_0} - 2k\right) = c^{-1}\lambda(Y)\left(\frac{r}{s_0} - \frac{2k}{T}\right)$$

Recall that $c, s_0, r > 0$ only depend on the topology of X and $k > 0$ is independent of T.
It remains to show the Theorem in the case that Y is a pair of pants.
Let $\alpha_\sigma, \beta_\sigma$ the intersecting hyperbolic geodesics as in Figure 6. The hyperbolic length of both closed curves is bounded from above by some uniform constant. Since the Theorem of Rafi only compares the length of simple closed curves, we have to show that the flat length of the geodesic representatives α_q, β_q is uniformly bounded. For both cases the line of argument is identical, so we only show this fact for α_q. We slightly homotope α_q such that it has exactly one point of self-intersection and the length only changes by some small factor. One can decompose α_q in two simple closed curves α_1,α_2 which are both in the homotopy class of boundary components. Therefore, the flat length of the geodesic representative $\alpha_{1,q}, \alpha_{2,q}$ is at most $\lambda(Y)$. By Theorem 3.14 the diameter of Y_q, with respect to the flat metric, is less than $c\lambda(Y)$ for some constant uniform constant c. Since α_q is the length-minimizing representative in the free homotopy class, $l_q(\alpha_q) \leq 2\lambda(Y)(1+c)$.

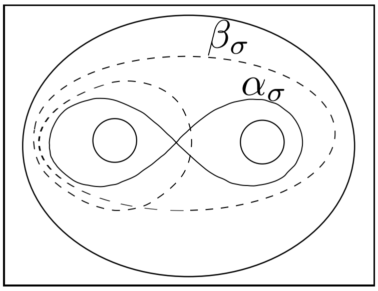

Figure 6: A pair of pants is homeomorphic to a twice-punctured disc. Since the boundary components are uniformly short, it is a consequence of hyperbolic geometry that the geodesics $\alpha_\sigma, \beta_\sigma$ are uniformly short.

We choose the 4-sheeted cover $\pi : Y' \to Y$ of the thick piece, so that up to passing through $\alpha_\sigma, \beta_\sigma$ twice, the lifts $\alpha'_\sigma, \beta'_\sigma$ are simple closed and intersecting. We can lift the hyperbolic metric to Y'. The hyperbolic length of $\alpha'_\sigma, \beta'_\sigma$ is twice the length of $\alpha_\sigma, \beta_\sigma$. We show, that the remaining part of the argument is analogous to the one in the first case:
Since the length of α'_σ is the uniformly bounded, there exists some hyperbolic collar $C_\alpha \subset Y'$ around α'_σ which has some definite width. As in the first case we choose the collar of directions $CD_{\alpha'}$ so that for each $v \in CD_{\alpha'}$ the flow $g_t(v), 0 < t < s_0$ remains in $C_{\alpha'}$ and intersects β'_σ a certain number of times m. We can choose CD_α that the volume of $CD_{\alpha'}$ has some definite amount. The projection of the collar of directions to the unit tangent bundle of the base surface T^1X has hyperbolic measure at least one fourth the hyperbolic measure of $CD_{\alpha'}$.
Let $v \in \pi(CD_{\alpha'}) \subset T^1X$ be a point. We can choose a lift of the flow $\tau(g_t(v)), 0 < t < s_0$ to T^1Y'. The lifted geodesic winds through the collar $C_{\alpha'}$ and can be closed up with a piece of β'_σ to a closed curve which is freely homotopic to $\alpha'^{m'}\beta'^{l'}$. The situation on the base surface is equal. For $v \in \pi(CD_{\alpha'})$ the geodesic flow $\tau(g_t(v)), 0 < t < t_0$ remains in $\pi(C_{\alpha'})$ and intersects β_σ at least m times. One closes up the piece with β_σ and obtains a closed curve which is freely homotopic to $\alpha^{m'}\beta^l, l \in \mathbb{N}, |m' - (m-2)| < 1$.
The remaining part of the argument is analogous to the one in the first case. □

5.3 Geodesic flow

We investigate how the typical geodesic ray winds through the flat surface. Therefore, we define the geodesic flow with respect to the flat metric.
Similar constructions have already been made by Kaimanovich [Kai94] in the case of Gromov hyperbolic spaces with additional conditions, by Bourdon [Bou95] for $Cat(-1)$

spaces and by Coornaert and Papadopoulos [CP94], [CP97] for metric trees resp. graphs.

5.3.1 Construction of the geodesic flow

The main work will be to define an appropriate measure for the geodesic flow.
We recall the necessary concepts and results from section 3.
Let $S = (X, d_q)$ be a flat surface and $\pi : \tilde{S} \to S$ the flat universal cover. Let $\nu_{\tilde{x}}$ be the Patterson-Sullivan measure on the compactification of $\tilde{S}$ with respect to some base point $\tilde{x} \in \tilde{S}$.
The set $sh_{\tilde{y}}(U)$ denotes the shadow of a set U with respect to some base point $\tilde{y}$ in the compactification of $\tilde{S}$. The set

$$\partial sh_{\tilde{y}}(U) := \partial\tilde{S} \cap sh_{\tilde{y}}(U)$$

is defined as the restriction of the shadow to the boundary. Assume that $sh_{\tilde{y}}(U)$ is Borel. The Patterson-Sullivan measure is supported on the boundary and therefore

$$\nu_{\tilde{x}}(sh_{\tilde{y}}(U)) = \nu_{\tilde{x}}(\partial sh_{\tilde{y}}(U))$$

If U is an open set, the shadow is open as well. Furthermore, if U is a singular point, the boundary shadow is a closed set containing an open set.
$\tilde{S}$ is a δ-hyperbolic space. Therefore, there exists a family of Gromov metrics $d_{\infty,\tilde{x}}$ on the boundary $\partial\tilde{S}$ which satisfy

$$\xi^{-(\eta\cdot\zeta)_{\tilde{x}}} \geq d_{\infty,\tilde{x}}(\eta,\zeta) \geq (1-\epsilon(\xi))(\xi^{-(\eta\cdot\zeta)_{\tilde{x}}})$$

It is our goal to understand the typical behavior of bi-infinite geodesics with respect to some measure which is strongly connected to the Patterson-Sullivan measure. First we show some criteria for non-typical boundary points.

Proposition 5.1. *Let $\tilde{x}$ be a point in the universal cover $\tilde{S}$ and let $B_{\tilde{x}}$ be the set of points $\eta \in \partial\tilde{S}$ such that $[\tilde{x},\eta]$ passes through finitely many singularities. Then $\nu_{\tilde{x}}(B_{\tilde{x}}) = 0$.*

Proof. Let $B_{\tilde{x},0} \subset B_{\tilde{x}}$ be the set of endpoints $\eta \in \partial\tilde{S}$ such that the connecting geodesic $[\tilde{x},\eta]$ does not pass through any singularity.
Since shadows of points are closed, $B_{\tilde{x},0}$ is the complement of countably many closed sets and therefore Borel.
Let $\Sigma_{\tilde{S}}$ be the set of singularities on $\tilde{S}$. Let $L \gg 0$ and $A_L \subset \Sigma_{\tilde{S}}$ be the set of singularities $\tilde{\varsigma}$ of distance at most L to $\tilde{x}$ so that $[\tilde{x},\varsigma]$ does not pass through any other singularity. It was shown by [Mas90, Theorem 1] that there is some bound $L_0 > 0$ such that for each

$L > L_0$ there are less than L^3 such singularities.

Clearly

$$\partial\tilde{S} - \partial sh_{\tilde{x}}(A_L) \supset B_{\tilde{x},0}$$

Let $k\pi$ be the cone angle at $\tilde{x}$. On the circle of directions at $\tilde{x}$, we choose a base direction θ_0 and a clockwise ordering. For simplicity we assume that θ_0 is the direction of the geodesic $[\tilde{x}, \tilde{\varsigma}]$ for some point $\tilde{\varsigma} \in A_L$.

To each point $\tilde{y} \neq \tilde{x}$ in the compactification of $\tilde{S}$, we associate a well-defined angle $\vartheta(\tilde{y}) \in [0, k\pi)$ which is the angle of $[\tilde{x}, \tilde{y}]$ at $\tilde{x}$ with respect to θ_0 and the choice of clockwise ordering. Let $\tilde{z} \neq \tilde{x}$ be a point in $[\tilde{x}, \tilde{y}]$. Clearly $\vartheta(\tilde{z}) = \vartheta(\tilde{y})$. We order the points $\tilde{\varsigma}_i \in A_L, i = 1, \ldots, n$ by their angle ϑ. After enlarging L, by Lemma 3.4 we can assume that

$$\begin{aligned} 0 &< \vartheta(\tilde{\varsigma}_{i+1}) - \vartheta(\tilde{\varsigma}_i) \leq \pi/3, i \leq n-1 \\ & k\pi - \vartheta(\tilde{\varsigma}_n) \leq \pi/3 \end{aligned}$$

We decompose $B_{0,\tilde{x}}$ into sets

$$\begin{aligned} S_i^L &:= \{\eta \in B_{0,\tilde{x}} | \vartheta(\tilde{\varsigma}_i) < \vartheta(\eta) < \vartheta(\tilde{\varsigma}_{i+1})\} \\ S_n^L &:= \{\eta \in B_{0,\tilde{x}} | \vartheta(\tilde{\varsigma}_n) < \vartheta(\eta)\} \end{aligned}$$

see Figure 7. We estimate the diameter of each set S_i^L with respect to the Gromov

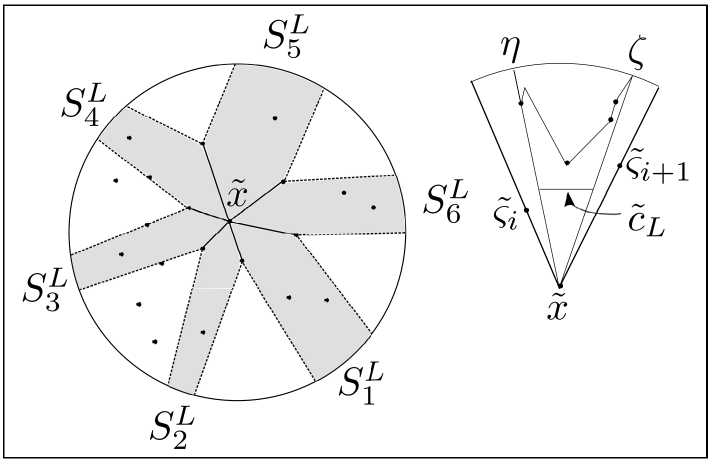

Figure 7: We cut out the shadows of all saddle connections of length at most L. The triangle formed of $\tilde{x}$ and the endpoints of $\tilde{c}_L$ does not contain a singularity and therefore it is euclidean. $\tilde{c}_L$ has distance at least $L/3$ to $\tilde{x}$.

metric $d_{\tilde{x},\infty}$.

Let η, ζ be points in S_i^L so that $\vartheta(\eta) \leq \vartheta(\zeta)$. We claim that

$$(\eta \cdot \zeta)_{\tilde{x}} \geq L/3 - 2\delta$$

Consider the triangle $\triangle(\tilde{x}, [\tilde{x}, \eta](s), [\tilde{x}, \zeta](s)), 0 < s < L$. The geodesic $[\tilde{x}, \eta](t)$ and $[\tilde{x}, \zeta](t), 0 < t \leq s$ do not pass through any singularity.
Denote by

$$\tilde{c}_s := [[\tilde{x}, \eta](s), [\tilde{x}, \zeta](s)], 0 < s < L$$

the third geodesic. $[\tilde{x}, \eta]$ and $[\tilde{x}, \zeta]$ sweep out a sector so that the angle at $\tilde{x}$ is less than $\pi/3$. The sector does not contain any singularity of distance less than L to $\tilde{x}$. $\tilde{c}_s, s < L$ remains in the sector and therefore does not pass through any singularity.
In the interior of a triangle there is no singularity either and consequently the triangle $\triangle(\tilde{x}, [\tilde{x}, \eta](s), [\tilde{x}, \zeta](s)), 0 < s < L$ is isometric to a euclidean triangle and the inner angle at $\tilde{x}$ is less than $\pi/3$. Therefore

$$d(\tilde{c}_s, \tilde{x}) > s/3, 0 < s < L$$

We can estimate the Gromov product

$$([\tilde{x}, \eta](s) \cdot [\tilde{x}, \zeta](s))_{\tilde{x}} > d(\tilde{x}, \tilde{c}_s) - 4\delta > s/3 - 2\delta, 0 < s < L$$

and therefore

$$(\eta \cdot \zeta)_{\tilde{x}} \geq L/3 - 2\delta \Rightarrow d_\infty(\eta, \zeta) \leq \xi^{-L/3+4\delta}$$

So

$$diam_\infty(S_i^L) \leq \xi^{-L/3+4\delta}$$

So, for each L, the sets $\bigcup S_i^L, i = 1, \dots n(L)$ cover $B_{\tilde{x},0}$. The diameter of S_i^L is bounded from above by a function $diam(L)$. Since $diam(L)$ decreases exponentially in L and $n(L)$ grows polynomially in L, the Hausdorff dimension of $B_{\tilde{x},0}$ has to be 0. Since the Hausdorff dimension of the whole boundary is at least 1, $B_{\tilde{x},0}$ has measure 0 with respect to each Patterson-Sullivan measure $\nu_{\tilde{y}}$.
For each geodesic ray $\tilde{r}$ from $\tilde{x}$ to a point in $B_{\tilde{x}}$ there is a last singularity $\tilde{r}$ passes through.
Therefore

$$B_{\tilde{x}} \subset \bigcup_{\tilde{\varsigma} \in \Sigma_{\tilde{S}}} B_{\tilde{\varsigma},0} \cup B_{\tilde{x},0}$$

Due to the fact that there are only countably many singularities, we deduce that

$$\nu_{\tilde{x}}(B_{\tilde{x}}) = 0$$

□

We need a stronger criterion for non-typical boundary points.
We recall that by Definition 3.3 and Remark 3.2 a geodesic ray $\tilde{r} \subset \tilde{S}$ is called quasi-straight if the sum of angles at one side is bounded:

$$\sum_{t>0} (\vartheta^+(\tilde{r}(t)) - \pi) < \infty$$

A bi-infinite geodesic is quasi-straight if the induced ray in positive direction is quasi-straight.
We called str_∂ the set of quasi-straight boundary points of geodesic rays.
We show that quasi-straight boundary points are of measure 0 with respect to each Patterson-Sullivan measure.

Proposition 5.2. $\nu_{\tilde{x}}(str_\partial) = 0$

The following Lemma is needed.

Lemma 5.3. *Let $\pi : \tilde{S} \to S$ be the flat universal cover of a closed flat surface. There is some positive function $R : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following holds: Let $\tilde{\varsigma} \in \tilde{S}$ be some singularity and let I be a closed interval of directions of length $l > 0$ at $\tilde{\varsigma}$. Let S_I be the set of boundary points $\eta \in \partial\tilde{S}$ such that the direction of $[\tilde{\varsigma}, \eta]$ is contained in I.*
Then the size of S_I is bounded from below the size of $R(l)$:

$$\nu_{\tilde{\varsigma}}(S_I) \geq R(l)$$

Proof. We first have to show that is S_I closed and therefore Borel. Let U be a standard neighborhood of $\tilde{\varsigma}$. Consider the continuous closest point projection $pr_U : \partial\tilde{S} \to U$. Since $S_I = pr^{-1}(V)$, where V is a closed sector in U, S_I is a closed subset of $\partial\tilde{S}$.
Let $\tilde{\varsigma}_i \in \Sigma_{\tilde{S}}, i = 1 \ldots n$ be finitely many singularities in the flat universal cover so that every singularity $\tilde{\varsigma} \in \Sigma_{\tilde{S}}$ is in the Γ-orbit of one $\tilde{\varsigma}_i$. We cover the circle of directions at $\tilde{\varsigma}_i$ with finitely many closed intervals $I_{i,j}$ of length $l/3$. We define the boundary intervals $S_{I_{i,j}} \subset \partial\tilde{S}$ as those points η so that the direction of $[\tilde{\varsigma}_i, \eta]$ is contained in $I_{i,j}$.
$S_{I_{i,j}}$ contains an open set and therefore has positive measure.
We define

$$R(l) := \min_{i,j} \ \nu_{\tilde{\varsigma}_i}(S_{I_{i,j}}) > 0$$

For every other singularity $\gamma(\tilde{\varsigma}_i)$ in the Γ-orbit of $\tilde{\varsigma}_i$ we can translate the covering of directions with γ and obtain the same measure.

$$\nu_{\gamma(\tilde{\varsigma}_i)}(S_{\gamma(I_{i,j})}) = \nu_{\tilde{\varsigma}_i}(S_{I_{i,j}})$$

Observe that each interval of directions I of length l at ς has to contain one of the smaller intervals of length $l/3$. $\square$

Proof of the Proposition. Recall that ∂str is the set of boundary points η of the universal cover such that for any point $\tilde{x} \in \tilde{S}$

$$\sum_{t>0} (\vartheta^+([\tilde{x}, \eta](t)) - \pi) < \infty$$

By Proposition 5.1 it suffices to consider only those quasi-straight points η such that $[\tilde{x}, \eta]$ passes through infinitely many singularities. Let $A_{\tilde{x}} \subset \partial\tilde{S}$ be the set of those quasi-straight endpoints η such that

$$\sum_{t>0} (\vartheta^+([\tilde{x}, \eta](t)) - \pi) \leq \pi/3$$

For each n let $A_{\tilde{x},n} \subset \Sigma_{\tilde{S}}$ be the set of singularities $\tilde{\varsigma}_n$ such that $[\tilde{x}, \tilde{\varsigma}_n]$ passes through n singularities and that at each singularity $\tilde{\varsigma} \in [\tilde{x}, \tilde{\varsigma}_n]$, the smaller angle between the two rays $[\tilde{x}, \tilde{\varsigma}]$ and $[\tilde{\varsigma}, \tilde{\varsigma}_n]$ is between π and $4\pi/3$, see figure 8.

For $\tilde{\varsigma}_{n+1} \in A_{n+1}$ let $\tilde{\varsigma}_n$ be the n-th singularity in $[\tilde{x}, \tilde{\varsigma}_{n+1}]$.

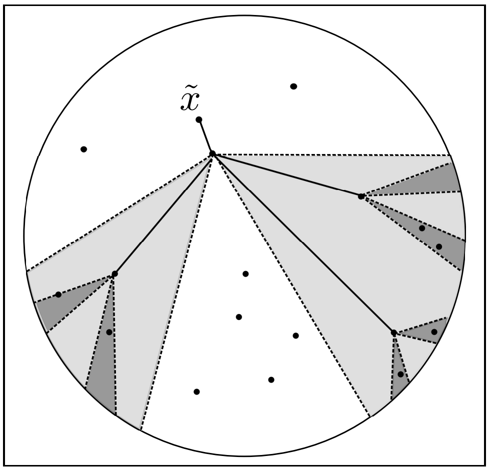

Figure 8: For any singularity in $A_{\tilde{x},1}$ the further points in $A_{\tilde{x},2}$ are contained in the light gray part. For the consecutive singularity in $A_{\tilde{x},3}$ we see a splitting in two parts, hence a Cantor construction.

$[\tilde{x}, \tilde{\varsigma}_n]$ passes through n singularities and the angle properties at each singularity are satisfied, so $\tilde{\varsigma}_n \in A_{\tilde{x},n}$.

So we can define the mapping

$$\phi_n : A_{\tilde{x},n+1} \to A_{\tilde{x},n}, \phi_n : \tilde{\varsigma}_{n+1} \to \tilde{\varsigma}_n$$

Consequently

$$sh_{\tilde{x}}(A_{\tilde{x},n}) \supset A_{\tilde{x},n+1}$$

By definition of $A_{\tilde{x}}$ it follows that

$$sh_{\tilde{x}}(A_{\tilde{x},n}) \supset A_{\tilde{x}}$$

Since $sh_{\tilde{x}}(A_{\tilde{x},n})$ is a countable union of closed sets, it is Borel. By uniqueness of geodesics it follows that the shadows are disjoint for two different points $\tilde{\varsigma}_n \neq \tilde{\varsigma}'_n \in A_{\tilde{x},n}$.

$$sh_{\tilde{x}}(\tilde{\varsigma}_n) \cap sh_{\tilde{x}}(\tilde{\varsigma}'_n) = \emptyset$$

Since the set of singularities is countable,

$$\nu_{\tilde{x}}(sh_{\tilde{x}}(A_{\tilde{x},n})) = \sum_{\tilde{\varsigma}_n \in A_{\tilde{x},n}} \nu_{\tilde{x}}(sh_{\tilde{x}}(\tilde{\varsigma}_n))$$

Let $\tilde{\varsigma}_{n+1}$ be a point in $A_{\tilde{\varsigma}_0,n+1}$ and denote by $\tilde{\varsigma}_n := \phi_n(\tilde{\varsigma}_{n+1})$ the last singularity on the geodesic $[\tilde{\varsigma}_0, \tilde{\varsigma_{n+1}}]$.
A point $\tilde{y}$ is contained in $sh_{\tilde{x}}(\tilde{\varsigma}_{n+1})$ only if the two geodesics $[\tilde{x}, \tilde{\varsigma}_n]$ and $[\tilde{\varsigma}_n, \tilde{y}]$ make angle between π and $4\pi/3$ at $\tilde{\varsigma}_n$. Therefore, the direction of $[\tilde{\varsigma}_n, \tilde{y}]$ is contained in two intervals of direction I_1, I_2 at $\tilde{\varsigma}_n$ which are both of length $\pi/3$.
On the other hand, the total angle at $\tilde{\varsigma}_n$ is at least 3π. The set of points $\tilde{y}$ in the whole shadow $sh_{\tilde{x}}(\tilde{\varsigma}_n)$ defines an interval of direction at $\tilde{\varsigma}_n$ of length at least π. Therefore, the shadow $sh_{\tilde{x}}(\phi_n^{-1}(\tilde{\varsigma}_n))$ misses a subinterval of directions of length at least $\pi/3$ at $\tilde{\varsigma}_n$.
Let R be the function of Lemma 5.3. It follows that

$$\begin{aligned} & \nu_{\tilde{\varsigma}_n}(sh_{\tilde{x}}(\tilde{\varsigma}_n)) > \nu_{\tilde{\varsigma}_n}(sh_{\tilde{x}}(\phi_n^{-1}(\tilde{\varsigma}_n))) + R(\pi/3) \\ \Leftrightarrow \quad & 1 - \frac{R(\pi/3)}{\nu_{\tilde{\varsigma}_n}(sh_{\tilde{x}}(\tilde{\varsigma}_n))} > \frac{\nu_{\tilde{\varsigma}_n}(sh_{\tilde{x}}(\phi_n^{-1}(\tilde{\varsigma}_n)))}{\nu_{\tilde{\varsigma}_n}(sh_{\tilde{x}}(\tilde{\varsigma}_n))} = \frac{\nu_{\tilde{x}}(sh_{\tilde{x}}(\phi_n^{-1}(\tilde{\varsigma}_n))}{\nu_{\tilde{x}}(sh_{\tilde{x}}(\tilde{\varsigma}_n))} \end{aligned}$$

The last equality is due to Corollary 2.1. Since $\nu_{\tilde{\varsigma}_n}(sh_{\tilde{x}}(\tilde{\varsigma}_n))$ is uniformly bounded by $\exp(e(\tilde{S}, \Gamma_S)diam)$, the left term has a uniform upper bound $c < 1$.
Therefore

$$\nu_{\tilde{x}}(sh_{\tilde{x}}(\phi_n^{-1}(\tilde{\varsigma}_n))) \leq c \cdot \nu_{\tilde{x}}(sh_{\tilde{x}}(\tilde{\varsigma}_n))$$

and so

$$\nu_{\tilde{x}}\left(sh_{\tilde{x}}(A_{\tilde{x},n+1})\right) \leq c\nu_{\tilde{x}}\left(sh_{\tilde{x}}(A_{\tilde{x},n})\right)$$

Since

$$A_{\tilde{x}} \subset \bigcap_{n \geq 0} sh_{\tilde{x}}(A_{\tilde{x},n})$$

$A_{\tilde{x}}$ is a subset of a measure 0 set. As we extended the Patterson-Sullivan measure to a complete measure, it follows that $\nu_{\tilde{x}}(A_{\tilde{x}}) = 0$ and therefore

$$\nu_{\tilde{y}}(A_{\tilde{x}}) = 0$$

for any $\tilde{y} \in \tilde{S}$.
It remains to show that the whole set ∂str has measure 0. Let η be a quasi-straight point which passes through infinitely many singularities.
Since $\vartheta^+([\tilde{x},\eta](t)) - \pi$ is non-negative, there is a time t_0 such that

$$\sum_{t>t_0} (\vartheta^+([\tilde{x},\eta](t)) - \pi) \leq \pi/3$$

As $[\tilde{x},\eta]$ passes through infinitely many singularities, there is some singularity $\tilde{\varsigma} \in [\tilde{x},\eta](t_0,\infty)$.
So η is contained in $A_{\tilde{\varsigma}}$ for some singularity $\tilde{\varsigma}$ where $A_{\tilde{\varsigma}}$ is defined as above. Therefore, the union of sets

$$\bigcup_{\tilde{\varsigma}\in\Sigma_{\tilde{S}}} A_{\tilde{\varsigma}} \supset str_{\partial}$$

covers the quasi-straight points up to measure 0.
Since the set of singularities Σ is countable, the covering has measure 0. □

We are able to define the geodesic flow and an appropriate measure as in [Bou95], [Hop71]. Since the flat metric is not smooth, we cannot make use of the unit tangent bundle for the geodesic flow. On the other hand, a point in the unit tangent bundle of a closed hyperbolic surface is in natural one-to-one correspondence with a parametrized bi-infinite unit speed geodesic. So the space of bi-infinite parametrized unit-speed geodesics plays the role of the unit tangent bundle.

Definition 5.2. *Let $\pi : \tilde{S} \to S$ be the flat universal cover of a closed flat surface.*
Let $\mathcal{G}\tilde{S}$ resp. $\mathcal{G}S$ be the set of all parametrized bi-infinite unit speed geodesics $\tilde{\alpha}$ resp. α in $\tilde{S}$ resp. S.
Both spaces are endowed with the metric:

$$d_{\mathcal{G}}(\tilde{\alpha},\tilde{\alpha}') := \int_{-\infty}^{\infty} d(\tilde{\alpha}(t),\tilde{\alpha}'(t))exp(-|t|)dt$$

One observes that the space $\mathcal{G}\tilde{S}$ is proper and $\mathcal{G}S$ is compact.
The group of Deck transformations Γ acts on $\mathcal{G}\tilde{S}$ as

$$\gamma : \tilde{\alpha} \mapsto \gamma(\tilde{\alpha})$$

Γ acts isometric properly discontinuously and freely on S. The same holds for the action of Γ on $\mathcal{G}\tilde{S}$. The natural mapping between the quotient space $\mathcal{G}\tilde{S}/\Gamma$ and the space $\mathcal{G}S$ is a homeomorphism.
Furthermore, we define the geodesic flow g_t acting on $\mathcal{G}\tilde{S}$ resp. $\mathcal{G}S$ as

$$g_t(\tilde{\alpha})(s) := \tilde{\alpha}(t+s)$$

Each geodesic $\tilde{\alpha} \in \mathcal{G}\tilde{S}$ can be projected to its endpoints and therefore we define

$$\tau : \mathcal{G}\tilde{S} \to \partial^2\tilde{S} - \triangle$$

By Proposition 2.5 τ is onto and equivariant with respect to Γ. g_t acts on the fibers of τ.

$$\tau \circ g_t = \tau$$

We will first define a measure on $\partial^2\tilde{S} - \triangle$.
Let $\tilde{x} \in \tilde{S}$ be a point and $\nu_{\tilde{x}}$ the Patterson-Sullivan measure.
Let $(\eta, \zeta) \in \partial^2\tilde{S} - \triangle$ be a pair of boundary points. There always exists a geodesic connecting η with ζ.
If the geodesic is unique up to reparametrization, we choose one such geodesic $[\eta, \zeta]$. Otherwise, by Proposition 3.8 the projection of each connecting geodesic to the base flat surface S is a core curve of the same maximal flat cylinder. So there are only countable many of those pairs (η, ζ). For simplicity reasons we choose a connecting geodesic $[\eta, \zeta]$ which projects to the central core curve of the maximal cylinder. Again $[\eta, \zeta]$ is uniquely defined up to reparametrization.
Let $\tilde{y}$ be a point on $[\eta, \zeta]$.
We define

$$\iota_{\tilde{x}} : (\partial^2\tilde{S} - \triangle) \to \mathbb{R}_+, \iota_{\tilde{x}}(\eta, \zeta) := \frac{d\nu_{\tilde{y}}}{d\nu_{\tilde{x}}}(\eta)\frac{d\nu_{\tilde{y}}}{d\nu_{\tilde{x}}}(\zeta)$$

It is a consequence of Corollary 2.1 that $\iota_{\tilde{x}}$ is independent of the choice of $\tilde{y} \in [\eta, \zeta]$.
We define the following Borel measure on $\partial^2\tilde{S} - \triangle$:

$$\tilde{\nu}_{\tilde{x}} := \iota_{\tilde{x}} * \nu_{\tilde{x}}^2$$

Proposition 5.3. *$\tilde{\nu}_{\tilde{x}}$ satisfies the following properties:*

i) $\tilde{\nu}_{\tilde{x}}$ is independent of the base point $\tilde{x}$.

ii) $\tilde{\nu}_{\tilde{x}}$ is Γ-invariant.

iii) $\tilde{\nu}_{\tilde{x}}$ is an infinite Radon measure without atoms.

Proof. Let $\nu_{\tilde{x}}$ be the Patterson-Sullivan measure with respect to some base point $\tilde{x}$.

i) Let $(\eta, \zeta) \in \partial^2\tilde{S} - \triangle$ be some pair of boundary points and $\tilde{y} \in [\eta, \zeta]$ some point. The following equation holds:

$$\frac{d\tilde{\nu}_{\tilde{x}}}{d\tilde{\nu}_{\tilde{y}}}(\eta, \zeta) = \frac{d\nu_{\tilde{x}}}{d\nu_{\tilde{y}}}(\eta)\frac{d\nu_{\tilde{x}}}{d\nu_{\tilde{y}}}(\zeta)\frac{d\nu_{\tilde{y}}}{d\nu_{\tilde{x}}}(\eta)\frac{d\nu_{\tilde{y}}}{d\nu_{\tilde{x}}}(\zeta) = 1$$

Consequently, it follows for arbitrary $\tilde{z} \in \tilde{S}$

$$\frac{d\tilde{\nu}_{\tilde{x}}}{d\tilde{\nu}_{\tilde{z}}}(\eta,\zeta) = \frac{d\tilde{\nu}_{\tilde{x}}}{d\tilde{\nu}_{\tilde{y}}}\frac{d\tilde{\nu}_{\tilde{y}}}{d\tilde{\nu}_{\tilde{z}}}(\eta,\zeta) = 1$$

Therefore, we can skip the index and abbreviate $\tilde{\nu}$.

ii) $\gamma^*\nu_{\gamma(\tilde{x})} = \nu_{\tilde{x}}$ and Γ preserves geodesics, so

$$\gamma^*\tilde{\nu}_{\gamma(\tilde{x})} = \tilde{\nu}_{\tilde{x}} = \tilde{\nu}_{\gamma(\tilde{x})}$$

iii) $\tilde{\nu}$ is absolutely continuous to $\nu_{\tilde{x}}^2$ and by Corollary 2.1, $\iota_{\tilde{x}}$ can be compared with the Busemann distance. Therefore, $\iota_{\tilde{x}}$ is locally bounded by positive constants from above and below. Consequently, $\tilde{\nu}$ is locally finite and atom free.
It remains to show that $\tilde{\nu}$ is infinite. Let $\eta \in \partial\tilde{S}$ be a point on the boundary, and let $B := B_\eta(\epsilon) \subset \partial\tilde{S}$ be a metric ball about η of radius ϵ with respect to the Gromov metric $d_{\infty,\tilde{x}}$. Up to making ϵ smaller we can assume that B is not dense on the boundary.
Each element of Γ acts on the boundary with north-south dynamics and the attracting fixed points are dense. Therefore, we can translate B with infinitely many elements $\gamma_i \in \Gamma, i \in \mathbb{N}$, so that $\gamma_i(B) \cap \gamma_j(B) = \emptyset, \forall i \neq j$. Since the ν_x is supported on the whole boundary and $\tilde{\nu}$ is absolutely continuous to $\nu_{\tilde{x}}^2$, $B \times B - \triangle$ is of positive $\tilde{\nu}$-measure. Since $\tilde{\nu}$ is Γ-invariant, $\tilde{\nu}$ is infinite.

$\square$

Let $(\eta,\zeta) \in \partial^2\tilde{S} - \triangle$ be a pair of distinct boundary points which is typical for $\tilde{\nu}$ and so, η,ζ are not quasi-straight. Moreover, let $[\eta,\zeta]$ be a connecting geodesic. The mapping $t \mapsto g_t([\eta,\zeta])$ defines an $\mathbb{R}$-parametrization of the fiber $\tau^{-1}(\eta,\zeta)$. We can pull back the Lebesgue measure ℓ on $\mathbb{R}$ to the fiber. This fiber measure is independent of the parametrization as the transition map is a translation.

Definition 5.3. *We define a measure on $\tilde{\mu}$ on $\mathcal{G}\tilde{S}$*

$$\tilde{\mu} := \tilde{\nu} \times \ell$$

$\tilde{\mu}$ is a Γ- invariant Radon measure.

Definition 5.4. *For each set $U \subset \tilde{S}$ we define:*

$$\mathcal{G}U := \{\tilde{\alpha} \in \mathcal{G}\tilde{S}, \tilde{\alpha}(0) \in U\}$$

The set $\mathcal{G}U$ is open resp. closed resp. Borel if and only if $U \subset \tilde{S}$ is open resp. closed resp. Borel.
Obviously $\gamma(\mathcal{G}U) = \mathcal{G}\gamma(U)$ for each $\gamma \in \Gamma$.
Let $F \subset \tilde{S}$ be a Borel fundamental domain of $\tilde{S}/\Gamma$ i.e. F is a Borel set so that, $\Gamma F = \tilde{S}, \gamma(F) \cap F = \emptyset, \forall \gamma \neq id$. $\mathcal{G}F$ naturally forms a fundamental domain for $\mathcal{G}\tilde{S}/\Gamma$.
We define the measure μ on $\mathcal{G}S$:
Let $U \subset \mathcal{G}S$ be Borel

$$\mu(U) := \tilde{\mu}(\pi^{-1}(U) \cap \mathcal{G}F)$$

Remark 5.1. *μ is independent of F. One can also choose directly a Borel fundamental domain in $\mathcal{G}\tilde{S}$ for Γ and also obtain the same measure μ. On the other hand, we will need special properties of the domain $\mathcal{G}F$ in section 5.3.2.*

Since $\tilde{\mu}$ is Radon and we can choose $F \subset \tilde{S}$ so that $\mathcal{G}F$ is bounded, μ is a finite Radon measure.

Proposition 5.4. *g_t acts μ-ergodically on $(\mathcal{G}S, \mu)$.*

Proof. We refer to the so-called Hopf Argument [Hop71] which we sketch here.
Let $f : \mathcal{G}S \to \mathbb{R}$ be a continuous function. By Birkhoff Ergodic Theorem $s^{-1}\int_0^s f(g_t)dt$ converges a.e. to a measurable g_t-invariant function f^*.
Recall that any two bi-infinite geodesics in the flat universal cover with a common positive not-quasi-straight endpoint η are asymptotic.
Therefore, up to a subset of μ-measure 0, for any two bi-infinite parametrized geodesics α, α' in $\mathcal{G}S$ one finds parametrized geodesics $\tilde{\alpha}_1, \tilde{\alpha}_1$, so that each of the pairs $(\tilde{\alpha}, \tilde{\alpha}_1)$, $(\tilde{\alpha}_1, \tilde{\alpha}_2)$, $(\tilde{\alpha}_2, \tilde{\alpha}')$ are asymptotic in positive or negative direction. Consequently, $f^*(\alpha) = f^*(\alpha')$ and so f^* is constant a.e. As μ is finite, continuous function are dense in $L^1(\mathcal{G}S, \mu)$ □

5.3.2 Typical behavior

On a flat surface, the infinite extension of a compact geodesic arc is typically not unique. It turns out that the set of geodesics, which pass through such an arc, typically has positive measure.
We estimate the size of shadows.

Definition 5.5. *Let $\pi : \tilde{S} \to S$ be the flat universal cover of a closed flat surface. Let $\tilde{x} \in \tilde{S}$ be a point and $\tilde{\varsigma}$ be some singularity.*
$\partial sh_{\tilde{x}}(\tilde{\varsigma})$ is closed and therefore measurable. We define

$$r_{\tilde{x}}(\tilde{\varsigma}) := \nu_{\tilde{x}}\left(sh_{\tilde{x}}(\tilde{\varsigma})\right) \exp(e(\tilde{S}, \Gamma_S) d(\tilde{x}, \tilde{\varsigma})) = \nu_{\tilde{\varsigma}}(sh_{\tilde{x}}(\tilde{\varsigma}))$$

The equation follows from Corollary 2.1. It is the goal to show, that $r_{\tilde{x}}(\tilde{\varsigma})$ is nearly constant and so one can roughly identify $\nu_{\tilde{x}}(\partial sh_{\tilde{x}}(\tilde{\varsigma}))$ with $\exp(-e(\tilde{S},\Gamma_S)d(\tilde{x},\tilde{\varsigma}))$.
Denote by $diam(S)$ the diameter of S. Let c_1, c_2 be local geodesics in S so that c_1 ends at a singularity and c_2 issues from some singularity. In Proposition 3.11 it was shown that there exists a local geodesic c of length at most $C_l(S) + l(c_1) + l(c_2)$ which first passes through c_1 and eventually passes through c_2.

Lemma 5.4. *Let $\tilde{S}$ be the flat universal cover of a closed flat surface S. The function $r_{\tilde{x}}(\varsigma)$ is bounded from above and below.*

$$\exp(e(\tilde{S},\Gamma_S)diam(S)) \geq r_{\tilde{x}}(\varsigma) \geq \exp(-e(\tilde{S},\Gamma_S)(2C_l(S) + diam(S)))/4$$

Proof. The first inequality follows from

$$r_{\tilde{x}}(\varsigma) = \nu_{\tilde{\varsigma}}(sh_{\tilde{x}}(\tilde{\varsigma})) \leq \nu_{\tilde{\varsigma}}(\partial\tilde{S}) \leq \exp(e(\tilde{S},\Gamma_S)diam(S))$$

The last estimation is due to Lemma 2.4.
On the other hand, let $\tilde{\varsigma}_0 \in \tilde{S}$ be some singularity. By Lemma 3.5 and Proposition 3.11 there is a set of at most 4 saddle connections $\tilde{s}_1, \dots \tilde{s}_4$ with endpoint $\tilde{\varsigma}_0$. The length of each $\tilde{s}_i$ is bounded by $C_l(S)$ and the shadows of all $\tilde{s}_i$ cover the whole boundary. To be more precise, let $\tilde{\varsigma}_i$ be the starting point of $\tilde{s}_i$.

$$\partial\tilde{S} = \bigcup_{i\leq 4} \partial sh_{\tilde{\varsigma}_i}(\tilde{\varsigma}_0)$$

As $\nu_{\tilde{\varsigma}_0}(\tilde{S}) \geq \exp(-e(\tilde{S},\Gamma_S)diam(S))$, it follows that

$$\sum_{i\leq 4} \nu_{\tilde{\varsigma}_0}(sh_{\tilde{\varsigma}_i}(\tilde{\varsigma}_0)) \geq \exp(-e(\tilde{S},\Gamma_S)diam(S))$$

Therefore, we can assume

$$r_{\tilde{\varsigma}_1}(\tilde{\varsigma}_0) = \nu_{\tilde{\varsigma}_0}(sh_{\tilde{\varsigma}_1}(\tilde{\varsigma}_0)) \geq \exp(-e(\tilde{S},\Gamma_S)diam(S))/4$$

We showed the existence of at least one uniformly short saddle connection $\tilde{s}_1$ with starting point $\tilde{\varsigma}_1$ and endpoint $\tilde{\varsigma}_0$ such that $r_{\tilde{\varsigma}_1}(\tilde{\varsigma}_0)$ is uniformly bounded from below.
Let $\tilde{x}$ be some point and let $\tilde{\varsigma}$ be some singularity. There exists a geodesic $\tilde{c}$ which first connects $\tilde{x}$ with $\tilde{\varsigma}$ and eventually passes through $\gamma(\tilde{s}_1)$ for some $\gamma \in \Gamma$. We can choose such a $\tilde{c}$ with length bounded from above by

$$l(\tilde{c}) \leq C_l(S) + l(\tilde{s}_1) + d(\tilde{x},\tilde{\varsigma}) \leq 2C_l(S) + d(\tilde{x},\tilde{\varsigma})$$

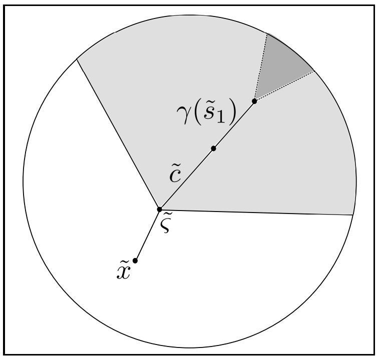

Figure 9: The shadow of $\tilde{\varsigma}$ with respect to $\tilde{x}$, marked light gray, contains the shadow of the endpoint of $\gamma(\tilde{s}_1)$, here dark gray.

see Figure 9.

The shadow of $\partial sh_{\tilde{x}}(\tilde{\varsigma})$ contains $\partial sh_{\tilde{x}}(\gamma(\tilde{\varsigma}_0))$, the shadow of the endpoint of $\gamma(\tilde{s}_1)$ with respect to the base point $\tilde{x}$. Therefore

$$\nu_{\tilde{x}}(\partial sh_{\tilde{x}}(\tilde{\varsigma})) \geq \nu_{\tilde{x}}(\partial sh_{\tilde{x}}(\gamma(\tilde{\varsigma}_0)))$$

Since

$$\begin{aligned} & \nu_{\tilde{x}}\left(\partial sh_{\tilde{x}}(\gamma(\tilde{\varsigma}_0))\right) \\ = \;& \nu_{\gamma(\tilde{\varsigma}_0)}(\partial sh_{\tilde{x}}(\gamma(\tilde{\varsigma}_0)))exp(-e(\tilde{S},\Gamma_S)d(\tilde{x},\gamma(\tilde{\varsigma}_0))) \\ = \;& \nu_{\gamma(\tilde{\varsigma}_0)}(\partial sh_{\gamma(\tilde{\varsigma}_1)}(\gamma(\tilde{\varsigma}_0)))exp(-e(\Gamma)d(\tilde{x},\gamma(\tilde{\varsigma}_0))) \\ = \;& \nu_{\tilde{\varsigma}_0}(\partial sh_{\tilde{\varsigma}_1}(\tilde{\varsigma}_0))exp(-e(\tilde{S},\Gamma_S)d(\tilde{x},\gamma(\tilde{\varsigma}_0))) \\ = \;& r_{\tilde{\varsigma}_1}(\tilde{\varsigma}_0)exp(-e(\tilde{S},\Gamma_S)d(\tilde{x},\gamma(\tilde{\varsigma}_0))) \\ \geq \;& \exp(-e(\tilde{S},\Gamma_S)diam(S) - e(\tilde{S},\Gamma_S)d(\tilde{x},\gamma(\tilde{\varsigma}_0)))/4 \end{aligned}$$

We conclude

$$\begin{aligned} r_{\tilde{x}}(\tilde{\varsigma}) \;&\geq\; \nu_{\tilde{x}}(sh_{\tilde{x}}(\gamma(\tilde{\varsigma}_0)))\exp(e(\tilde{S},\Gamma_S)d(\tilde{x},\tilde{\varsigma})) \\ &\geq\; \exp(e(\tilde{S},\Gamma_S)(d(\tilde{x},\tilde{\varsigma}) - diam(S) - d(\tilde{x},\gamma(\tilde{\varsigma}_0)))/4 \\ &=\; \exp(-e(\tilde{S},\Gamma_S)(2C_l(S) + diam(S)))/4 \end{aligned}$$

□

As a side note, the developed tools allow to estimate the entropy.

Corollary 5.1. *Let $\tilde{\varsigma} \in \tilde{S}$ be a singularity in the universal cover. Let $\Sigma_{\tilde{\varsigma},0}$ be the set of singularities $\tilde{\varsigma}' \neq \tilde{\varsigma}$ so that the connecting geodesic $[\tilde{\varsigma}, \tilde{\varsigma}']$ is a saddle connection. Denote* *by $C_l(S)$ the constant as in Lemma 5.4. The entropy satisfies the following inequalities:*

$$4\exp(e(\tilde{S},\Gamma_S)(2C_l(S)+diam)) \geq \sum_{\tilde{\varsigma}'\in\Sigma_{\tilde{\varsigma},0}} \exp(-e(\tilde{S},\Gamma_S)d(\tilde{\varsigma},\tilde{\varsigma}')) \geq \exp(-e(\tilde{S},\Gamma_S)diam)$$

Proof. We compute the Patterson-Sullivan measure $\nu_{\tilde{\varsigma}}$ with respect to the orbit $\Gamma\tilde{\varsigma}$. The total measure $\nu_{\tilde{\varsigma}}(\partial\tilde{S})$ is 1.
Let $\varsigma_1 \neq \varsigma_2 \in \Sigma_{\tilde{\varsigma},0}$. The shadows $sh_{\tilde{\varsigma}}(\varsigma_i), i = 1,2$ are disjoint and the complement of all the shadows of saddle connections has measure 0. It follows that

$$1 = \nu_{\tilde{\varsigma}}(\partial\tilde{S}) = \sum_{\tilde{\varsigma}'\in\Sigma_{\tilde{\varsigma},0}} \exp(-e(\tilde{S},\Gamma_S)d(\tilde{\varsigma},\tilde{\varsigma}'))r_{\tilde{\varsigma}}(\tilde{\varsigma}')$$

By Lemma 5.4 $r_{\tilde{\varsigma}}(\tilde{\varsigma}')$ is bounded by the constants,

$$\exp(e(\tilde{S},\Gamma_S)diam) \geq r_{\tilde{\varsigma}}(\tilde{\varsigma}') \geq \exp(-e(\tilde{S},\Gamma_S)(2C_l(S)+diam))/4$$

□

Remark 5.2. *It is not clear if the first inequality is redundant, compare the example in section 6.*

We return to the geodesic flow.

Theorem 5.4. *There is a constant $C(S) > 0$ which depends on the geometry of S such that the following holds: For any local geodesic $c : [0,s] \to S$ in S of positive finite length, let c_{ext} be the maximal extension of c with the property that the extension in unique. A typical geodesic passes through the geodesic arc c with a frequency F which is bounded from above and below by*

$$C(S)^{-1}exp(-e(\tilde{S},\Gamma_S)l(c_{ext})) \leq F \leq C(S)exp(-e(\tilde{S},\Gamma_S)l(c_{ext}))$$

Proof. Let $c : [0,s] \to S, s > 0$ be a local geodesic in S. Denote by A_c the set

$$A_c := \{\alpha \in GS : \exists 0 < t \leq 1 : g_t(\alpha)|_{[0,s]} = c\}$$

The set A_c is Borel.
As g_t acts ergodically with respect to μ, for any typical geodesic $\alpha \in \mathcal{G}S$

$$\lim_t \frac{1}{2t}\int_{-t}^{t} 1_{A_c}(g_t(\alpha))dt = \frac{\mu(A_c)}{\mu(\mathcal{G}S)}$$

Choose some lift $\tilde{c}$ of c in the universal cover. Let $F \subset \tilde{S}$ be a Borel fundamental domain of S. Furthermore, we can assume that F contains $\tilde{c}(0)$ in its interior. Let $F' \subset F$ be

the subdomain of F with the removed sides.
There is some $\epsilon > 0$ such that

$$\tilde{\alpha}(0) = \tilde{c}(0), |t| < \epsilon \Rightarrow \tilde{\alpha}(t) \in \overset{\circ}{F}$$

and

$$d(\gamma(\tilde{c}(0)), F) > \epsilon, \forall \gamma \in \Gamma - id$$

For simplicity reasons we can choose ϵ so that ϵ^{-1} is an integer..
We decompose $A_c \subset \mathcal{G}S$ in sets

$$A_{c,i} := \{\alpha \in \mathcal{G}S : g_t(\alpha)|_{[0,s]} = c, i\epsilon < t \leq (i+1)\epsilon\}$$

Since the geodesic flow translates the sets $g_\epsilon(A_{c,i}) = A_{c,i+1}$, all the sets $A_{c,i}$ have the same measure.
Since A_c equals the disjoint union $\overset{\circ}{\bigcup}_{i \leq \epsilon^{-1}} A_{c,i}$,

$$\mu(A_c) = \frac{s}{\epsilon}\mu(A_{c,0})$$

Let

$$\tilde{A}_{\tilde{c},0} := \{\tilde{\alpha} \in G\tilde{S} : g_t(\tilde{\alpha})|_{[0,s]} = \tilde{c}, 0 < t \leq \epsilon\}$$

One observes that $\pi^{-1}(A_{c,0}) \cap \mathcal{G}F' = \tilde{A}_{\tilde{c},0}$.
Therefore

$$\mu(A_{c,0}) = \tilde{\mu}(\tilde{A}_{\tilde{c},0})$$

Let $\tilde{x} := \tilde{c}(s/2)$ be the midpoint of $\tilde{c}$. Each geodesic in $\tilde{\alpha} \in \tilde{A}_{\tilde{c},0}$ has the property that its endpoints are contained in the following shadow:

$$\tau(\tilde{\alpha}) \in sh_{\tilde{x}}(\tilde{c}(0)) \times sh_{\tilde{x}}(\tilde{c}(s)) \subset \partial^2\tilde{S} - \triangle$$

On the other hand, for each pair of points

$$(\eta, \zeta) \in sh_{\tilde{x}}(\tilde{c}(0)) \times sh_{\tilde{x}}(\tilde{c}(s))$$

the concatenation of $[\eta, \tilde{x}]$ and $[\tilde{x}, \zeta]$ is locally geodesic outside $\tilde{x}$ and coincides with $\tilde{c}$ about $\tilde{x}$. Therefore, the concatenation is a geodesic passing through $\tilde{c}$. So

$$\tau(\tilde{A}_{\tilde{c},0}) = sh_x(\tilde{c}(0)) \times sh_x(\tilde{c}(s))$$

Furthermore, for each pair of points $(\eta, \zeta) \in \tau(\tilde{A}_{\tilde{c},0})$ there exists a unique connecting geodesic $[\eta, \zeta]$ which coincides with $\tilde{c}$ on the interval $[0, s]$. Consequently

$$g_t([\eta, \zeta]) \in \tilde{A}_{\tilde{c},0} \Leftrightarrow t \in [0, \epsilon]$$

Therefore, the Lebesgue measure of the intersection of the fiber with $\tilde{A}_{\tilde{c},0}$ is ϵ. We deduce

$$\mu(A_c) = \frac{1}{\epsilon}\tilde{\mu}(\tilde{A}_{\tilde{c},0}) = \tilde{\nu}(sh_{\tilde{x}}(\tilde{c}(0)) \times sh_{\tilde{x}}(\tilde{c}(s)))$$

It remains to compute the measure of the shadow product of $\tilde{c}$.
Recall that $\nu_{\tilde{x}}$ is the Patterson-Sullivan measure with respect to the base point $\tilde{x}$.
Let $(\eta, \zeta) \in \tilde{\nu}(sh_{\tilde{x}}(\tilde{c}(0)) \times sh_{\tilde{x}}(\tilde{c}(s)))$ be a pair of not quasi-straight points.

Each geodesic connecting η with ζ has to pass through $\tilde{x}$ and so

$$\tilde{\nu}(sh_{\tilde{x}}(\tilde{c}(0)) \times sh_{\tilde{x}}(\tilde{c}(s))) = \nu_{\tilde{x}}(sh_{\tilde{x}}(\tilde{c}(0))) * \nu_{\tilde{x}}(sh_{\tilde{x}}(\tilde{c}(s)))$$

Assume that the endpoint $\tilde{c}(s)$ of $\tilde{c}$ is regular. Since geodesics are straight line segments outside the singularities, there is locally only one possibility of extending $\tilde{c}$ in positive direction so that it remains geodesic. On the other hand, at each singularity there is a one-parameter family of locally geodesic extensions. We extend $\tilde{c}$ as far as possible in positive as well as negative direction as long as the extension is unique, i.e. until the extension either hits a singularity or tends to infinity. We call the extended geodesic $\tilde{c}_{ext}$.
If $\tilde{c}_{ext}$ is infinite, one of the two factors in the product shadow is a point and therefore the product has $\tilde{\nu}$-measure 0.
Assume that the extension is finite.
We parametrize the extended geodesic $\tilde{c}_{ext} : [-s_1, s_2] \to S$ so that $\tilde{c}_{ext}|_{[0,s]} = \tilde{c}$. By uniqueness of the extension

$$\partial sh_{\tilde{x}}(\tilde{c}(0)) = \partial sh_{\tilde{x}}(\tilde{c}_{ext}(-s_1))$$

and

$$\partial sh_{\tilde{x}}(\tilde{c}(s)) = \partial sh_{\tilde{x}}(\tilde{c}_{ext}(s_2))$$

Since the extension is finite, both endpoints of $\tilde{c}_{ext}$ are singularities. It follows from Lemma 5.4 that there is a universal constant $C(S)$ such that

$$C(S)^{-1}exp(-e(\tilde{S}, \Gamma_S)(s_1 + s/2)) \leq \nu_{\tilde{x}}(sh_{\tilde{x}}(\tilde{c}_{ext}(-s_1))) \leq C(S)exp(-e(\tilde{S}, \Gamma_S)(s_1 + s/2))$$

in the same way

$$C(S)^{-1}exp(-e(\tilde{S}, \Gamma_S)(s_2 - s/2)) \leq \nu_{\tilde{x}}(sh_{\tilde{x}}(\tilde{c}_{ext}(s_2))) \leq C(S)exp(-e(\tilde{S}, \Gamma_S)(s_2 - s/2))$$

since $l(\tilde{c}_{ext}) = s_2 + s_1$

$$C(S)^{-2}exp(-e(\tilde{S}, \Gamma_S)l(\tilde{c}_{ext})) \leq \mu(A_c) \leq C(S)^2 exp(-e(\tilde{S}, \Gamma_S)l(\tilde{c}_{ext}))$$

Let be $\alpha \in \mathcal{G}S$ be a μ-typical geodesic. It follows that the time α spends in c is proportional $exp(-e(\tilde{S}, \Gamma_S)l(c_{ext}))$. Consequently, the frequency α enters c is proportional to

$$exp(-e(\tilde{S}, \Gamma_S)l(c_{ext}))$$

. □

6 Example branched cover of the torus

We estimate the entropy and Hausdorff dimension for a family of well-studied examples. It is the family of branched n-sheeted coverings over the standard unit torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ which branches over one point with maximal ramification. The smallest example is the so-called square tiled L-surface S_3, see Figure 10.

It is a consequence of the Riemann Hurrewicz formula that such branched coverings

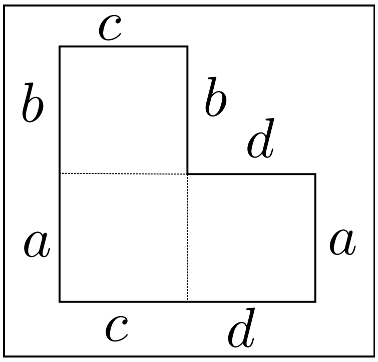

Figure 10: The figure indicates the square tiled L-surface. Each boundary side has length 1. The dashed lines indicate that there is a 3-sheeted branched covering map from S_3 to the torus with one branch point of maximal ramification.

only exist if the number of sheets is odd. Let $\pi : S_n \to T^2$ be such an n-sheeted flat branched covering. There is only one singularity $\varsigma \in S_n$ on the flat surface, which is the branch point.
The distance of any point to ς is at most $\frac{\sqrt{2}}{2}$ and therefore the diameter is bounded

$$diam(S_n) = \sqrt{2}$$

It follows that $\delta_{inf}(\tilde{S}_n)$, the minimal Gromov hyperbolic constant of the flat universal cover, can be estimated.

$$\sqrt{2} \geq \delta_{inf}(\tilde{S}_n) \geq \frac{1}{2\sqrt{2}}$$

Therefore, the base ξ for the Gromov metric $d_{\infty,\tilde{x}}$ on the boundary of the flat universal cover $\partial\tilde{S}_n$ is bounded from above

$$\xi := \frac{1}{2}\xi(\delta_{inf}(\tilde{S})) \leq 2^{\sqrt{2}}$$

Let c_1,c_2 be any two locally geodesic arcs on S_n so that c_1 ends at ς and c_2 issues from ς. One observes that any seperatrix emanating from the singularity in vertical resp. horizontal direction is a saddle connection of length 1. Moreover, there exists one of these vertical, resp. horizontal saddle connections s such that the concatenation $c_1 * s * c_2$ is geodesic. Therefore, the constant $C_l(S)$ from Proposition 3.11 is bounded from above by 1.
On the other hand, any saddle connection has length at least 1 and therefore $C_l(S) = 1$.
We make use of the formula in Corollary 5.1 to compute the entropy $e(\tilde{S}_n, \Gamma_{S_n})$ of S_n which we abbreviate:

$$e(\Gamma_n) := e(\tilde{S}_n, \Gamma_{S_n})$$

Let $\tilde{\varsigma}$ be a singularity in the universal cover. Denote by $\Sigma_{\tilde{\varsigma},0}$ the set of singularities ς' on $\tilde{S}$ so that $[\tilde{\varsigma}, \tilde{\varsigma}']$ is a saddle connection. We showed that

$$\begin{aligned} & 4\exp(e(\Gamma_n)(2C_l(S_n) + diam(S_n))) \\ \geq & \sum_{\tilde{\varsigma}' \in \Sigma_{\tilde{\varsigma},0}} \exp(-e(\Gamma_n) d(\tilde{\varsigma}, \tilde{\varsigma}')) \\ \geq & \exp(-e(\Gamma_n) diam(S_n)) \end{aligned}$$

Therefore, we have to compute the length of each saddle connection.
Let $x_0 \in T^2$ be the image of the branch point. Since the isometric universal cover of the flat torus can be identified with $\mathbb{R}^2$, we can choose the unit square as a fundamental domain of T^2 such that the origin projects to x_0.
Every saddle connection s in S_n projects to a straight line l on T^2. It connects the x_0 with itself and meanwhile never runs through x_0.
We can lift l to a straight line $\tilde{l}$ in $\mathbb{R}^2$ which emanates from the origin. $\tilde{l}$ ends at a point in $\mathbb{Z}^2$ and never passes through a point in $\mathbb{Z}^2$ in the meantime. Therefore, the endpoint has coordinates $(p, q) \in \mathbb{Z}^2$ which are relatively prime, denoted as $p \perp q$.
On the other hand, let $\tilde{l}$ be a straight line in $\mathbb{R}^2$ which emanates at the origin and ends at a point $(p, q) \in \mathbb{Z}^2$, such that $p \perp q$. $\tilde{l}$ projects a geodesic line l on T^2 which starts and ends at x_0 and does not pass through x_0 in the meantime. For each of these lines l, there are n distinct lifts in S_n. Therefore, there is an n-to-1 map of saddle connections on S_n to points $(p, q) \in \mathbb{Z}^2, p \perp q$. This map is length-preserving.

We make use of mirror symmetries in $\mathbb{R}^2$ along the coordinate axes and the diagonal. There is a length-preserving 8-to-1 map from relatively prime points.

$$\left\{(p,q) \in \mathbb{Z}^2 - \{-1,0,1\}^2, p \perp q\right\} \to \left\{(p,q) \in \mathbb{N}^2, 1 \leq p < q, p \perp q\right\}$$

Consequently, for any pair of points $(p,q) \in \mathbb{N}^2, 1 \leq p < q, p \perp q$, there exist $8n$ corresponding saddle connections in S_n of the same length. Additionally, we have to count the n horizontal and n vertical saddle connections and the $2n$ diagonal saddle connections in S_n.
We apply the values of C_l and $diam$ to the formula.

$$4\exp\left(e\left(\Gamma_n\right)\left(2+\sqrt{2}\right)\right) \geq \sum_{\tilde{\varsigma}' \in \Sigma_{\tilde{\varsigma},0}} \exp\left(-e\left(\Gamma_n\right) d\left(\tilde{\varsigma}, \tilde{\varsigma}'\right)\right) \geq \exp\left(-e\left(\Gamma_n\right)\sqrt{2}\right)$$

Therefore

$$\begin{aligned} & \exp\left(-e\left(\Gamma_n\right)\sqrt{2}\right) \\ \leq\ & 2n\left(\exp\left(-e\left(\Gamma_n\right)\right) + \exp\left(-e\left(\Gamma_n\right)\sqrt{2}\right)\right) + 8n \sum_{2 \leq p < q, p \perp q} \exp\left(-e\left(\Gamma_n\right)\sqrt{p^2+q^2}\right) \end{aligned}$$

The inequality is redundant. So we cannot find an upper bound for $e(\Gamma_n)$.
On the other hand,

$$\begin{aligned} & 4\exp\left(e\left(\Gamma_n\right)\left(2+\sqrt{2}\right)\right) \\ \geq\ & 2n\left(\exp\left(-e\left(\Gamma_n\right)\right) + \exp\left(-e\left(\Gamma_n\right)\sqrt{2}\right)\right) + 8n \sum_{2 \leq p < q, p \perp q} \exp\left(-e\left(\Gamma_n\right)\sqrt{p^2+q^2}\right) \end{aligned}$$

We find lower bounds for the entropy and Hausdorff dimension

n	$e(\Gamma_n)$	$dim > \frac{e(\Gamma_n)}{\sqrt{2}log(2)}$
3	> 0.64	> 0.65
5	> 0.72	> 0.73
7	> 0.77	> 0.78
9	> 0.82	> 0.84
11	> 0.86	> 0.88
13	> 0.89	> 0.91
15	> 0.92	> 0.94
17	> 0.94	> 0.96
19	> 0.96	> 0.98
21	> 0.98	> 1
23	> 1	> 1.02

Since the Hausdorff dimension is always at least 1, this is only remarkable for $n \geq 23$. Moreover, in this family the entropy grows logarithmically in the combinatorics of the covering i.e. in n. We already showed in section 4.2 that the entropy cannot grow faster and therefore the formula is asymptotically sharp.
Recall that the area of S_n is n. We already showed in Remark 2.1 that the Hausdorff dimension of the boundary is invariant under scaling the metric on the flat surface S_n.

7 Periodic points of the Arnoux-Yoccoz diffeomorphism

After analyzing geometric properties of flat metrics, we investigate the behavior of a special affine diffeomorphism. The methods used here evolve from symbolic dynamics and therefore differ from the ones in the previous sections. We introduce new notations.

7.1 Basic concepts

Definition 7.1. *Let S be some set and $f : S \to S$ be some self-map.*
A point $x \in S$ is called f-periodic if $f^k(x) = x$ for some $k > 0$. We call x f-preperiodic if the f-orbit of x contains a periodic point.
The set of periodic resp. preperiodic points is denoted by $Per(f)$ resp. $PPer(f)$.

For a finite alphabet $\mathcal{A}$, we denote by $\mathcal{A}^n$ the set of words of length n with letters in $\mathcal{A}$. If $n = \mathbb{N}$ resp. $n = \mathbb{Z}$, $\mathcal{A}^n$ is the set of one-sided infinite resp. bi-infinite words.
There exists the right shift map s acting on the set of one-sided infinite as well as on the set of bi-infinite words, independent of the alphabet.

$$s : ((w_i)_i) \mapsto (w_{i+1})_i$$

Definition 7.2. *We call a word $w \in \mathcal{A}^n, n = \mathbb{N}, \mathbb{Z}$ periodic resp. preperiodic if it is periodic resp. preperiodic under the shift map. We call the set of all periodic resp. preperiodic words $Per(\mathcal{A})$ resp. $PPer(\mathcal{A})$.*

Convention: We require that $0 \in \mathbb{N}$. An interval $I \subset \mathbb{R}$ is called half-open if it is closed to the left and open to the right.

Let $u = (u_i)_i \in A^n$ be a finite word and $v = (v_i)_i$ be a one-sided infinite or finite word. We introduce the notation $w = uv$ as the concatenation of u and v, $w_i := u_i, i \leq n, w_i := v_{i-n}, i > n$.
If $u, v \in \mathcal{A}^{\mathbb{N}}$ are both one-sided infinite, the concatenation $w = uv \in \mathcal{A}^{\mathbb{Z}}$ is defined as $w_i := u_{-i+1}, i < 0, w_i := v_i, i \geq 0$.

On the set of one-sided infinite words there is a family of metrics, so-called word metrics, defined via a bijective map: $en : \mathcal{A} \to \{1, \ldots, |\mathcal{A}|\}$

$$d(u, v) := \sum_i |en(u_i) - en(v_i)| 2^{-i}$$

Two different metrics evolving from different choices of en are bilipschitz. The distances of any two infinite words u, v is small if and only if the first letters of u, v equal.
We are interested in words arising from expansions. So we introduce the term of $(\mathcal{A}, f)$-expansions:
Let $S_a, a \in \mathcal{A}$ be a finite partition of some set S and let $f : S \to S$ be some self-map. We define $d_{f,\mathcal{A}} : S \to \mathcal{A}^{\mathbb{N}}$, an expansion of a point, as a word with letters in $\mathcal{A}$. The letter a_i corresponds to the set where $f^i(x)$ is situated.

$$d_{f,\mathcal{A}} : S \to \mathcal{A}^{\mathbb{N}}, x \mapsto (a_i)_i \Leftrightarrow f^i(x) \in S_{a_i}, \forall i \in \mathbb{N}$$

f acts on the $(f, \mathcal{A})$-expansion as a shift map s.

$$d_{f,\mathcal{A}} \circ f = s \circ d_{f,\mathcal{A}}$$

In our case $S = [0, 1)$ is the half-open unit interval. Furthermore, the partitioning sets S_a are half-open intervals. f is always a right continuous mapping, and therefore $d_{f,\mathcal{A}}$ is right continuous with respect to each word metric.
Let $\alpha \in [0, 1)$ be some fixed number. We construct two kinds of partitions:
We define the binary partition as $\mathcal{B} = \{0, 1\}, S_0 = [0, \alpha), S_1 = [\alpha, 1)$ and $\mathcal{M}$ as some refinement which meets combinatorial properties i.e. it is Markov with respect to f. As the $\mathcal{M}$-partition is a refinement of the $\mathcal{B}$-partition, there exists a projection

$$\tau : \mathcal{M} \to \mathcal{B}, \tau(m) = b \Leftrightarrow S_m \subset S_b$$

τ extends to $\tau_n : \mathcal{M}^n \to \mathcal{B}^n$ where n is finite, $\mathbb{N}$ or $\mathbb{Z}$. Whenever n is clear from the context, we skip the index and abbreviate τ.

Remark 7.1. *In the literature there is an ambiguity for the notation. A point whose orbit is finite under some map f is commonly denoted as f-preperiodic, whereas f-periodic points are characterized by the property that they are fixed points of some iterate of f. On the other hand, what we call a preperiodic resp. periodic expansion appears in the literature as a periodic resp. purely periodic expansion.*

7.2 Geometrical setting

Convention: Let $g \in \mathbb{N}, g \geq 3$ be a fixed constant for the rest this work.

Let α be the real root of $1 - \sum_{i=1}^{g} x^i$.

Following [Arn88], [AY81] and [Bow09] there is a fundamental domain F of a flat surface S of genus g.

$$F = [0,1)^2 \cap \left\{ (x,y) : x \geq \sum_{i=1}^{n} \alpha^i \Rightarrow y < \sum_{j=1}^{g-n} \alpha^j \right\} \cap \left\{ (x,y) : y \geq \sum_{i=1}^{n} \alpha^i \Rightarrow x < \sum_{j=1}^{g-n} \alpha^j \right\}$$

We have an explicit description of the Arnoux-Yoccoz diffeomorphism Φ acting on S. We describe Φ as a map on F.

$$\Phi : (x,y) \mapsto \begin{cases} (\alpha^{-1}x - 1, & \alpha(y+1)) \quad \text{if} \quad x \geq \alpha \\ (\alpha^{-1}(x+1/2) - 3/2, & \alpha y) \quad \text{if} \quad \alpha > x \geq (3\alpha - 1)/2 \\ (\alpha^{-1}(x+1/2) - 1/2, & \alpha y) \quad \text{if} \quad (3\alpha - 1)/2 > x \geq 0 \end{cases}$$

Let $\pi_1, \pi_2 : F \to [0,1)$ be the canonical projections of the fundamental domain onto the

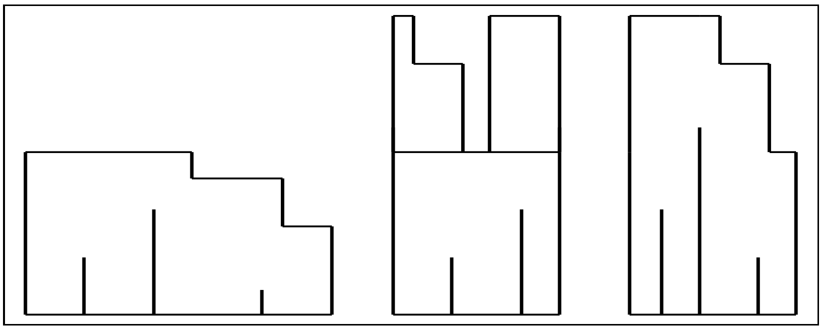

Figure 11: The gluing sequence indicates how the Arnoux Yoccoz diffeomorphism acts on F.

coordinate axes and let

$$\Phi_i := \pi_i \circ \Phi : F \to [0,1)$$

We consider $\Phi_1 : F \to [0,1]$. By definition of Φ, the projection Φ_1 is independent of the vertical coordinate and therefore descends to

$$\Phi_1 : [0,1) \to [0,1), x \mapsto \begin{cases} \alpha^{-1}x - 1 & \text{if} \quad x \geq \alpha \\ \alpha^{-1}(x+1/2) - 3/2, & \text{if} \quad \alpha > x \geq (3\alpha - 1)/2 \\ \alpha^{-1}(x+1/2) - 1/2 & \text{if} \quad (3\alpha - 1)/2 > x \geq 0 \end{cases}$$

7.3 Expansion

Let

$$\mathcal{B} := \{0,1\}, S_0 := [0,\alpha), S_1 := [\alpha, 1)$$

be a partition of the unit interval. We consider the following maps:

$$T_\alpha :\to [0,1) : x \mapsto \alpha^{-1}x \bmod 1$$

and the map $\Phi_1 : [0,1) \to [0,1)$ as described in the previous section. With respect to both mappings we define the expansions of the interval: The standard α-expansion:

$$d := d_{T_\alpha,\mathcal{B}} : [0,1) \to \mathcal{B}^{\mathbb{N}}$$

and the $(\Phi_1, \mathcal{B})$-expansion

$$d_{\Phi_1,\mathcal{B}} : [0,1) \to \mathcal{B}^{\mathbb{N}}$$

We define the σ-expansion $\sigma : F \to \mathcal{B}^{\mathbb{Z}}$ of the whole fundamental domain F which is the concatenation of the words $d(y)$ and $d_{\Phi_1,\mathcal{B}}(x)$.

$$\sigma : F \to \mathcal{B}^{\mathbb{Z}} : (x,y) \mapsto d(y)d_{\Phi_1,\mathcal{B}}(x)$$

The σ-expansion is not arbitrary. With respect to σ, one observes that the action of the Arnoux-Yoccoz diffeomorphism Φ commutes with the shift map s.

$$\sigma \circ \Phi = s \circ \sigma$$

Therefore, a point (x,y) is periodic under Φ only if $\sigma(x,y)$ is a periodic word.
So we have to investigate the expansions d and $d_{\Phi_1,\mathcal{B}}$.
The standard α-expansion d was studied extensively. We deal with the situation that α is a pisot unit.

Definition 7.3. *Let n be a finite number, $\mathbb{N}$ or $\mathbb{Z}$. A word $b \in \mathcal{B}^n$ is admissible if and only if it does not contain the subword $\underbrace{1\dots1}_{g}$.*

We recall standard facts about standard α-expansions and refer to [Aki98], [Sch80].

Proposition 7.1. *i) The image $d([0,1)) \subset \mathcal{B}^{\mathbb{N}}$ is the set of admissible words.*

ii) d is injective. Moreover, $d(x) = (b_i)_i \Leftrightarrow x = \sum_{i\geq 0} b_i\alpha^{i+1}$.

iii) $PPer(T_\alpha) = \mathbb{Q}[\alpha] \cap [0,1)$ where $\mathbb{Q}[\alpha]$ denotes the polynomial ring over α with rational coefficients. Equivalently, the set of preperiodic admissible one-sided infinite words equals the set $d(\mathbb{Q}[\alpha] \cap [0,1))$.

iv) $d(\alpha x) = 0d(x)$*, where* $0d(x)$ *denotes the concatenation. In the same way,* $d(\alpha x + \alpha) = 1d(x)$ *if* $\alpha x + \alpha < 1$.

Lemma 7.1. *If* y *is an algebraic integer, then the standard* α*-expansion of a point* $y \in [0, 1) - \{0\}$ *is not periodic .*
On the other hand, if y *is rational and smaller than some constant* $c > 0$*, then the standard* α*-expansion of* y *is periodic.*

Proof. [Aki98, Theorem 1, Theorem 2] □

Statement *ii)* of Proposition 7.1 is central. It implies that for any admissible periodic word $b \in \mathcal{B}^{\mathbb{N}}$ one can explicitly compute its preimage $d^{-1}(b)$. In section 7.4 we will use this fact to give an Algorithm for computing Φ-periodic points.
Lemma 7.1 allows to associate number theoretical properties to the vertical coordinate of Φ-periodic points. By definition of σ, and the fact that Φ commutes with the shift map, one observes, that there is no Φ-periodic point (x, y) so that y is an algebraic integer. On the other hand, a rational numbers $y < c$ may occur as second coordinate of a Φ-periodic point (x, y). We will show that this is indeed true for all but a finite set of such rational numbers.

7.3.1 Non-standard Expansion

The standard α-expansion d is well-understood. In this section we show that the $(\Phi_1, \mathcal{B})$-expansion shares various properties with the standard α-expansion.
Namely, there exists a partition $S_m, m \in \mathcal{M}$ which is a refinement of the binary partition $[0, \alpha), [\alpha, 1)$ and which satisfies the Markov-property. Such a partition is called Markov. The Markov partition is obtained directly from the construction of the Arnoux-Yoccoz diffeomorphism [Arn88]. The $(\Phi_1, \mathcal{M})$-expansion of a Markov partition $d_{\Phi_1, \mathcal{M}}$, a so-called Markov-expansion, was already studied extensively. The action of Φ_1 on the Markov partition induces a subshift of finite type.
Since the Markov partition is a refinement of the binary partition, we are able to pull back properties of the Markov-expansion to the $(\Phi_1, \mathcal{B})$-expansion.
We state the main results and refer for the computations to the appendix.

i) We explicitely construct a countable set of words

$$D(\mathcal{B}, \Phi_1) \subset \mathcal{B}^{\mathbb{N}}$$

which contains only finitely many periodic words. We call a word $b \in D(\mathcal{B}, \Phi_1)$ *exceptional.*

ii) As in the case of standard α-expansions, we call a word $b \in \mathcal{B}^n$ *admissible* if and only if it does not contain the subword $\underbrace{1 \ldots 1}_{g}$.

We show that the $(\Phi_1, \mathcal{B})$-expansion misses at most a set of exceptional words and is injective. Moreover, we connect periodic words to Φ_1-periodic points.

Corollary (Corollary 7.2). *$d_{\Phi_1,\mathcal{B}}$ is injective.*

Proposition (Proposition 7.7). *Denote by $D(\mathcal{B}, \Phi_1)$ the set of exceptional words.*

i) The image $d_{\Phi_1,\mathcal{B}}([0,1))$ consists of admissible words. On the other hand, each admissible word, which is not the $(\Phi_1, \mathcal{B})$-expansion of some point, is contained in the set of exceptional words $D(\mathcal{B}, \Phi_1)$.

ii) Let $x \in [0,1)$ and let $(b_i)_i = d_{\Phi_1,\mathcal{B}}(x)$ be the $(\Phi_1, \mathcal{B})$-expansion of x. Then

$$x > \sum_{i=1}^{n} \alpha^i \Leftrightarrow b_i = 1, i < n$$

Lemma (Lemma 7.6). *Each admissible periodic word which is not contained in $D(\mathcal{B}, \Phi_1)$ is the $(\Phi_1, \mathcal{B})$-expansion of some Φ_1-periodic point x.*

We characterize the points which are Φ_1-preperiodic and indicate an invariant for finite Φ_1-orbits.

Proposition (Proposition 7.4). *Denote by $\mathbb{Q}[\alpha]$ the polynomial ring over α with rational coefficients.*

- $PPer(\Phi_1) = \mathbb{Q}[\alpha] \cap [0,1)$
- *Let $x \in \mathbb{Q}[\alpha] \cap [0,1)$ be a point. x can be uniquely written as*

$$x = \sum_{i=1}^{g} a_i \alpha^i, a_i \in \mathbb{Q}$$

Let q be the greatest common denominator of a_i. Let

$$x' := \Phi_1^n(x), x' = \sum_{i=1}^{g} a_i' \alpha^i, a_i' \in \mathbb{Q}$$

and let q' be the greatest common denominator of a_i'. Then q' is a divisor of $2q$.

7.3.2 Expansion of the fundamental domain

The information about the expansions d and $d_{\Phi_1,\mathcal{B}}$ allow to compute Φ-periodic points $(x,y) \in F$.
Let $\sigma : F \to \mathcal{B}^{\mathbb{Z}}$ be the σ-expansion as defined in section 7.1.

$$\sigma : F \to \mathcal{B}^{\mathbb{Z}}, (x,y) \mapsto d(y)d_{\Phi_1,\mathcal{B}}(x)$$

Φ acts on the bi-infinite word as the right shift.

$$\sigma \circ \Phi = s \circ \sigma$$

As Φ is bijective, a point is periodic under Φ if and only if it is preperiodic under Φ.
Recall, that we defined a word $b \in \mathcal{B}^{\mathbb{Z}}$ to be admissible if and only if it does not contain the subword $\underbrace{1\dots1}_{g}$.

Proposition 7.2. *The σ-expansion satisfies the following properties:*

i) Each image under σ is an admissible word in $\mathcal{B}^{\mathbb{Z}}$.
Moreover, let $b = (b_i)_i \in \mathcal{B}^{\mathbb{Z}}$ be an admissible word. If there exists some $x \in d^{-1}_{\Phi_1,\mathcal{B}}((b_i)_{i\geq 0})$, then there exists some $(x,y) \in \sigma^{-1}(b)$.

ii) σ is injective.

iii) $(x,y) \in F$ is periodic under Φ if and only if $\sigma(x,y)$ is periodic.

iv) For all but a finite set of periodic admissible words $b \in \mathcal{B}^{\mathbb{Z}}$ there is a Φ-periodic preimage in F.

Proof. The proofs follow directly from the results in the previous section.

i) Let $(b_i)_i := \sigma(x,y)$ be the image of some point $(x,y) \in F$. It suffices to show that $b_i = \ldots = b_{g+i} \neq \underbrace{1\dots1}_{g}, \forall i \in \mathbb{Z}$.
For $i < -g$ and $i \geq 0$ this is a consequence of Proposition 7.7 and Proposition 7.1. For $-g < i < 0$ observe that $\sigma(\Phi^g(x,y)) = s^g(\sigma(x,y))$. So, if $\sigma(x,y)$ contains such a subword, $\sigma(\Phi^g(x,y))$ contains the subword at some position $i \geq 0$ what is impossible.
One the other hand, let b be an admissible word and $x \in d^{-1}_{\Phi_1,\mathcal{B}}((b_i)_{i\geq 0})$. By Proposition 7.1 there exists some $y \in d^{-1}(b_{-i+1})_{i\geq 0}$.
So, $(x,y) \in [0,1)^2$. It remains to show that $(x,y) \in F$ which follows from of Proposition 7.7 and Proposition 7.1.

ii) Follows from Proposition 7.1 and Corollary 7.2.

iii) Follows from *ii* and the fact that Φ commutes with the shift map.

iv) Let $b = (b_i)_i \in \mathcal{B}^{\mathbb{Z}}$ be a periodic admissible word. The truncated one-sided infinite word $(b_i)_{i\geq 0} \in \mathcal{B}^{\mathbb{N}}$ is also admissible and periodic. By Lemma 7.6, for all but a finite number of such truncated $(b_i)_{i\geq 0}$, there exists some

$$x \in d^{-1}_{\Phi_1,\mathcal{B}}((b_i)_{i\geq 0}) \subset [0,1)$$

Furthermore, since $(b_{-i-1})_{(i\geq 0)}$ is admissible, there exists some $y \in d^{-1}((b_{-i-1})_{i\geq 0})$. Since the concatenated word is admissible, $(x,y) \in F$.
Due to *ii*) (x,y) is Φ-periodic.

□

Characterization of periodic points We showed that periodic points $(x,y) \in Per(\Phi)$ are closely connected to periodic admissible bi-infinite words under the map σ.
We show an easier connection: The projection $\pi_1 : F \to [0,1), (x,y) \mapsto x$ preserves the property to be periodic for Φ resp. Φ_1. We explicitly describe the inverse mapping which maps a Φ_1-periodic point x to a Φ-periodic point (x,y).

Theorem 7.1. *i) Let $x \in [0,1)$ be a point which is periodic under Φ_1.
Then there exists exactly one point $y \in [0,1)$ such that $(x,y) \in Per(\Phi)$. Precisely, denote by $d_{\Phi_1,\mathcal{B}}(x) = \overline{b_1 b_2 \ldots b_n}$ the $(\Phi_1,\mathcal{B})$-expansion of x. We define*

$$y := \frac{\sum_{i=1}^{n} b_{n-i+1}\alpha^i}{1-\alpha^{n+1}}$$

Then the point (x,y) is periodic under Φ.

ii) The map $d \circ \pi_2 : Per(\Phi) \to \mathcal{B}^{\mathbb{N}}, (x,y) \mapsto d(y)$ projects a periodic point to a periodic admissible word. It misses only a finite set of periodic admissible words and is injective.

Proof. i) Let $x \in Per(\Phi_1)$ be a periodic point and let $d_{\Phi_1,\mathcal{B}}(x) = b = \overline{b_1 \ldots b_n}$ be its $(\Phi_1,\mathcal{B})$-expansion.
We define $\hat{b} := \overline{b_n \ldots b_1}$ and

$$y := d^{-1}(\hat{b})$$

Such a point y exists as $\hat{b}$ is admissible if and only if b is admissible. (x, y) is periodic under Φ as $\sigma(x, y) = \hat{b}b$ is periodic and x periodic under Φ_1.
By Proposition 7.1

$$y = \frac{\sum_{i=1}^{n} b_{n-i+1}\alpha^i}{1 - \alpha^{n+1}}$$

On the other hand, there is only one way of extending the one-sided infinite word $d_{\Phi_1,\mathcal{B}}(x)$ in backward direction such that the resulting bi-infinite word is periodic. By Proposition 7.2, the expansion map σ is injective, so there exists at most one point $(x, y) \in \sigma^{-1}(\hat{b}b)$.

ii) Let y be a point so that the standard α-expansion $d(y) =: b = \overline{b_1 \dots b_n}$ is a periodic admissible word. Again, there is exactly one word $\hat{b} := \overline{b_n \dots b_1}$ so that the concatenation $b\hat{b}$ is periodic and admissible.
By Proposition 7.2 only a finite set of periodic admissible words $\hat{b}$ is exceptional. If $\hat{b}$ is not exceptional, by Proposition 7.7 at least one such Φ_1-periodic point $x \in d^{-1}_{\mathcal{B},\Phi_1}(\hat{b})$ exists. So, (x, y) is Φ-periodic. Again by injectivity of σ, x is unique.

□

Remark 7.2. *Clearly no two Φ-periodic points can be contained on the same stable resp. unstable leaf of a Pseudo-Anosov diffeomorphism Φ on compact surface. So, for any two Φ-periodic points the horizontal coordinates in F are distinct.*
This is not necessarily true, for the vertical coordinate. As one can see in figure 11, F is endowed with vertical slits. So, points with same vertical coordinate in F are not necessarily contained in the same stable leaf on the resulting surface.

Corollary 7.1. *If y is an algebraic integer, then there is no x such that the point (x, y) is Φ-periodic. For all but a finite set of rational numbers y which are smaller than some constant $c > 0$, there exists an x such that $(x, y) \in Per(\Phi)$.*

Proof. By Lemma 7.1 the standard α-expansion of y is not periodic if y is an algebraic integer. Therefore, $\sigma(x, y)$ cannot be periodic for any x.
On the other hand, by Lemma 7.1 the standard α-expansion of y is periodic if y is rational and smaller than some constant c. By Theorem 7.1 for all but a finite set of such points y, there exists some x so that (x, y) is Φ-periodic □

7.4 Algorithm

We give an algorithm to compute points which are Φ-periodic.
Take $x \in \mathbb{Q}[\alpha] \cap [0,1)$. By Proposition 7.4 the Φ_1-orbit of x is finite so there exist $k, l > 0$

$$\hat{x} := \Phi_1^k(x) = \Phi_1^{k+l}(x)$$

By Theorem 7.1 we can explicitly compute the Φ-periodic point $(\hat{x}, \hat{y}) \in F$. We have to find some criterion to ensure that different numbers $x, x' \in \mathbb{Q}[\alpha] \cap [0,1)$ do not lead to the same periodic point $(\hat{x}, \hat{y})$.
Each number $x \in \mathbb{Q}[\alpha] \cap [0,1)$ can be uniquely written as $x = \sum_{i=1}^{g} a_i \alpha^i, a_i \in \mathbb{Q}$. Let q be the greatest common denominator of a_i.
Let $x' := \sum_{i=1}^{g} a'_i \alpha^i, a'_i \in \mathbb{Q}$ be a different point so that the rational numbers a'_i have greatest common denominator $q' > q$, where q is not a divisor of $2q'$. By Proposition 7.4 the resulting Φ_1-periodic number $\hat{x}'$ is different from $\hat{x}$.

7.5 Example genus 3

We explicitly compute the constants for genus $g = 3$. The set of exceptional words is characterized as $D(\mathcal{B}, \Phi_1) = \{b\overline{001}\}$ for all finite words b.

Infinite index subgroups of the Veech group It is a result of [HLM09] that the Veech group of the Arnoux Yoccoz surface S in the case $g = 3$ is not virtually cyclic. There exists a second pseudo-Anosov affine diffeomorphism $\Psi : S \to S$ with derivative

$$d\Psi = \begin{pmatrix} 23 + 18\alpha + 12\alpha^2 & -29 - 24\alpha - 16\alpha^2 \\ 74 + 62\alpha + 40\alpha^2 & -95 - 80\alpha - 52\alpha^2 \end{pmatrix}$$

Proposition 7.3. *There exist points $(x, y) \in F$ which are periodic with respect to Φ but not with respect to $\Psi^{-1}\Phi\Psi$.*

The proof is based on computer search. The algorithm in section 7.4 allows to find Φ-periodic points. Let (x, y) be such a periodic point and let $(x', y') := \Psi(x, y)$. To show, that (x', y') is not Φ-periodic it suffices to show, that x' is not Φ_1-periodic.

Proof. Let $(x, y) = (\frac{21+7\alpha}{44}, \frac{-63+96\alpha+56\alpha^2}{202})$ be a point in F. One computes that $\Phi^{21}(x, y) = (x, y)$. The image $(x', y') = \Psi(x, y)$ has coordinates

$$\begin{aligned} x' &= \frac{2707 + 1911\alpha - 72\alpha^2}{4444} \\ y' &= \frac{-366 + 1914\alpha + 763\alpha^2}{2222} \end{aligned}$$

One computes that x' is Φ_1-preperiodic but not periodic with respect to Φ_1. Therefore, (x', y') cannot be Φ-periodic. $\square$

7.6 Appendix

We show the relevant facts for the $(\Phi_1, \mathcal{B})$-expansion $d_{\Phi,\mathcal{B}} : [0, 1) \to \mathcal{B}^{\mathbb{N}}$.
Let $\mathbb{Q}[\alpha]$ be the polynomial ring over α with rational coefficients .
As Φ_1 is a piecewise affine mapping defined over $\mathbb{Q}[\alpha]$, the preperiodic points of Φ_1 are necessarily contained in $\mathbb{Q}[\alpha] \cap [0, 1)$. We show that each point in $\mathbb{Q}[\alpha] \cap [0, 1)$ is in fact Φ_1-preperiodic.

Proposition 7.4. $PPer(\Phi_1) = \mathbb{Q}[\alpha] \cap [0, 1)$

The methods to show the proposition are close to [Sch80]. We need some Lemmas. We define the evaluation mapping $eval : \mathbb{C}^g \to \mathbb{C}$ which maps the standard basis element b_i to $\alpha^i, i = 1, \dots g$. It can also be written as

$$eval : v \mapsto < v, eval^{\tau} >, eval^{\tau} := (\alpha, \alpha^2, \dots, \alpha^g)^{\tau}$$

$eval^{-1}([0, 1))$ is situated between the two hyperplanes $Kern(eval)$ and $Kern(eval) + \frac{1}{||eval||^2} eval^{\tau}$.
The matrix A naturally acts on $\mathbb{C}^g$.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ & & \ddots & & & \\ 1 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

The action of A commutes with the action of α^{-1} on $\mathbb{C}$ with respect to the evaluation mapping, so A leaves the subspace $Kern(eval)$ invariant.
The characteristic polynomial of A is $x^g - \sum_{i=0}^{g-1} x^i$ which has only simple roots. All but one eigendirection has absolute value of eigenvalue smaller than 1. The only eigenvalue with absolute value larger than 1 is α^{-1}.
Furthermore, the eigendirection corresponding to the eigenvalue α^{-1} is not contained in $Kern(eval)$.
We choose a basis $e_1, \dots, e_g$ of unit eigenvectors for $\mathbb{C}^g$. Let $\lambda_i, 1 \leq i \leq g$ be the eigenvalues of the direction e_i where $|\lambda_i| < 1$ for all $i > 1$ and $\lambda_1 = \alpha^{-1}$.
Let $|| * ||_A$ be the norm on $\mathbb{C}^g$ which is the pull-back of the standard norm on $\mathbb{C}^g$ under

the endomorphism that maps the basis of eigenvectors $\{e_1 \dots e_g\}$ to the standard basis of $\mathbb{C}^g$.

Lemma 7.2. *A acts as a contraction on* $Kern(eval)$ *with respect to* $|| * ||_A$ *with a contraction factor* $0 < \lambda < 1$.

Proof. Since the eigenvalues of A are distinct and the eigenvectors form a basis, each A-invariant subspace is the direct sum of eigenspaces. As $e_1 \notin Kern(eval)$ and $Kern(eval)$ is a $g-1$ dimensional subspace

$$Kern(eval) = span(e_2, \dots, e_g)$$

So, A acts as a contraction on $Kern(eval)$ with respect to $||*||_A$. The contraction factor is not bigger than

$$\lambda := \max_{i>1} |\lambda_i|$$

□

We return to the original mapping $\Phi_1 : [0,1) \to [0,1)$.
Recall that Φ_1 is a piecewise affine mapping with a stretching factor α^{-1} and a piecewise translation by constants $c_i \in \mathbb{Z}/2[\alpha]$.
We can choose a lift of Φ_1 to a map $\Psi : eval^{-1}([0,1)) \to eval^{-1}([0,1))$ so that Ψ satisfies the following properties:

- $eval \circ \Psi = \Phi_1 \circ eval$
- Ψ is piecewise affine: There is some finite partition $S_i, i = 1,2,3$ of $eval^{-1}([0,1])$ and a mapping $v(x) = v_i \Leftrightarrow x \in S_i$. Each vector v_i has coordinates in $(\mathbb{Z}/2)^g$ and Ψ is of the form:
$$\Psi : x \mapsto Ax + v(x)$$

Lemma 7.3. *For any point* $x \in eval^{-1}([0,1)) \cap \mathbb{Q}^g$*, the* Ψ*-orbit of* x *is discrete and bounded and therefore finite.*

Proof. Let $x \in eval^{-1}([0,1)) \cap \mathbb{Q}^g$ be a rational point and let $2q$ be a common divisor of the coordinates. The image of x under Ψ is again contained in $(\mathbb{Z}/2q)^g$. Therefore, the orbit of each point with rational coefficients is discrete.
It remains to show that the Ψ-orbit of each point is bounded in $\mathbb{C}^g$ with respect to the norm $|| * ||_A$.
Recall that for each point $x \in eval^{-1}([0,1))$ the distance between x and $Kern(eval))$ is uniformly bounded. We write x as a linear combination of eigenvectors $x = \sum a_i e_i$.

Since $Kern(eval) = span(e_2, \dots, e_g)$, there is uniform constant $c' > 0$ so that $|a_1| < c'$. Let $c := |\alpha^{-1}c'| + \max_i ||Av_i||_A$ and let λ be the contraction factor for the action of A on $Kern(eval)$ as in Lemma 7.2.
We conclude

$$||\Psi(x)||_A = ||Ax + v(x)||_A < \lambda||x||_A + c$$

If $||x||_A$ is larger than some uniform constant, $||\Psi(x)||_A < ||x||_A$. One concludes that the Ψ-orbit of each point x is bounded. □

Proof of the Proposition. Observe that each point in $[0,1) \cap \mathbb{Q}[\alpha]$ has a unique preimage in $y \in eval^{-1}(x) \cap \mathbb{Q}^g$.
We showed that the Ψ-orbit of y is finite. Since Ψ commutes with Φ_1, x is Φ_1-preperiodic. □

Recall the binary partition of $[0,1)$

$$\mathcal{B} = \{0,1\}, S_0 = [0,\alpha), S_1 = [\alpha,1)$$

Let

$$d_{\Phi_1,\mathcal{B}} : [0,1) \to \mathcal{B}^{\mathbb{N}}$$

be the $(\mathcal{B}, \Phi_1)$-expansion.
We construct a finer partition which satisfies the Markov property.

Proposition 7.5. *There exists a finite partition $S_m, m \in \mathcal{M}$ of $[0,1)$ in half-open intervals. It meets the Markov property i.e. the image of each interval S_m under Φ_1 is the union of other intervals $S_{m'}$. Moreover, it is a refinement of the binary partition.*

Proof. The interval $S_1 = [\alpha,1)$ can be decomposed in the following half-open intervals:

$$[\alpha,1) = \bigcup_{k=1}^{g-1} S_{1,k}, S_{1,k} := [\sum_{i=1}^{k} \alpha^i, \sum_{i=1}^{k+1} \alpha^i)$$

We observe, that Φ_1 acts on $S_{1,k}$ in descending order.

$$\Phi_1(S_{1,k}) = S_{1,(k-1)}, \Phi_1(S_{1,1}) = [0,\alpha) = S_0, \Phi_1(S_0) = [0,1)$$

So, the partition S_0 together with $S_{1,k}$ form the Markov partition $\mathcal{M}$ of $[0,1)$. We define $\mathcal{M} := \{0, (1,1), (1,2) \dots (1, g-1)\}$. □

We also consider the $(\Phi_1, \mathcal{M})$-expansion or Markov-expansion.

$$d_{\Phi_1,\mathcal{M}} : [0,1) \to \mathcal{M}^{\mathbb{N}}$$

Definition 7.4. *Let n be either a finite number $\mathbb{N}$ or $\mathbb{Z}$.*

- *A word $m = (m_i)_i \in \mathcal{M}^n$ is called admissible if and only if*

$$\Phi_1(S_{m_i}) \cap S_{m_{i+1}} \neq \emptyset, \forall i$$

By the Markov property this is equivalent to $\Phi_1(S_{m_i}) \supset S_{m_{i+1}}$.

- *As in the case of standard α-expansion, we call a word $b = (b_i)_i \in \mathcal{B}^n$ admissible if and only if*

$$b_i b_{i+1} \dots b_{i+g-1} \neq \underbrace{1 \dots 1}_{g}, \forall i \geq 0$$

Proposition 7.6. *The Markov-expansion has the following properties*

i) $d_{\Phi_1,\mathcal{M}}$ *is injective.*

ii)

$$d_{\Phi_1,\mathcal{M}}(Per(\Phi_1)) = Per(\mathcal{M}) \cap d_{\Phi_1,\mathcal{M}}([0,1))$$

$$d_{\Phi_1,\mathcal{M}}(PPer(\Phi_1)) = PPer(\mathcal{M}) \cap d_{\Phi_1,\mathcal{M}}([0,1)$$

Proof. *i*) is well-known since the action of Φ_1 is induced by Pseudo-Anosov diffeomorphism. We refer to [Bow70].
ii) is a consequence of *i* □

We show that $d_{\Phi_1,\mathcal{B}}$ misses at most a countable set of admissible words.
Let $S_a, a \in \mathcal{A}$ be some partition of the half-open unit interval. $\mathcal{A}$ is either $\mathcal{M}$ or $\mathcal{B}$. The partitioning sets $S_a, a \in \mathcal{A}$ are half-open intervals. Moreover, Φ_1 is right continuous and consequently $d_{\Phi_1,\mathcal{A}}$ is right continuous for some word metric.
We define the left limit expansion of a point $x \in [0,1)$.

Definition 7.5. *Let $x_k \in [0,1)$ be a strictly increasing sequence of points with limit $x \in [0,1)$. We define the limit sequence*

$$l_{\Phi_1,\mathcal{A}}(x) := \lim_{x_k \nearrow x} d_{\Phi_1,\mathcal{A}}(x_k)$$

which is independent of the sequence x_k. We call $l_{\Phi_1,\mathcal{A}}(x)$ the left limit expansion of x.

Denote by s the shift map acting on the set words.

Definition 7.6. *Let $\mathcal{A}$ be either $\mathcal{M}$ or $\mathcal{B}$. Let $D' \subset [0,1)$ be the discontinuities of Φ_1 and the right boundary points of $S_a, a \in \mathcal{A}$. We call*

$$D(\mathcal{A}, \Phi_1) = \bigcup_{d \in D'} \bigcup_{i \geq 0} s^{-i}(l_{\Phi_1, \mathcal{A}}(d))$$

the set of exceptional words.

Lemma 7.4. *$D(\mathcal{A}, \Phi_1)$ contains at most a finite set of periodic words.*

Proof. Observe that $D(\mathcal{A}, \Phi_1)$ consists of a finite set of one-sided infinite words concatenated with all finite words. Let $a \in \mathcal{A}^{\mathbb{N}}$ be some one-sided infinite word.
Assume first that a is not periodic. For any finite word u, the concatenation ua is not periodic either.
Assume next that a is a periodic word. Consider the set of words ua arising from a concatenation a with some finite word u. The subset of those words which are periodic is finite.
As a consequence, $D(\mathcal{A}, \Phi_1)$ contains only a finite set of periodic words. □

Remark 7.3. *Recall the projection $\tau : \mathcal{M}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$. One observes that $\tau(D(\mathcal{M}, \Phi_1)) = D(\mathcal{B}, \Phi_1)$*

The following Lemma gives a sufficient criterion under which conditions an infinite word is expansion of a point.

Lemma 7.5. *Let $S_a, a \in \mathcal{A}$ be a partition of $[0,1)$ in half-open intervals. Let $(a_i)_i, \in \mathcal{A}^{\mathbb{N}}$ be a one-sided infinite word with the following properties:*
There exists a strictly increasing sequence $k_i \in \mathbb{N}, k_0 = 0$ and a sequence of sets $U_{k_i} \subset S_{a_{k_i}}$ so that

- *each U_{k_i} is a finite disjoint union of half-open intervals.*
- $$\Phi_1^t(U_{k_i}) \subset S_{a_{(k_i+t)}} \forall 0 \leq t < k_{i+1} - k_i, \Phi_1^{k_{(i+1)}-k_i}(U_{k_i}) = S_{a_{k_{(i+1)}}}$$

Then the word is either the $(\Phi_1, \mathcal{A})$-expansion of some point or contained in the exceptional set.

$$(a_i)_i \in d_{\Phi_1, \mathcal{A}}([0,1)) \cup D(\mathcal{A}, \Phi_1)$$

Proof. This is a consequence of the nested interval Theorem.
Denote by D' the discontinuities of Φ_1 together with the right boundary point of the partitioning sets $S_a, a \in \mathcal{A}$. One uses the sets U_{k_i} to construct a sequence of sets $V_{k_i} \subset S_{a_0}$ so that

- $$\Phi_1^j(V_{k_i}) \subset S_{a_j}, \forall j < k_i, \Phi_1^{k_i}(V_{k_i}) = S_{a_{k_i}}$$

- Each set V_{k_i} is a finite union of half-open intervals. The Φ_1-orbit of a right boundary point of V_{k_i} contains a point in D'.

- The sequence V_{k_i} is nested $V_{k_{i+1}} \subset V_{k_i}$.

By the nested interval Theorem, there exists a point x in $\bigcap_i \overline{V_{k_i}}$. If x is already contained in $\bigcap_i V_{k_i}$, the claim is shown as

$$d_{\Phi_1,\mathcal{A}}(x) = (a_i)_i$$

On the other hand, assume that the point x is not contained in some $V_{k_{i_0}}$ and so x is a right boundary point for all $V_{k_j}, j \geq i_0$. There is a sequence

$$x_j \in V_{k_j}, x_j \nearrow x$$

Again by construction

$$\lim_j d_{\Phi_1,\mathcal{A}}(x_j) = (a_i)_i$$

Therefore, $(a_i)_i$ is the left limit expansion of x. The Φ_1-orbit of x contains a point in D' and so

$$(a_i)_i \in D(\mathcal{A}, \Phi_1)$$

□

We have to show that each admissible word with letters in $\mathcal{B}$ resp. $\mathcal{M}$ satisfies the conditions of Lemma 7.5.

Proposition 7.7. *With respect to the Markov partition as well as to the binary partition any admissible word is the expansion of a point $x \in [0,1)$ or an element of the exceptional set:*

i) Each word in $d_{\Phi_1,\mathcal{M}}([0,1))$ is admissible. On the other hand, each admissible word, which is not the Markov-expansion of some point, is contained in $D(\mathcal{M}, \Phi_1)$.

ii) The same holds for the $(\Phi_1, \mathcal{B})$-expansion.

iii) Let $x \in [0,1)$ and $(b_i)_i := d_{\Phi_1,\mathcal{B}}(x)$. Then

$$x > \sum_{i=1}^{n} \alpha^i \Leftrightarrow b_i = 1, \forall i < n$$

Proof. i) Recall that a word $(m_i)_i \in \mathcal{M}^{\mathbb{N}}$ is called admissible if and only if

$$\Phi_1(S_{m_i}) \cap S_{m_{i+1}} \neq \emptyset$$

Therefore, each word appearing as a Markov-expansion of a point is admissible. It remains to show that the image of the Markov-expansion map only misses a subset of $D(\mathcal{M}, \Phi_1)$.
Let $m = (m_i)_i$ be an admissible word and define

$$k_i := i, U_i := \Phi_1(S_{m_{(i+1)}})^{-1} \cap S_{m_i}$$

The conditions of Lemma 7.5 are satisfied and therefore either $(m_i)_i$ is contained in $D(\mathcal{M}, \Phi_1)$ or $(m_i)_i$ is the Markov-expansion of some point x.

ii) Recall that a word $(b_i)_i$ is admissible if it does not contain the subword $\underbrace{1 \ldots 1}_{g}$. We defined the projection $\tau : \mathcal{M}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$.
Φ_1 acts on the Markov partition in the following way:

$$\Phi_1(S_0) = [0, 1), \Phi_1(S_{1,k}) = S_{1,(k-1)}, \Phi_1(S_{1,1}) = S_0$$

So, the Markov-expansion $d_{\Phi_1,\mathcal{M}}(x)$ of a point x cannot contain the subword $\underbrace{m_1 \ldots m_g}_{g}, m_i \neq 0$. Since $\tau(d_{\Phi_1,\mathcal{M}}(x)) = d_{\Phi_1,\mathcal{B}}(x)$ each image of the map $d_{\Phi_1,\mathcal{B}}(x)$ is an admissible word.
It remains to show that any admissible word is either contained in $D(\mathcal{B}, \Phi_1)$ or is the image of some point x under the $(\mathcal{B}, \Phi_1)$-expansion map.
Let $(b_i)_i \in \mathcal{B}^{\mathbb{N}}$ be an admissible word. We define, $k_0 = 0, k_{i+1} := \min\{j > k_i, b_j = 0\}$.
One checks that there is a corresponding sequence of sets U_{k_i} so the conditions of Lemma 7.5 are satisfied.

iii) Follows from the structure of the Markov partition and the fact, that Φ_1 acts on $S_{1,k}$ in descending order.

$$x > \sum_{i=1}^{n} \alpha^i \Leftrightarrow x \in S_{1,k}, k \geq n \Leftrightarrow \Phi_1^i(x) \notin S_0, \forall i < n$$

□

Remark 7.4. *In interval exchange transformations the existence of the set $D(\mathcal{A}, f)$ is a well-known phenomena which is due to the fact, that at a discontinuity one might have two expansions and only one is appropriate.*

We show that $d_{\Phi_1,\mathcal{B}}$ is also injective. Recall the mapping $\tau : \mathcal{M}^n \to \mathcal{B}^n, \tau(0) = 0, \tau((1,k)) = 1$

Proposition 7.8. *i) Let $(b_i)_i \in \mathcal{B}^n$ be a finite word and $m \in \mathcal{M}$ be some letter. Then there exists at most one admissible word $(m_i)_i \in \tau^{-1}((b_i)_i)$ with the property that $m_n = m$.*

ii) Let $(b_i)_i \in \mathcal{B}^{\mathbb{N}}$ be a one-sided infinite word. Then the cardinality of admissible words in $\tau^{-1}((b_i)_i)$ is at most 1.

Proof. i) Recall that Φ_1 maps $S_0 = [0, \alpha)$ injectively into $[0,1)$. So for all letters $m_i \in \mathcal{M}$ there is at most one $m_{i-1} \in \tau^{-1}(0)$ such that $\Phi_1(S_{m_{i-1}}) \cap S_{m_i} \neq \emptyset$.
The same holds for $S_1 = [\alpha, 1)$.
That is why for any letter m_i there exists at most one letter $m_{i-1} \in \tau^{-1}(b_{i-1})$ such that $m_{i-1}m_i$ is admissible. Consequently, there is at most one combinatorial possibility to construct an admissible word $(m_i)_i$ such that $\tau((m_i)_i) = (b_i)_i$ for fixed last letter m_n.

ii) Assume on the contrary that there are different admissible words $(m_i)_i, (m'_i)_i \in \tau^{-1}((b_i)_i) \subset \mathcal{M}^{\mathbb{N}}$. Fix some i_0 such that $m_{i_0} \neq m'_{i_0}$. There is some $i \geq i_0$ so that $b_i = 0$. So $m_i = m'_i = 0$. But m_i determines its predecessors what is a contradiction.

□

Corollary 7.2. *The map $d_{\Phi,\mathcal{B}}$ is injective.*

Proof. Each image point of $d_{\Phi_1,\mathcal{M}}$ is an admissible word and $d_{\Phi_1,\mathcal{M}}$ is injective. Therefore, the Corollary is a consequence of Proposition 7.8. □

We show that all but a finite set of periodic admissible words in $\mathcal{B}^{\mathbb{N}}$ admit a periodic preimage.

Lemma 7.6. *Let $\tau : \mathcal{M} \to \mathcal{B}$ be the canonical projection. For each admissible periodic word $b \in \mathcal{B}^{\mathbb{N}}$ which is not contained in $D(\mathcal{B}, \Phi_1)$ there exists a periodic preimage $x \in Per(\Phi_1) \cap d^{-1}_{\Phi_1,\mathcal{B}}(b)$.*

Proof. Let $b \in \mathcal{B}^{\mathbb{N}}$, $b_{i+n} = b_i$ be a periodic admissible word not contained in $D(\mathcal{B}, \Phi_1)$. By Proposition 7.7, b is the $(\Phi_1, \mathcal{B})$-expansion of some point $x \in [0,1)$. Since $d_{\Phi_1,\mathcal{B}}$ is injective, x is Φ_1-periodic. □

References

[Aki98] Shigeki Akiyama. Pisot numbers and greedy algorithm. pages 9–21, 1998.

[Arn88] Pierre Arnoux. Un exemple de semi-conjugaison entre un échange d'intervalles et une translation sur le tore. *Bull. Soc. Math. France*, 116(4):489–500 (1989), 1988.

[AY81] Pierre Arnoux and Jean-Christophe Yoccoz. Construction de difféomorphismes pseudo-Anosov. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(1):75–78, 1981.

[Ber85] Lipman Bers. An inequality for Riemann surfaces. pages 87–93, 1985.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BM96] M. Burger and S. Mozes. CAT(-1)-spaces, divergence groups and their commensurators. *J. Amer. Math. Soc.*, 9(1):57–93, 1996.

[Bou95] Marc Bourdon. Structure conforme au bord et flot géodésique d'un CAT(-1)-espace. *Enseign. Math. (2)*, 41(1-2):63–102, 1995.

[Bow70] Rufus Bowen. Markov partitions for Axiom A diffeomorphisms. *Amer. J. Math.*, 92:725–747, 1970.

[Bow71] Rufus Bowen. Periodic points and measures for axiom a diffeomorphisms. *Transactions of the American Mathematical Society*, 154:377–397, 1971.

[Bow09] Joshua Bowman. *Flat Structures And Complex Structures In Teichmuller Theory.* PhD thesis, http://hdl.handle.net/1813/13979, 2009.

[BP92] Riccardo Benedetti and Carlo Petronio. Lectures on hyperbolic geometry. pages xiv+330, 1992.

[BS07] Alexander I. Bobenko and Boris A. Springborn. A discrete Laplace-Beltrami operator for simplicial surfaces. *Discrete Comput. Geom.*, 38(4):740–756, 2007.

[Coo93] Michel Coornaert. Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov. *Pacific J. Math.*, 159(2):241–270, 1993.

[CP93] Michel Coornaert and Athanase Papadopoulos. *Symbolic dynamics and hyperbolic groups*, volume 1539 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1993.

[CP94] M. Coornaert and A. Papadopoulos. Une dichotomie de Hopf pour les flots géodésiques associés aux groupes discrets d'isométries des arbres. *Trans. Amer. Math. Soc.*, 343(2):883–898, 1994.

[CP97] Michel Coornaert and Athanase Papadopoulos. Upper and lower bounds for the mass of the geodesic flow on graphs. *Math. Proc. Cambridge Philos. Soc.*, 121(3):479–493, 1997.

[DLR09] M. Duchin, C.J. Leininger, and K. Rafi. Length spectra and degeneration of flat metrics. *Arxiv preprint arXiv:0907.2082*, 2009.

[EKS84] Allan L. Edmonds, Ravi S. Kulkarni, and Robert E. Stong. Realizability of branched coverings of surfaces. *Transactions of the American Mathematical Society*, 282(2):pp. 773–790, 1984.

[Fal03] Kenneth Falconer. *Fractal geometry*. John Wiley & Sons Inc., Hoboken, NJ, second edition, 2003. Mathematical foundations and applications.

[HLM09] Pascal Hubert, Erwan Lanneau, and Martin Möller. The Arnoux-Yoccoz Teichmüller disc. *Geom. Funct. Anal.*, 18(6):1988–2016, 2009.

[Hop71] Eberhard Hopf. Ergodic theory and the geodesic flow on surfaces of constant negative curvature. *Bull. Amer. Math. Soc.*, 77:863–877, 1971.

[HS06] Pascal Hubert and Thomas A. Schmidt. An introduction to veech surfaces. pages 501–526, 2006.

[Hub06] John Hamal Hubbard. Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1. pages xx+459, 2006. Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle.

[Hus62] Dale H. Husemoller. Ramified coverings of Riemann surfaces. *Duke Math. J.*, 29:167–174, 1962.

[Jac70] William Jaco. On certain subgroups of the fundamental group of a closed surface. *Proc. Cambridge Philos. Soc.*, 67:17–18, 1970.

[Kai94] Vadim A. Kaimanovich. Ergodicity of harmonic invariant measures for the geodesic flow on hyperbolic spaces. *J. Reine Angew. Math.*, 455:57–103, 1994.

[Kat92] Svetlana Katok. *Fuchsian groups*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.

[Kee74] Linda Keen. Collars on Riemann surfaces. pages 263–268. Ann. of Math. Studies, No. 79, 1974.

[KMS86] Steven Kerckhoff, Howard Masur, and John Smillie. Ergodicity of billiard flows and quadratic differentials. *Ann. of Math. (2)*, 124(2):293–311, 1986.

[Kre85] Ulrich Krengel. Ergodic theorems. 6:viii+357, 1985. With a supplement by Antoine Brunel.

[Mas79] Howard Masur. The Jenkins-Strebel differentials with one cylinder are dense. *Comment. Math. Helv.*, 54(2):179–184, 1979.

[Mas85] Bernard Maskit. Comparison of hyperbolic and extremal lengths. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 10:381–386, 1985.

[Mas86] H. Masur. Closed trajectories for quadratic differentials with an application to billiards. *Duke Math. J*, 53(2):307–314, 1986.

[Mas90] Howard Masur. The growth rate of trajectories of a quadratic differential. *Ergodic Theory Dynam. Systems*, 10(1):151–176, 1990.

[Mas92] Howard Masur. Hausdorff dimension of the set of nonergodic foliations of a quadratic differential. *Duke Math. J.*, 66(3):387–442, 1992.

[Min92] Yair N. Minsky. Harmonic maps, length, and energy in Teichmüller space. *J. Differential Geom.*, 35(1):151–217, 1992.

[MM00] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000.

[MS85] Albert Marden and Kurt Strebel. On the ends of trajectories. pages 195–204, 1985.

[MS91] Howard Masur and John Smillie. Hausdorff dimension of sets of nonergodic measured foliations. *Ann. of Math. (2)*, 134(3):455–543, 1991.

[MT02] Howard Masur and Serge Tabachnikov. Rational billiards and flat structures. In *Handbook of dynamical systems, Vol. 1A*, pages 1015–1089. North-Holland, Amsterdam, 2002.

[Pat76] S. J. Patterson. The limit set of a Fuchsian group. *Acta Math.*, 136(3-4):241–273, 1976.

[Raf07] Kasra Rafi. Thick-thin decomposition for quadratic differentials. *Math. Res. Lett.*, 14(2):333–341, 2007.

[Rud87] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[Sch80] Klaus Schmidt. On periodic expansions of Pisot numbers and Salem numbers. *Bull. London Math. Soc.*, 12(4):269–278, 1980.

[Str84] Kurt Strebel. *Quadratic differentials*, volume 5 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.

[Sul79] Dennis Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.*, (50):171–202, 1979.

[Thu80] W.P. Thurston. *The Geometry and Topology of Three-manifolds*. Princeton University, 1980.

[Vee90] William A. Veech. Moduli spaces of quadratic differentials. *J. Analyse Math.*, 55:117–171, 1990.

[Vor96] Ya. B. Vorobets. Plane structures and billiards in rational polygons: the Veech alternative. *Uspekhi Mat. Nauk*, 51(5(311)):3–42, 1996.

Curriculum vitae

name	Klaus Dankwart
address	Vorgebirgsstrasse 80
	53119 Bonn
phone	0228/2400690
email	klaus@math.uni-bonn.de
birth	January 27th, 1982/ Böblingen
Nationality	German

scientific und work experience

08/2006-05/2007	student assistant at the Collaborative Research Center (SFB) 611 - Singular Phenomena and Scaling in Mathematical Models
10/2004-10/2005	running a tutorial of the course analysis at mathematical institut at Bonn university
08/2005-10/2005	research internship at Institut für Sedimentäre Syteme of the Forschungszentrum Jülich

scholarships

10/2007-10/2010	doctoral scholarship of Bonn graduate school in Mathematics
02/2005-10/2007	scholarship of German National Academic Foundation

academic education

since 10/2007	doctoral studies at Bonn University advisor: Pr. Ursula Hamenstädt
09/2007	received the diploma in mathematics
10/2005-06/2006	year abroad at Lund university (Sweden)
10/2004	intermediate diploma examination
10/2002	start of studies of mathematics and operation research at Bonn university.

school education and community service

07/2001-05/2002	community service Johanniterstift Erkelenz
10/1992-10/2001	Cornelius-Burgh-Gymnasium Erkelenz
09/1988-09/1992	Katholische Grundschule Erkelenz

Summary

This thesis deals with the geometry of flat surfaces.
A flat surface is a Riemann surface which is endowed with a singular flat metric. Such a metrics arises from a distinguished set of charts on X so that outside a finite set of marked points, the transition maps are half-translations. In the marked points the metric is of cone type with cone angle $k\pi, k \geq 3$.
For each flat metric on a Riemann surface X there exists a hyperbolic metric in the same conformal class. It is the goal of the first part of this work to investigate the asymptotic geometry and dynamics of flat metrics and to compare the structure of such metrics with the corresponding hyperbolic metrics.
Therefore, we lift the singular flat metric to the universal cover $\tilde{X}$ of X which is a proper geodesic Gromov hyperbolic space. The boundary of $\tilde{X}$ can be naturally endowed with a Gromov metric. We investigate the volume entropy of the lifted metric on $\tilde{X}$ and the Hausdorff dimension of the boundary. On a hyperbolic surface these quantities are constant 1.
This does not hold in the case of flat surfaces. We estimate the volume entropy and the Hausdorff dimension for a sequence of flat surfaces and show that both quantities tend to infinity if and only if the flat surfaces degenerate in the moduli space of flat structures.
Branched coverings form a central concept in the theory of Riemann surfaces. We claim the compatibility of the covering with the metric, that means that the covering surface and the base surface are both flat surfaces and away from the branch points the covering map is a local isometry. We estimate the volume entropy of the covering surface by the geometry of the covering map and by the volume entropy of the base surface.
For each flat metric, there is a unique hyperbolic metric σ in the same conformal class. We measure the asymptotic length quotient between the hyperbolic and the flat geodesics which are in the same homotopy class with fixed endpoints.
The geodesic flow g_t on the unit tangent bundle of X acts ergodically with respect to the Lebesgue measure defined by σ. For $T > 0, v \in T^1X$ let $c : [0,T] \to X, c'(0) = v$ be a σ-geodesic arc of length T on X. The function $F_{0,T}(v)$ measures the length of the shortest arc in the homotopy class of arcs with fixed endpoints $[c]$ for the flat metric.
By ergodic theory the limit $\lim \frac{1}{T} F_{0,T}(v)$ converges to a constant F a.e. We show that the volume entropy is an upper bound for F^{-1}.
Let Y be a component of the hyperbolically thick part of the surface with respect to the Margulis constant ϵ. [Raf07] compared the length of a free homotopy class $[\alpha]$ of simple closed curves for the flat and hyperbolic metric in the same conformal class which can be

realized in Y. There is a constant $\lambda(Y)$ so that the quotient of flat length and hyperbolic length of $[\alpha]$ is comparable to $\lambda(Y)$. We show that $F \geq A\lambda(Y)$ for some constant $A > 0$ which only depends on the topology of X.
In addition we define a geodesic flow on a flat surface S. Each locally geodesic segment which terminates at a cone point admits a one-parameter family of possible locally geodesic extensions. Therefore, a definition similar to the one for Riemannian metrics on the unit tangent bundle cannot be given.
Let $\mathcal{G}S$ be the set of all parametrized bi-infinite geodesics for the flat metric. The geodesic flow g_t acts as a reparametrization $g_t\alpha(s) := \alpha(t+s)$ on $\mathcal{G}S$. We define a natural measure on $\mathcal{G}S$. Let c be a compact geodesic arc on S. Let c_{ext} be the maximal extension of c with the property that the extension is unique.

Theorem. *(Theorem 5.4) There is a constant $C(S) > 0$ which depends on the geometry of S but not on c such that the following holds:*
A typical geodesic passes through c with a frequency f which is bounded from above and below by

$$C(S)^{-1}exp(-e(\tilde{S},\Gamma_S)l(c_{ext})) \leq f \leq C(S)exp(-e(\tilde{S},\Gamma_S)l(c_{ext}))$$

Finally we deal with a different object on a flat surface, the group of orientation preserving affine diffeomorphisms. Away from the singularities, each diffeomorphism descends to a differentiable mapping $U \subset \mathbb{R}^2 \to \mathbb{R}^2$ with a constant derivative which we interpret as a matrix $A \in GL_+(2,\mathbb{R})$. A is independent of the choice of charts up to multiplication with $\pm id$. Therefore, there is a well-defined map of each affine diffeomorphism to its projectivized differential in $PGL_+(2,\mathbb{R}) = PSL(2,\mathbb{R})$. The image of the group of affine diffeomorphisms is the so-called Veech group which is a non-cocompact fuchsian group.
We investigate one of the most prominent examples of flat surfaces with a non-trivial Veech group, the family of Arnoux Yoccoz surfaces in all genera with a distinguished affine diffeomorphism Φ. The flat surface arises from a so-called Markov partition $F \subset \mathbb{R}^2$ for Φ. The expansion factor α of Φ is a pisot number i.e. an algebraic number with all complex conjugates having absolute value less than 1.
We make use of so-called α-expansions of points in the Markov partition, a technique similar to continued fraction expansions, to compute periodic points under Φ. We can show that the coordinates of periodic points meet number theoretical conditions:
For all but a finite set of rational points y there is a periodic point in F with vertical coordinate y. On the other hand, there is no such periodic point if is y an algebraic integer.

[HLM09] showed that in the case of genus $g = 3$ the Veech group is not virtually cyclic. In their work they explicitly found a second pseudo-Anosov element Ψ. We find points which are periodic for Φ but not periodic for the conjugate of Φ with Ψ. Thus we construct Veech groups which still contain the original pseudo-Anosov element up to finite index but have infinite index in the original Veech group.