On Generalised $\mathcal{D}$-Shtukas

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Contents

Introduction

Let F be a global field of positive characteristic, i.e. the function field of a smooth projective curve X over a finite field $\mathbb{F}_q$.

In a series of works [Dri87], [Dri88], [Dri89] Drinfeld introduced the notion of shtukas (F-sheaves in his terminology) and used it to prove the Langlands correspondence for GL_2 over function fields by showing that this correspondence is realised in the cohomology of the moduli space of shtukas of rank 2. Roughly speaking a shtuka is a vector bundle on $X \otimes \overline{\mathbb{F}}_q$ which differs from its Frobenius twist by the simplest non-trivial modification. Previously Drinfeld already had shown that for a certain class of automorphic representations the Langlands correspondence is realised in the cohomology of the moduli space of elliptic modules ('Drinfeld modules') or, equivalently, elliptic sheaves of rank 2, which may be considered as a special case of shtukas.

In a wide generalisation of Drinfeld's approach Lafforgue [Laf02] proved the Langlands correspondence for GL_d over function fields, which in a certain sense is realised in the cohomology of the moduli space of shtukas of rank d. The main difficulty is that like in case $d = 2$ this space is not proper and not even of finite type.

The situation changes when GL_d is replaced by an anisotropic inner form, i.e. the multiplicative group of a central division algebra D over F.

Laumon, Rapoport and Stuhler [LRS] showed that $\mathcal{D}$-elliptic sheaves have a proper moduli space and calculated its cohomology. They proved that for certain automorphic representations π there exists an irreducible d-dimensional Galois representation $\sigma(\pi)$ which is unramified outside a fixed infinite place, the ramification locus of D and the ramification locus of π such that on this open set the L-functions of π and $\sigma(\pi)$ coincide. From their global results they can deduce the local Langlands correspondence in positive characteristic.

$\mathcal{D}$-shtukas have first been studied by Lafforgue [Laf97]. Their moduli space is of finite type, but contrary to the expectation not proper in all cases. For $d = 2$ it is proper if and only if D is ramified in at least 4 places. Lafforgue's calculation of the cohomology assumes a proper moduli space and shows that in this case for any automorphic representation there exists at least a multiple of the expected corresponding d-dimensional Galois representation (see below). In the non-proper cases the only difference is that the assertions related to purity of cohomology get lost.

Already in [Dri87] Drinfeld indicated that one can consider shtukas with more general modifications as well. Moduli spaces of generalised $\mathcal{D}$-shtukas in this sense are the object of the present work. They are always of finite type, and they are proper if D is ramified at sufficiently many places with respect to the size of the modifications. We will restrict ourselves to a sufficient criterion for properness following a remark of Lafforgue. In the case $d = 2$ we show that this criterion is optimal and return to the general case in a later work. Moduli spaces of generalised $\mathcal{D}$-shtukas have also been studied by Ngô Bao Châu [Ngô03] (see below). Varshavsky [Var] considered similar moduli spaces for an arbitrary split reductive group over $\mathbb{F}_q$.

Our main interest is the description of the l-adic cohomology of moduli spaces of generalised $\mathcal{D}$-shtukas as a common module of the Hecke algebra, the product of some copies of the Galois group of F, and certain symmetric groups. For precise statements we need a few notations.

We choose an idele $a \in \mathbb{A}_F^*$ of degree 1 which is concentrated in a finite set of places. The type of the modification of a generalised $\mathcal{D}$-shtuka in one point is given by a sequence $\lambda = (\lambda^{(1)} \geq \ldots \geq \lambda^{(d)}) \in \mathbb{Z}^d$ (elementary divisors theorem). These are the dominant coweights for GL_d. For any sequence of such types $\underline{\lambda} = (\lambda_1 \ldots \lambda_r)$ of total degree zero and for a non-empty finite closed subscheme $I \subset X$ there is a quasiprojective moduli space of $\mathcal{D}$-shtukas with modifications bounded by $\underline{\lambda}$ and with a level-I-structure modulo twisting with multiples of a line bundle $\mathscr{L}(a)$

$$\pi_I^{\underline{\lambda}} \;:\; \mathrm{Sht}_I^{\leq \underline{\lambda}} / a^{\mathbb{Z}} \longrightarrow (X' \setminus I)^r.$$

Properness of the moduli space means that this morphism is projective. In the case $\underline{\lambda} = (\mu^+, \mu^-)$ with $\mu^+ = (1, 0 \ldots 0)$ and $\mu^- = (0 \ldots 0, 1)$ we obtain Lafforgue's $\mathcal{D}$-shtukas.

The direct images of the intersection cohomology sheaves by this morphism naturally give rise to representations $H^n_{I,\underline{\lambda}}$ of the product

$$\mathcal{H}_I \times \big(\pi_1(X' \setminus I)\big)^r \rtimes \mathrm{Stab}(\underline{\lambda})\,.$$

Here $\mathcal{H}_I$ is the subalgebra of the Hecke algebra of $D^*_{\mathbb{A}}$ defined by I, and $\mathrm{Stab}(\underline{\lambda})$ is the subgroup of the symmetric group on r elements which stabilises $\underline{\lambda}$. Let $H_{I,\underline{\lambda}}$ be their alternating sum in an appropriate Grothendieck group.

Lafforgue has shown that (in the proper case)

$$H_{I,(\mu^+,\mu^-)} \;=\; \bigoplus_{\pi} m'_\pi \cdot \pi^{K_I} \boxtimes \sigma'(\pi) \boxtimes \sigma'(\pi)^\vee \tag{1}$$

with π running through the irreducible automorphic representations of $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ having non-trivial K_I-invariants. Here m'_π divides the automorphic multiplicity m_π of π, and $\sigma'(\pi)$ is a semisimple l-adic representation of G_F of dimension $d \cdot \sqrt{m_\pi/m'_\pi}$ which is unramified outside the ramification locus of D and I. Moreover on a dense open subset of X the L-function of $\sigma'(\pi)$ coincides with the $\sqrt{m_\pi/m'_\pi}$-th power of the L-function of π.

Our main results are the following. Assume that the moduli space of ordinary $\mathcal{D}$-shtukas is proper.

(a) In equation (1) one can choose $m'_\pi = m_\pi$. That is, for any irreducible automorphic representation π of $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ there exists a d-dimensional l-adic Galois representation $\sigma(\pi)$ which is unramified outside the ramification loci of π and of D such that on a dense open subset of X the L-functions of π and of $\sigma(\pi)$ coincide.

(b) If the moduli space of $\mathcal{D}$-shtukas with modifications bounded by $\underline{\lambda}$ is proper as well, then the virtual $\mathcal{H}_I \times \pi_1(X' \setminus I)^r$-module $H_{I,\underline{\lambda}}$ takes the form

$$H_{I,\underline{\lambda}} = \bigoplus_{\pi} m_\pi \cdot \pi^{K_I} \boxtimes \left(\rho_{\lambda_1} \circ \sigma(\pi)\right) \boxtimes \ldots \boxtimes \left(\rho_{\lambda_r} \circ \sigma(\pi)\right) \qquad (2)$$

with π running through the irreducible automorphic representations of $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ having non-trivial K_I-invariants. Here $\rho_\lambda : \mathrm{GL}_d(\overline{\mathbb{Q}}_l) \to \mathrm{GL}(V_\lambda)$ denotes the irreducible representation with highest weight λ.

(c) If $\sigma(\pi)$ does not contain any irreducible factor with multiplicity greater than 1 (this would follow from a Jacquet-Langlands correspondence), then the description of the π-isotypic components

$$H_{I,\underline{\lambda}}(\pi) = m_\pi \cdot \left(\rho_{\lambda_1} \circ \sigma(\pi)\right) \boxtimes \ldots \boxtimes \left(\rho_{\lambda_r} \circ \sigma(\pi)\right)$$

corresponding to (2) holds including the action of the group $\mathrm{Stab}(\underline{\lambda})$, which acts on the right hand side by permutation of the factors.

The usual approach to prove these assertions is to compare the trace of a Hecke correspondence times a Frobenius on $H_{I,\underline{\lambda}}$ with the trace of an associated Hecke function on the space of automorphic forms. In simplified notation:

$$\mathrm{Tr}\left(\text{Hecke} \times \text{Frob}^{\underline{s}}, H_{I,\underline{\lambda}}\right) \stackrel{!}{=} \mathrm{Tr}\left(\text{assoc.Hecke}, \text{Aut}\right) \qquad (3)$$

Since $D^* \backslash D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ is compact, the calculation of the right hand side via the Selberg trace formula is trivial and gives a certain sum over the conjugation classes in D^* of adelic orbital integrals. By the Lefschetz formula, in order to calculate the left hand side we have to count fixed points. This results in the same sum, where at finitely many places the local orbital integrals are replaced by twisted orbital integrals. That these coincide is the assertion of the fundamental lemma for GL_d in positive characteristic, for which presently only in two cases a published proof exists [Lau96]: for the unit element of the Hecke Algebra (Kottwitz) and for the coweights μ^+ and μ^- (Drinfeld).

However, for the proof of equation (2) the general case of the fundamental lemma is not needed: if the existence of $\sigma(\pi)$ is settled, it is sufficient to show the equality of the traces of Hecke $\times$ Frob$^{\underline{s}}$ on both sides of (2) in the case $\underline{s} = (1 \dots 1)$. For those places where the L-functions of π and of $\sigma(\pi)$ coincide, this is equivalent to equation (3), which in this case contains no twisted integrals.

That $\sigma'(\pi)$ is the multiple of a d-dimensional Galois representation follows from a simple argument related to the cases $\underline{\lambda} = (\mu^+ \dots \mu^+, \mu^- \dots \mu^-)$ with sequences of arbitrary length. Since this exceeds the range of proper moduli spaces, the Lefschetz formula can be applied only for sufficiently large exponents $\underline{s}$, which means that here Drinfeld's case of the fundamental lemma must be used. This had also been necessary for Lafforgue's calculation of the L-function of $\sigma'(\pi)$ underlying our reasoning.

One might expect that the L-functions of π and of $\sigma(\pi)$ coincide at all places $x \in X' \setminus I$. Since for this question the idele a may be changed, we can assume that a is supported outside x. In that case a violation of the equation $L_x(\pi, T) = L_x(\sigma(\pi), T)$ is equivalent to the existence of two different irreducible automorphic representations of $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ which are unramified at x and which coincide outside x. A closer investigation shows that the occurrence of this phenomenon would imply that in the case $\underline{\lambda} = (\mu^+ + \mu^-)$ equation (3) does not always hold. Thus conversely the fundamental lemma for $\mu^+ + \mu^-$ would imply that the local L-functions in question coincide, which in turn would imply equation (3) in all cases.

We remark that the fundamental lemma would also permit to calculate the cohomology with compact support of the non-proper moduli spaces. In this case twisted orbital integrals cannot be avoided, because the local terms at infinity in the Lefschetz formula are in general known to vanish only for sufficiently high powers of Frobenius. However, the assertions related to the actions of symmetric groups do not carry over to the non-proper cases directly, for it is not clear on which open set the direct images are smooth sheaves.

The proof of (c) relies on the following decomposition, which is a relatively formal consequence of the Springer correspondence for gl_m. We consider the sequence $\underline{\widetilde{\lambda}} = (\mu^+ \dots \mu^+, \mu^- \dots \mu^-)$ of length $2m$. The restriction of the representation $H^n_{I,\underline{\widetilde{\lambda}}}$ to the square of the diagonal of length m is a representation of

$$\mathcal{H}_I \times \pi_1(X' \setminus I)^2 \times (\mathfrak{S}_m)^2.$$

Then with respect to a certain bijection $\chi \leftrightarrow \underline{\lambda}(\chi)$ between the irreducible representations of the group $(\mathfrak{S}_m)^2$ and a subset of the sequences $\underline{\lambda}$ of length 2 there is a natural isomorphism of $\mathcal{H}_I \times \pi_1(X' \setminus I)^2$-modules

$$H^n_{I,\underline{\widetilde{\lambda}}}(\chi) = H^n_{I,\underline{\lambda}(\chi)}\,. \tag{4}$$

In view of the classical relation between the irreducible representations of the symmetric groups and polynomial representations of the general linear groups this is compatible with equation (2).

An immediate consequence of this decomposition is the description of $H_{I,\widetilde{\underline{\lambda}}}(\pi)$ as a virtual $\pi(X' \setminus I)^2 \times (\mathfrak{S}_m)^2$-module. Though this does not determine its structure as a $\pi(X' \setminus I)^{2m} \rtimes (\mathfrak{S}_m)^2$-module in general, it does so under the assumption on $\sigma(\pi)$ we made in (c).

For sequences $\widetilde{\underline{\lambda}}$ consisting only of μ^+ and μ^- with more than than two blocks, there is a decomposition analogous to (4) in which each block corresponds to one component of $\underline{\lambda}(\chi)$. Moreover this is compatible with the action of certain symmetric groups. Using this, (c) can be proved by decomposing one sequence $\widetilde{\underline{\lambda}}$ in two different ways.

Many results on generalised $\mathcal{D}$-shtukas can also be found in a recent preprint of Ngô Bao Châu [Ngô03] which appeared while I was working on the final chapters of the present work. As above, Ngô shows that the number of fixed points of a Hecke correspondence times a Frobenius on moduli spaces of generalised $\mathcal{D}$-shtukas can be expressed as a sum of partially twisted orbital integrals, but he restricts his attention to the subalgebra of those functions in $\mathcal{H}_I$ which are the unit element at points of I. This enables him to handle the modifications of the shtukas and the Hecke correspondences in a symmetric way.

Ngô's aim is to avoid the fundamental lemma completely in the calculation of the cohomology of the moduli spaces. To explain his approach we consider $\mathcal{D}$-shtukas with only one modification, i.e. $\underline{\lambda} = (\lambda)$. For any integer $s \geq 1$ let $s \cdot \underline{\lambda}$ be the sequence $(\lambda \dots \lambda)$ of length s, and let τ be the cyclic permutation on s letters. Ngô proves the equation

$$\operatorname{Tr}\left(\text{Hecke} \times \text{Frob}^s, H_{I,\underline{\lambda}}\right) = \operatorname{Tr}\left(\text{Hecke} \times \text{Frob} \times \tau, H_{I,s\cdot\underline{\lambda}}\right) \tag{5}$$

without using the fundamental lemma and shows that the right hand side equals the desired automorphic trace using a sheaf theoretic description of cyclic base change for the Hecke algebra [Ngô99]. Equation (5) can also be viewed as a relatively direct consequence of (2) and the general form of (4).

The restriction of this approach is that it depends on the properness of the moduli space of $\mathcal{D}$-shtukas with modifications bounded by $s \cdot \underline{\lambda}$, such that in each situation only finitely many s are allowed. Thus a direct calculation of the L-functions of the occurring Galois representations is not possible in this way. As pointed out above, using Drinfeld's case of the fundamental lemma we can calculate these L-functions outside some finite set of places. Moreover one additional instance of the fundamental lemma (the case $\mu^+ + \mu^-$) would give the L-functions at all unramified places.

Nevertheless, Ngô states that his results imply the fundamental lemma for GL_d and sketches a proof for semisimple regular elements. This is not affected by the problems with properness, because for base change of any fixed degree one can choose an appropriate division algebra D. In addition he writes that Henniart has proved the fundamental lemma for GL_d in positive characteristic (unpublished) using similar methods as Clozel and Labesse used in characteristic zero.

Using this, the computation of the cohomology of the moduli spaces is a standard matter. Thus the interest of the present work must be seen in showing how far one can get without or with little use of local harmonic analysis.

This work is divided into three parts.

The first part deals with the geometric properties of stacks of different kinds of generalised $\mathcal{D}$-shtukas. Logically this includes appendix A on properness in the case $d = 2$.

Our calculation of the number of fixed points in the second part closely follows the presentations in [Lau96] and [Laf97]. It is preceded by a short exposition of the statement of the fundamental lemma and by a discussion of the category of (D, φ)-spaces.

In the third part we construct the cohomology as a representation of the Hecke algebra, the Galois groups, and the symmetric groups. Starting from the computation of the fixed points we describe these representations as announced above.

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Notations

The following notations are in use in the entire work unless they are replaced by different conventions in some single sections.

Let X be a smooth, projective and geometrically irreducible curve over the finite field $\mathbb{F}_q$ with q elements and let F be its function field. Let $\mathcal{D}$ be an $\mathcal{O}_X$-algebra which is locally free of rank d^2 as an $\mathcal{O}_X$-module and such that $D = \mathcal{D} \otimes F$ is a division algebra with centre F. We denote by $X' \subseteq X$ the dense open subset where $\mathcal{D}$ is an Azumaya algebra.

For any closed point $x \in X$ we denote by $\mathcal{O}_x$ the completion of the local ring $\mathcal{O}_{X,x}$ and by F_x its quotient field, and we choose a generator of the maximal ideal $\varpi_x \in \mathcal{O}_x$. We assume that $\mathcal{D}_x = \mathcal{D} \otimes \mathcal{O}_x$ is a maximal order in $D_x = D \otimes F_x$ for all x, which implies that the closed points of X' are characterised by the condition $\mathrm{inv}_x(D) = 0$.

For a finite set of closed points $T \subset |X'|$ the scheme $X'_{(T)}$ is defined to be the intersection of all open subsets $U \subseteq X'$ containing T.

Let $\bar{\mathbb{F}}_q$ be a fixed algebraic closure of $\mathbb{F}_q$. An overbar generally denotes the corresponding base change:

$$\begin{aligned} \bar{X} &= X \times \mathrm{Spec}\, \bar{\mathbb{F}}_q & \qquad \bar{\mathcal{D}} &= \mathcal{D} \boxtimes \bar{\mathbb{F}}_q \\ \bar{F} &= F \otimes \bar{\mathbb{F}}_q & \bar{D} &= D \otimes \bar{\mathbb{F}}_q \\ \bar{\mathcal{O}}_x &= \mathcal{O}_x \mathbin{\widehat{\otimes}} \bar{\mathbb{F}}_q & \bar{\mathcal{D}}_x &= \mathcal{D}_x \mathbin{\widehat{\otimes}} \bar{\mathbb{F}}_q \\ \bar{F}_x &= F_x \mathbin{\widehat{\otimes}} \bar{\mathbb{F}}_q & \bar{D}_x &= D_x \mathbin{\widehat{\otimes}} \bar{\mathbb{F}}_q \end{aligned}$$

For a given embedding $\bar{x} : k(x) \subset \bar{\mathbb{F}}_q$ over $\mathbb{F}_q$ we use the notation:

$$\begin{aligned} \mathcal{O}_{\bar{x}} &= \mathcal{O}_x \mathbin{\widehat{\otimes}}_{k(x)} \bar{\mathbb{F}}_q & \qquad \mathcal{D}_{\bar{x}} &= \mathcal{D}_x \mathbin{\widehat{\otimes}}_{k(x)} \bar{\mathbb{F}}_q \\ F_{\bar{x}} &= F_x \mathbin{\widehat{\otimes}}_{k(x)} \bar{\mathbb{F}}_q & D_{\bar{x}} &= D_x \mathbin{\widehat{\otimes}}_{k(x)} \bar{\mathbb{F}}_q \end{aligned}$$

For any positive integer r let $F_{x,r}$ be the unramified extension of F_x of degree r, let $\mathcal{O}_{x,r} \subset F_{x,r}$ be its ring of integers, and accordingly:

$$\begin{aligned} \mathcal{D}_{x,r} &= \mathcal{D}_x \otimes_{\mathcal{O}_x} \mathcal{O}_{x,r} \\ D_{x,r} &= D_x \otimes_{F_x} F_{x,r} \end{aligned}$$

Let $T \subset |X|$ be a finite set of places. We write

$$\begin{aligned} \mathbb{A}^T &= \textstyle\prod'_{x \in X \setminus T} (F_x, \mathcal{O}_x) & \qquad D^T_{\mathbb{A}} &= D \otimes \mathbb{A}^T \\ \mathcal{O}^T_{\mathbb{A}} &= \textstyle\prod_{x \in X \setminus T} \mathcal{O}_x & \mathcal{D}^T_{\mathbb{A}} &= \mathcal{D} \otimes \mathcal{O}^T_{\mathbb{A}} \end{aligned}$$

Let μ^T be the invariant measure on the unimodular topological group $(D^T_{\mathbb{A}})^*$ which is normalised by $\mu^T((\mathcal{D}^T_{\mathbb{A}})^*) = 1$. For a finite closed subscheme $I \subset X$ with $I \cap T = \emptyset$ let

$$K^T_I = \mathrm{Ker}\big((\mathcal{D}^T_{\mathbb{A}})^* \to (\mathcal{D}_I)^*\big) .$$

The corresponding Hecke algebra is the space of K^T_I-biinvariant rational functions on $(D^T_{\mathbb{A}})^*$ with compact support, $\mathcal{H}^T_I = \mathcal{C}_0((D^T_{\mathbb{A}})^*//K^T_I, \mathbb{Q})$. The multiplication is given by convolution with respect to μ^T. We write $\mathcal{H}_I = \mathcal{H}^{\emptyset}_I$.

We choose an idele $a \in \mathbb{A}^*$ of degree 1 such that $a_x = 1$ outside a finite set of places $T(a) \subset |X|$ (this is possible because the Pic^0_X-torsor Pic^1_X is trivial).

Let P^+ be the set of dominant coweights for the group GL_d

$$P^+ = \big\{\lambda = (\lambda^{(1)} \geq \ldots \geq \lambda^{(d)}) \in \mathbb{Z}^d\big\}$$

and let $P^{++} \subset P^+$ be the subset of those λ for which all $\lambda^{(i)}$ have the same sign. The degree of $\lambda \in P^+$ is $\deg(\lambda) = \sum \lambda^{(i)}$, and for every integer m we denote by $P^+_m \subset P^+$ and $P^{++}_m \subset P^{++}$ the subsets of elements of degree m. Let $\mu^+ \in P^{++}_1$ and $\mu^- \in P^{++}_{-1}$ be the unique elements, i.e.

$$\mu^+ = (1, 0, \ldots, 0) \qquad \mu^- = (0, \ldots, 0, -1) .$$

The set P^+ will be given the partial order in which $\lambda_1 \leq \lambda_2$ if and only if $\lambda_1^{(1)} + \ldots + \lambda_1^{(i)} \leq \lambda_2^{(1)} + \ldots + \lambda_2^{(i)}$ for all $1 \leq i \leq d-1$ and $\deg(\lambda_1) = \deg(\lambda_2)$.

For a central simple algebra A over an arbitrary field k we denote by $d(A)$ its Schur index, which means $d(A)^2$ is the k-dimension of the division algebra equivalent to A. If k is a global field, this is the least common denominator of the local invariants of A.

S will always be a (variable) scheme over $\mathbb{F}_q$.

Part I
Geometry of the Moduli Spaces

Basically, a $\mathcal{D}$-shtuka over S is a diagram $[\,\mathcal{E} \to \mathcal{E}' \leftarrow {}^{\tau}\mathcal{E}\,]$ of locally free $\mathcal{D}\boxtimes\mathcal{O}_S$-modules of rank 1 with injective maps such that the length of both cokernels is d, i.e. $\mathcal{E}$ and its Frobenius twist ${}^{\tau}\mathcal{E}$ differ by the simplest nontrivial modification. The moduli space of these objects has been studied by Lafforgue in [Laf97].

Already in [Dri88] Drinfeld indicated that one can consider more general modifications as well. However there are a number of possibilities for a precise definition of the moduli problem. Diagrams as above without the condition on the lengths are generalised $\mathcal{D}$-shtukas in the sense of [Laf97] II.1, Definition 6. These will be discussed in the first section of the present work. One can also consider longer chains of such modifications (section 2).

As stated in the introduction, our main interest is the l-adic cohomology of moduli spaces of generalised $\mathcal{D}$-shtukas with modifications in a given set of points which are bounded by a given set of dominant coweights. These will be introduced in section 3. They could be studied independently of the previously defined $\mathcal{D}$-shtukas with free modifications, but in section 10 these will play a role.

1 Modifications of Arbitrary Length

In this section we will consider the simplest definition of generalised $\mathcal{D}$-shtukas, which already shows most interesting geometric phenomena. Many arguments are similar to those in [Laf97] for the case $m = 1$, which might not be well documented in all cases.

1.1 Definitions

It is clear that the following definitions yield fpqc-stacks over $\mathbb{F}_q$.

Definition 1.1.1. Let an integer $m \geq 0$ be given. For any $\mathbb{F}_q$-scheme S we denote by $\mathrm{Coh}^m_{\mathcal{D}}(S)$ the groupoid of coherent $\mathcal{O}_{X\times S}$-modules K with the structure of a $\mathcal{D}\boxtimes\mathcal{O}_S$-module and with the following properties:

– The support $\mathrm{Supp}(K)$ is finite over S and contained in $X' \times S$,
– $(pr_2)_* K$ is a locally free $\mathcal{O}_S$-module of rank dm.

Definition 1.1.2. For any integer $m \geq 0$ let $\mathrm{Sht}^m(S) = \mathrm{Sht}^m_{\mathcal{D}}(S)$ be the groupoid of diagrams

$$\left[\begin{array}{ccc} \mathcal{E} & \xrightarrow{j} & \\ & & \mathcal{E}' \\ {}^{\tau}\mathcal{E} & \xrightarrow{t} & \end{array} \right]$$

on $X \times S$ of the following type:
– $\mathcal{E}$ and $\mathcal{E}'$ are locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank 1,
– ${}^{\tau}\mathcal{E} = (\mathrm{id} \times \mathrm{Frob}_q)^* \mathcal{E}$,
– j and t are injective homomorphisms,
– $\mathrm{Coker}(j)$ and $\mathrm{Coker}(t) \in \mathrm{Coh}^m_{\mathcal{D}}(S)$.

These objects will be called $\mathcal{D}$-shtukas of length m (and of rank 1) over S.

Definition 1.1.3. For a given finite closed subscheme $I \subset X$, a level-I-structure for a $\mathcal{D}$-shtuka over S is a pair of two isomorphisms $\iota : \mathcal{D}_I \boxtimes \mathcal{O}_S \cong \mathcal{E}_I$ and $\iota' : \mathcal{D}_I \boxtimes \mathcal{O}_S \cong \mathcal{E}'_I$ such that the following diagram commutes.

$$\begin{array}{ccccc} \mathcal{E}_I & \xrightarrow[j]{\sim} & \mathcal{E}'_I & \xleftarrow[t]{\sim} & {}^{\tau}\mathcal{E}_I \\ \uparrow \iota & & \uparrow \iota' & & \uparrow {}^{\tau}\iota \\ \mathcal{D}_I \boxtimes \mathcal{O}_S & = & \mathcal{D}_I \boxtimes \mathcal{O}_S & = & \mathcal{D}_I \boxtimes \mathcal{O}_S \end{array}$$

We denote by $\mathrm{Sht}^m_I(S)$ the groupoid of $\mathcal{D}$-shtukas of length m over S with a level-I-structure.

The group $\mathrm{Pic}_I(X)$ of invertible $\mathcal{O}_X$-modules $\mathscr{L}$ equipped with a level-I-structure, i.e. with an isomorphism $\mathcal{O}_I \cong \mathscr{L}_I$, acts on Sht^m_I by

$$\left[\, \mathcal{E} \to \mathcal{E}' \leftarrow {}^{\tau}\mathcal{E} \,\right] \longmapsto \left[\, \mathcal{E} \otimes \mathscr{L} \to \mathcal{E}' \otimes \mathscr{L} \leftarrow {}^{\tau}\mathcal{E} \otimes \mathscr{L} \,\right]$$

with the obvious new level structure. We fix the homomorphism $\mathbb{A}^* \to \mathrm{Pic}_I(X)$ given by $b \mapsto \mathcal{O}(-b)$ with the multiplication $b : \mathcal{O}_I \cong \mathcal{O}(-b)_I$ as level structure. This gives an action of the chosen idele $a \in \mathbb{A}^*$ of degree 1 on Sht^m_I. If we decompose this stack according to $\deg(\mathcal{E}) \in \mathbb{Z}$, the quotient

$$\mathrm{Sht}^m_I / a^{\mathbb{Z}}$$

can be identified with the union of finitely many of the pieces.

We have the following commutative diagram.

$$\begin{array}{cccccc} \pi & : & \mathrm{Sht}^m/a^{\mathbb{Z}} & \xrightarrow{\alpha} \mathrm{Coh}^m_{\mathcal{D}} \times \mathrm{Coh}^m_{\mathcal{D}} \xrightarrow{N\times N} & X'^{(m)} \times X'^{(m)} \\ & & \uparrow{\scriptstyle\beta} & & \uparrow \\ \pi_I & : & \mathrm{Sht}^m_I/a^{\mathbb{Z}} & \longrightarrow & (X'\setminus I)^{(m)} \times (X'\setminus I)^{(m)} \end{array}$$

Here α is given by the quotients $\mathcal{E}'/{}^{\tau}\mathcal{E}$ and $\mathcal{E}'/\mathcal{E}$, and β is defined by forgetting the level structure. The norm $N : \mathrm{Coh}^m_{\mathcal{D}} \to X'^{(m)}$ will be explained below. Our next aim is to prove the following basic facts.

1. $\mathrm{Coh}^m_{\mathcal{D}}$ is an algebraic stack of finite type over $\mathbb{F}_q$ and smooth of relative dimension zero.

2. The morphism α is smooth of relative dimension $2md$.

3. The morphism β is representable and étale and is a $\mathcal{D}^*_I$-torsor outside I.

4. The morphism π_I is of finite type and is representable quasiprojective if $I \neq \emptyset$.

5. If the division algebra D is sufficiently ramified, then π_I is proper.

The second and third statement and their proofs remain valid when D is replaced by any central simple algebra over F.

1.2 Norm and determinant

We begin with a general remark. For an Azumaya algebra $\mathcal{A}$ defined over an open subset $U \subseteq X$ let $\mathrm{Coh}^m_{\mathcal{A}}$ be given by the obvious variant of Definition 1.1.1. In the following sense this stack only depends on the equivalence class of $\mathcal{A}$ in the Brauer group.

Lemma 1.2.1. *Any equivalence of Azumaya algebras $\mathcal{A}\otimes\mathcal{E}nd(\mathcal{E}) \cong \mathcal{A}'\otimes\mathcal{E}nd(\mathcal{E}')$ with locally free $\mathcal{O}_U$-modules $\mathcal{E}$ and $\mathcal{E}'$ induces a 1-isomorphism $\mathrm{Coh}^m_{\mathcal{A}} \cong \mathrm{Coh}^m_{\mathcal{A}'}$. Two of these are locally in U 2-isomorphic.*

Proof. The functor $K \mapsto K\otimes\mathcal{E}$ gives a 1-isomorphism $u_{\mathcal{E}} : \mathrm{Coh}^m_{\mathcal{A}} \cong \mathrm{Coh}^m_{\mathcal{A}\otimes\mathcal{E}nd(\mathcal{E})}$, and the composition $u^{-1}_{\mathcal{E}'} \circ u_{\mathcal{E}}$ is the asserted 1-isomorphism. For its local independence of the given equivalence we have to show that any 1-automorphism $u : \mathrm{Coh}^m_{\mathcal{A}} \cong \mathrm{Coh}^m_{\mathcal{A}}$ induced by an isomorphism $\alpha : \mathcal{A}\otimes\mathcal{E}nd(\mathcal{E}) \cong \mathcal{A}\otimes\mathcal{E}nd(\mathcal{E}')$ is locally in U 2-isomorphic to the identity. By the theorem of Skolem-Noether over local rings, cf. [Gr68] Theorem 5.10, the isomorphism α comes from an isomorphism of right $\mathcal{A}$-modules $\mathcal{A}\otimes\mathcal{E}\otimes\mathscr{L} \cong \mathcal{A}\otimes\mathcal{E}'$ with an invertible $\mathcal{O}_X$-module $\mathscr{L}$. Using this, u is 2-isomorphic to the functor $K \mapsto K\otimes\mathscr{L}^{-1}$. □

Now we return to our situation. By Tsen's theorem the Azumaya algebra $\mathcal{D} \otimes \bar{\mathbb{F}}_q$ is trivial on $X' \otimes \bar{\mathbb{F}}_q$. This holds already over some finite field $\mathbb{F}_{q^n}$, i.e. there are two vector bundles $\mathcal{E}$ and $\mathcal{E}'$ on $X' \otimes \mathbb{F}_{q^n}$ and an isomorphism $(\mathcal{D} \otimes \mathbb{F}_{q^n}) \otimes \mathcal{E}nd(\mathcal{E}) \cong \mathcal{E}nd(\mathcal{E}')$. The induced 1-isomorphism

$$\mathrm{Coh}^m_{\mathcal{D}} \otimes \mathbb{F}_{q^n} \cong \mathrm{Coh}^m_{X'} \otimes \mathbb{F}_{q^n} \tag{1.2.1}$$

is unique up to an automorphism of $\mathrm{Coh}^m_{\mathcal{D}} \otimes \mathbb{F}_{q^n}$ which locally in X' is 2-isomorphic to the identity.

Remark. If we assume $d \geq 2$ then X' is affine, and a multiple of any vector bundle on X' is trivial (over a Dedekind domain any finitely generated projective module is isomorphic to a direct sum of an ideal and a free module, the direct sum of projective modules corresponding to the product of ideals). Thus for some integer $r \geq 1$ there even is an isomorphism $M_r(\mathcal{D} \otimes \mathbb{F}_{q^n}) \cong M_{rd}(\mathcal{O}_{X'} \otimes \mathbb{F}_{q^n})$.

Proposition 1.2.2. *The stack* $\mathrm{Coh}^m_{\mathcal{D}}$ *is algebraic, of finite type and smooth over* $\mathbb{F}_q$ *of dimension zero.*

Proof. The isomorphism (1.2.1) reduces the assertion to the case $\mathcal{D} = \mathcal{O}_X$, which can be found in [Lau87], page 317 (the stack Coh^m_X is isomorphic to the quotient $\mathrm{Quot}^0 / \mathrm{GL}_m$ with a certain open substack $\mathrm{Quot}^0 \subseteq \mathrm{Quot}^m_{\mathcal{O}^m_X/X/\mathbb{F}_q}$). □

Following [Laf97] I.1, Lemma 3, the reduced norm $M_r(D) \to F$ has a unique extension to a homomorphism of monoids

$$\mathrm{nrd} \;:\; M_r(\mathcal{D}) \longrightarrow \mathcal{O}_X \,.$$

This allows the definition of a functor $\mathcal{E} \mapsto \det(\mathcal{E})$ (determinant) from the category of locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank r into the category of locally free $\mathcal{O}_{X \times S}$-modules of rank 1 along with a natural isomorphism $\det(\mathcal{E})^{\otimes d} \cong \Lambda^{rd^2} \mathcal{E}$.

Definition 1.2.3. Let $\mathrm{Inj}^{r,m}_{\mathcal{D}}(S)$ be the groupoid of injective maps $\mathcal{E} \supseteq \mathcal{E}'$ of locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank r with quotient $\mathcal{E}/\mathcal{E}' \in \mathrm{Coh}^m_{\mathcal{D}}(S)$. These objects will be called (upper) modifications of length m and of rank r.

In the case $\mathcal{D} = \mathcal{O}_X$ and $r = 1$, $\mathcal{E}$ and $\mathcal{E}'$ are invertible sheaves, and the quotient $\mathcal{E}/\mathcal{E}'$ is locally free of rank 1 over a uniquely determined relative divisor of degree m. We denote by

$$\mathrm{div} \;:\; \mathrm{Inj}^{1,m}_{\mathcal{O}} \longrightarrow X^{(m)}$$

the thereby defined morphism.

Proposition 1.2.4. *The above defined determinant is a morphism*

$$\det \ : \ \mathrm{Inj}_{\mathcal{D}}^{r,m} \longrightarrow \mathrm{Inj}_{\mathcal{O}}^{1,m} .$$

Moreover there is a unique morphism N (norm) which makes the following diagram commutative for all r.

$$\begin{array}{ccc} \mathrm{Inj}_{\mathcal{D}}^{r,m} & \xrightarrow{\det} & \mathrm{Inj}_{\mathcal{O}}^{1,m} \\ \downarrow{\scriptstyle \mathcal{E}/\mathcal{E}'} & & \downarrow{\scriptstyle \mathrm{div}} \\ \mathrm{Coh}_{\mathcal{D}}^{m} & \xrightarrow{N} X'^{(m)} \subseteq & X^{(m)} \end{array} \tag{1.2.2}$$

The norm is additive in exact sequences. In the case $m = 1$ it can be characterised by the fact that $K \in \mathrm{Coh}_{\mathcal{D}}^{1}(S)$ is a locally free sheaf of rank d over the graph of $N(K) : S \to X'$.

Lemma 1.2.5. *Let $\mathcal{E}$ be a $\mathcal{D} \boxtimes \mathcal{O}_S$-module which is locally free of finite rank over $\mathcal{O}_{X\times S}$. Then $\mathcal{E}$ is locally free over $\mathcal{D} \boxtimes \mathcal{O}_S$ if and only if for any geometric point $\bar{y} \in X \times S$ the fibre $\mathcal{E}_{\bar{y}}$ is locally free over $(\mathcal{D} \boxtimes \mathcal{O}_S)_{\bar{y}}$.*

Proof. This is an immediate consequence of [Laf97] I.2, Lemma 4. □

Lemma 1.2.6. *Let $\mathcal{E}$ be a locally free $\mathcal{D}\boxtimes\mathcal{O}_S$-module of finite rank and let $\mathcal{E}' \subseteq \mathcal{E}$ with $\mathcal{E}/\mathcal{E}' \in \mathrm{Coh}_{\mathcal{D}}^{m}(S)$. Then $\mathcal{E}'$ is locally free over $\mathcal{D} \boxtimes \mathcal{O}_S$ as well.*

Proof. Since $\mathcal{E}/\mathcal{E}'$ is flat over S, $\mathcal{E}'$ is flat over S and for any point $s \in S$ the fibre $\mathcal{E}'_s$ is torsion free, thus flat over $X \times \{s\}$. By EGA IV, Corollary 11.3.11 this implies that $\mathcal{E}'$ is locally free over $\mathcal{O}_{X\times S}$. That $\mathcal{E}'$ is also locally free over $\mathcal{D} \boxtimes \mathcal{O}_S$ must only be proved over $X' \times S$ and may in view of Lemma 1.2.5 be checked over the geometric points of S. This means we have to show that $\mathcal{E}_{\bar{s}}$ and $\mathcal{E}'_{\bar{s}}$ are locally isomorphic over $X' \times \{\bar{s}\}$. There the algebra $\mathcal{D}$ is trivialisable, and for $\mathcal{D} = \mathcal{O}_X$ the assertion is clear. □

Proof of Proposition 1.2.4. To prove the first assertion, let a modification $f : \mathcal{E}' \to \mathcal{E}$ (with S-flat cokernel) be given. Then for any point $s \in S$ the fibre f_s is injective, which implies $\det(f)_s = \det(f_s)$ is injective, too. Thus $\det(f)$ is injective with S-flat cokernel. The degree of the divisor can be seen at the geometric points of S, where a trivialisation of $\mathcal{D}\,|_{X'}$ is available.

In order to construct N, for a given $K \in \mathrm{Coh}_{\mathcal{D}}^{m}(S)$ we chose locally in S a presentation $0 \to \mathcal{E}' \to \mathcal{E} \to K \to 0$ with a locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-module $\mathcal{E}$. Then $\mathcal{E}'$ is locally free as well by Lemma 1.2.6. So locally in S the module K comes from a modification, and diagram (1.2.2) determines $N(K)$. One can show that this is independent of the presentation, which also implies that $N(K)$ is defined globally.

The norm is additive because a short exact sequence of K's locally in S admits a simultaneous presentation. The last assertion for $m = 1$ needs to be proved only over $\overline{\mathbb{F}}_q$, i.e. only in the case $\mathcal{D} = \mathcal{O}_X$. Then any $K \in \mathrm{Coh}^1_X(S)$ is locally in S a quotient of $\mathcal{O}_{X\times S}$, and the assertion follows from the definition of div. $\square$

Remark. The reduced norm for $r = 1$ induces a morphism $\det : \mathrm{Sht}^m_{\mathcal{D}} \to \mathrm{Sht}^m_{\mathcal{O}}$ which fits into the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Sht}^m_{\mathcal{D}} & \xrightarrow{\det} & \mathrm{Sht}^m_{\mathcal{O}} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\pi} \\ \mathrm{Coh}^m_{\mathcal{D}} \times \mathrm{Coh}^m_{\mathcal{D}} & \xrightarrow{N\times N} & X'^{(m)} \times X'^{(m)} \end{array}$$

Here π is given by the divisors of ${}^\tau\mathcal{E} \subseteq \mathcal{E}'$ and $\mathcal{E} \subseteq \mathcal{E}'$.

1.3 Smoothness

Now we consider the geometric properties of the maps

$$\mathrm{Sht}^m_I/a^{\mathbb{Z}} \xrightarrow{\beta} \mathrm{Sht}^m/a^{\mathbb{Z}} \xrightarrow{\alpha} \mathrm{Coh}^m_{\mathcal{D}} \times \mathrm{Coh}^m_{\mathcal{D}}\,.$$

Definition 1.3.1. Let $\mathrm{Vect}^r_{\mathcal{D}}(S)$ be the groupoid of locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules $\mathcal{E}$ of rank r.

By [Laf97] I.2, Lemma 5, $\mathrm{Vect}^r_{\mathcal{D}}$ is algebraic, locally of finite type and smooth over $\mathbb{F}_q$.

Lemma 1.3.2. *The two morphisms*

$$\mathrm{Inj}^{r,m}_{\mathcal{D}} \longrightarrow \mathrm{Vect}^r_{\mathcal{D}} \times \mathrm{Coh}^m_{\mathcal{D}}$$

given by $(\mathcal{E}, \mathcal{E}/\mathcal{E}')$ and by $(\mathcal{E}', \mathcal{E}/\mathcal{E}')$ are representable quasiaffine of finite type and smooth of relative dimension rdm. Consequently $\mathrm{Inj}^{r,m}_{\mathcal{D}}$ is algebraic, locally of finite type and smooth over $\mathbb{F}_q$.

Proof. A $\mathcal{D}$-modification $0 \to \mathcal{E}' \to \mathcal{E} \to K \to 0$ over S is equivalent to the dual $\mathcal{D}^{\mathrm{op}}$-modification

$$0 \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{E}'^\vee \longrightarrow \mathcal{E}xt^1_{\mathcal{D}\boxtimes\mathcal{O}_S}\big(K, \mathcal{D} \boxtimes \mathcal{O}_S\big) \longrightarrow 0$$

with $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{D}\boxtimes\mathcal{O}_S}(\mathcal{E}, \mathcal{D}\boxtimes\mathcal{O}_S)$. Thus we have to consider only one of the morphisms, for example the first one defined by $\mathcal{E}$ and $\mathcal{E}/\mathcal{E}'$. Its fibre over a given pair $(\mathcal{E}, K)$ is the open subsheaf of surjective homomorphisms in $(p_2)_*\mathcal{H}om_{\mathcal{D}\boxtimes\mathcal{O}_S}(\mathcal{E}, K)$ (their kernel is automatically locally free by Lemma 1.2.6). $\square$

Definition 1.3.3. Let $\mathrm{Hecke}^m(S)$ be the groupoid of diagrams on $X \times S$

$$[\,\mathcal{E} \to \mathcal{E}' \leftarrow \mathcal{E}''\,]$$

such that $\mathcal{E}' \supseteq \mathcal{E}$ and $\mathcal{E}' \supseteq \mathcal{E}''$ are upper modifications of length m and of rank 1.

This definition and the following consequence of Lemma 1.3.2 could also have been stated for arbitrary rank.

Lemma 1.3.4. *The three morphisms*

$$\mathrm{Hecke}^m_{\mathcal{D}} \longrightarrow \mathrm{Vect}^1_{\mathcal{D}} \times \mathrm{Coh}^m_{\mathcal{D}} \times \mathrm{Coh}^m_{\mathcal{D}}$$

given by $\mathcal{E}$, $\mathcal{E}'$ or $\mathcal{E}''$ in the first component and by $(\mathcal{E}'/\mathcal{E}'', \mathcal{E}'/\mathcal{E})$ in the second and third component are representable quasiaffine of finite type and smooth of relative dimension $2dm$. Consequently $\mathrm{Hecke}^m_{\mathcal{D}}$ is algebraic, locally of finite type and smooth over $\mathbb{F}_q$. □

Proposition 1.3.5. *The morphism*

$$\alpha \;:\; \mathrm{Sht}^m \longrightarrow \mathrm{Coh}^m_{\mathcal{D}} \times \mathrm{Coh}^m_{\mathcal{D}}$$

given by $\mathcal{E}'/{}^\tau\mathcal{E}$ and $\mathcal{E}'/\mathcal{E}$ is algebraic, locally of finite type and smooth of relative dimension $2md$. Consequently the stack Sht^m is algebraic, locally of finite type and smooth over $\mathbb{F}_q$ with dimension $2md$.

Proof. The following 2-cartesian diagram is a restatement of the definition of Sht^m.

$$\begin{array}{ccc}
\mathrm{Sht}^m & \longrightarrow & \mathrm{Vect}^1_{\mathcal{D}} \\
\big\downarrow & \square & \big\downarrow{\scriptstyle (\mathrm{Frob}_q,\mathrm{id})} \\
\mathrm{Hecke}^{1,m}_{\mathcal{D}} & \xrightarrow[(\mathcal{E}'',\mathcal{E})]{} & \mathrm{Vect}^1_{\mathcal{D}} \times \mathrm{Vect}^1_{\mathcal{D}}
\end{array}$$

Denoting $Y = \mathrm{Coh}^m_{\mathcal{D}} \times \mathrm{Coh}^m_{\mathcal{D}}$ and $U = \mathrm{Vect}^1_{\mathcal{D}} \times Y$, the assertion follows from Lemma 1.3.6 below. Here both $f = (\mathcal{E}'', \mathcal{E}'/\mathcal{E}'', \mathcal{E}'/\mathcal{E})$ and $g = (\mathcal{E}, \mathcal{E}'/\mathcal{E}'', \mathcal{E}'/\mathcal{E})$ are representable smooth morphisms. □

Lemma 1.3.6. *Let Y be an algebraic stack over $\mathbb{F}_q$ and let $U \to Y$, $V \to Y$ be two algebraic morphisms which are locally of finite type. Denote by ${}^\tau U$ the base change of U by the absolute Frobenius $\mathrm{Frob}_q : Y \to Y$ and by $F : U \to {}^\tau U$ its relative Frobenius. We consider the following 2-cartesian diagram of stacks in which $f : V \to {}^\tau U$ is assumed to be representable.*

$$\begin{array}{ccc}
W & \longrightarrow & U \\
\big\downarrow & \square & \big\downarrow{\scriptstyle (F,\mathrm{id})} \\
V & \xrightarrow[(f,g)]{} & {}^\tau U \times_Y U
\end{array}$$

Then the morphism $W \to Y$ is algebraic and locally of finite type, and the diagonal $W \to W \times_Y W$ (automatically representable, separated, and of finite type) is everywhere unramified, thus quasifinite.

If in addition $U \to Y$ is smooth and f is smooth of relative dimension n, then $W \to Y$ is smooth of relative dimension n as well.

Proof. Any fibred product of algebraic morphisms which are locally of finite type has the same property. The remaining assertions are reduced to the case that Y is a scheme by a smooth presentation $Y' \to Y$. In this case the lemma is (almost) literally [Laf97] I.2, Proposition 1. □

Definition 1.3.7. For a finite closed subscheme $I \subset X$ let $\mathrm{Sht}^0_{\mathcal{D}_I}(S)$ be the groupoid of locally free $\mathcal{D}_I \boxtimes \mathcal{O}_S$-modules $\mathcal{E}$ of rank 1 plus an isomorphism ${}^\tau\mathcal{E} \cong \mathcal{E}$.

In [Laf97] I.3 this stack is called $\mathrm{Tr}^1_{\mathcal{D}_I}$ (trivial shtukas) and it is proved that the morphism

$$\operatorname{Spec} \mathbb{F}_q / \mathcal{D}_I^* \longrightarrow \mathrm{Sht}^0_{\mathcal{D}_I}$$

given by $\mathcal{D}_I \in \mathrm{Sht}^0_{\mathcal{D}_I}(\mathbb{F}_q)$ is an isomorphism. (Since the stack $\mathrm{Sht}^0_{\mathcal{D}_I}$ is étale over $\mathbb{F}_q$ by Lemma 1.3.6, this follows from Drinfeld's lemma 8.1.1 below.) The definition of a level-I-structure can be expressed by the following 2-cartesian diagram in which the lower arrow is given by $[\mathcal{E} \to \mathcal{E}' \leftarrow {}^\tau\mathcal{E}] \mapsto [\mathcal{E}_I \xrightarrow{\sim} \mathcal{E}'_I \xleftarrow{\sim} {}^\tau\mathcal{E}_I]$.

$$\begin{array}{ccc}
\mathrm{Sht}^m_I & \longrightarrow & \operatorname{Spec} \mathbb{F}_q \\
\downarrow & \square & \downarrow \\
\mathrm{Sht}^m\big|_{(X' \setminus I)^{(m)} \times (X' \setminus I)^{(m)}} & \longrightarrow & \mathrm{Sht}^0_{\mathcal{D}_I}
\end{array}$$

This implies

Proposition 1.3.8. *With respect to the right action of the finite group $\mathcal{D}_I^*$ on Sht^m_I by twisting the level structure, the morphism*

$$\mathrm{Sht}^m_I \longrightarrow \mathrm{Sht}^m\big|_{(X' \setminus I)^{(m)} \times (X' \setminus I)^{(m)}}$$

given by forgetting the level structure is a $\mathcal{D}_I^*$-Torsor. □

1.4 Quasiprojectivity

The following is mainly a reproduction of the corresponding arguments from [Laf97] for the case $m = 1$.

We begin with some general remarks. Let $\mathrm{Vect}^r_{\mathcal{O},I}$ be the stack of vector bundles $\mathcal{E}$ on $X \times S$ with a level-I-structure, i.e. with an isomorphism $i : \mathcal{O}^r_{I\times S} \cong \mathcal{E}_I$. A pair $(\mathcal{E}, i)$ is called I-stable if for any geometric point $\bar{s} \in S$ and any proper submodule $0 \neq \mathcal{F} \subset \mathcal{E}_{\bar{s}}$ the following inequality holds:

$$\frac{\deg(\mathcal{F}) - \deg(I)}{\mathrm{rk}(\mathcal{F})} < \frac{\deg(\mathcal{E}_{\bar{s}}) - \deg(I)}{\mathrm{rk}(\mathcal{E}_{\bar{s}})}$$

This is a special case of [Sesh] 4.I, Definition 2. A vector bundle $\mathcal{E}$ on $X \otimes k$ with an algebraically closed field k admits arbitrary level structures and becomes I-stable as soon as the degree of I is sufficiently large.

Theorem 1.4.1 (Seshadri). *The open substack of I-stable vector bundles with fixed rank r and degree d*

$$\mathrm{Vect}^{r,d,\mathrm{stab}}_{\mathcal{O},I} \subset \mathrm{Vect}^r_{\mathcal{O},I}$$

is a smooth quasiprojective scheme with dimension $r^2(g-1+\deg I)$. In particular it is of finite type.

Proof. See [LRS] 4.3 or [Sesh] 4.III. □

Lemma 1.4.2. *The two morphisms*

$$\mathrm{Inj}^{r,m}_{\mathcal{D}} \longrightarrow \mathrm{Vect}^r_{\mathcal{D}}$$

given by $\mathcal{E}$ or $\mathcal{E}'$ are representable quasiprojective, in particular of finite type.

Proof. As in the proof of Lemma 1.3.2 we only need to consider the map given by $\mathcal{E}$. Its fibre in a given $\mathcal{E} \in \mathrm{Vect}^r_{\mathcal{D}}(S)$ is a locally closed subscheme of the relatively projective S-scheme $\mathrm{Quot}^{dm}_{\mathcal{E}/X\times S/S}$. □

Lemma 1.4.3. *The two morphisms*

$$\mathrm{Sht}^m \longrightarrow \mathrm{Vect}^{d^2}_{\mathcal{O}_X}$$

given by $\mathcal{E}$ or $\mathcal{E}'$ considered as $\mathcal{O}_{X\times S}$-modules are representable quasiprojective, in particular of finite type.

Proof. Both morphisms can be written as compositions $\mathrm{Sht}^m \to \mathrm{Hecke}^{1,m}_{\mathcal{D}} \to \mathrm{Vect}^1_{\mathcal{D}} \to \mathrm{Vect}^{d^2}_{\mathcal{O}_X}$. The first map is a closed immersion, and the remaining two maps are representable quasiprojective by Lemma 1.4.2 and by [Laf97] I.2, Lemma 2, respectively. □

Lemma 1.4.4. *Let $Y \subset X' \times S$ be a closed subscheme which is finite over S, and let $\mathcal{E}$ be a $\mathcal{D} \boxtimes \mathcal{O}_S$-module which is locally free over $\mathcal{O}_{X\times S}$. If there is an isomorphism*

$$\mathcal{E}\,|_{\,X\times S-Y} \cong {}^{\tau}\mathcal{E}\,|_{\,X\times S-Y}$$

then $\mathcal{E}$ is locally free over $\mathcal{D} \boxtimes \mathcal{O}_S$.

Proof. Though the statement of the lemma slightly differs from [Laf97] I.4, Proposition 7, the proof is literally the same. □

Proposition 1.4.5. *The stack $\mathrm{Sht}^m/a^{\mathbb{Z}}$ is of finite type over $\mathbb{F}_q$.*

Proof. Cf. [Lau97], Lemma 4.2. For any Harder-Narasimhan polygon P, the open substack in $\mathrm{Vect}^{d^2}_{\mathcal{O}_X}$ of vector bundles with polygon $\leq P$ and fixed degree is of finite type over $\mathbb{F}_q$. (This follows from Theorem 1.4.1.) Hence by Lemma 1.4.3 it suffices to show that for $\mathcal{D}$-shtukas with modifications of fixed length m over an algebraically closed field k the polygon of the $\mathcal{O}_{X\otimes k}$-module $\mathcal{E}$ is bounded.

Assume it is not bounded. Then there are $\mathcal{D}$-shtukas $[\,\mathcal{E} \to \mathcal{E}' \leftarrow {}^{\tau}\mathcal{E}\,]$ plus $\mathcal{O}_{X\otimes k}$-submodules $\mathcal{F} \subset \mathcal{E}$ with locally free quotient and $0 \neq \mathcal{F} \neq \mathcal{E}$ such that the jump of the slopes $\delta = \mu^-(\mathcal{F}) - \mu^+(\mathcal{E}/\mathcal{F})$ becomes arbitrary large. Here μ^+ denotes the maximal slope of a subsheaf and μ^- denotes the minimal slope of a quotient. Since the $\mathcal{O}_X$-module $\mathcal{D}$ can be written as a quotient of $\mathcal{O}(n)^N$, the map $\mathcal{F} \otimes \mathcal{D} \to \mathcal{E}/\mathcal{F}$ vanishes for sufficiently large δ, which means $\mathcal{F} \subset \mathcal{E}$ is a $\mathcal{D} \boxtimes k$-submodule.

Let $\mathcal{F}' \subset \mathcal{E}'$ be the inverse image of the torsion part of $\mathcal{E}'/\mathcal{F}$. The difference $\mu^+(\mathcal{E}'/\mathcal{F}') - \mu^+(\mathcal{E}/\mathcal{F})$ is bounded by md, so the map ${}^{\tau}\mathcal{F} \to \mathcal{E}'/\mathcal{F}'$ vanishes for large δ as well. Thus $[\,\mathcal{F} \to \mathcal{F}' \leftarrow {}^{\tau}\mathcal{F}\,]$ is a diagram of $\mathcal{D} \boxtimes k$-modules with $\mathcal{F}'/\mathcal{F} \subseteq \mathcal{E}'/\mathcal{E}$ and $\mathcal{F}'/{}^{\tau}\mathcal{F} \subseteq \mathcal{E}'/{}^{\tau}\mathcal{E}$. Then by Lemma 1.4.4 $\mathcal{F}$ is locally free over $\mathcal{D} \boxtimes k$, contradicting the assumption. □

Proposition 1.4.6. *In the case $I \neq \emptyset$ the stack $\mathrm{Sht}^m_I/a^{\mathbb{Z}}$ is a quasiprojective scheme over $\mathbb{F}_q$.*

Proof. Let $\mathrm{Sht}^{m,\mathrm{stab}}_I/a^{\mathbb{Z}} \subseteq \mathrm{Sht}^m_I/a^{\mathbb{Z}}$ be the open substack where $\mathcal{E}$ considered as an $\mathcal{O}_{X\times S}$-module with the induced level structure is I-stable. Then $\mathrm{Sht}^m_I/a^{\mathbb{Z}}$ is the ascending union of the open substacks

$$\mathrm{Sht}^{m,\mathrm{stab}}_{nI}/a^{\mathbb{Z}}K_n\,, \qquad K_n = \mathrm{Ker}\big(\mathcal{D}^*_{nI} \to \mathcal{D}^*_I\big)\,,$$

each of which is a quotient of a quasiprojective scheme modulo a finite group by Theorem 1.4.1 and Lemma 1.4.3. From Proposition 1.4.5 it follows that this chain of substacks becomes stationary at some finite level n.

So it remains to prove that the group K_n acts without fixed points. An equivalent statement is that the only automorphism of a $\mathcal{D}$-shtuka $\mathcal{E}^\bullet = [\mathcal{E} \to \mathcal{E}' \leftarrow {}^{\tau}\mathcal{E}]$

with non-trivial level-I-structure over an algebraically closed field k is the identity. Denoting by $\mathcal{I} \subset \mathcal{O}_X$ the ideal of I, such an automorphism u induces a homomorphism

$$1 - u \; : \; \mathcal{E}^\bullet \longrightarrow \mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I} \, .$$

From Lemma 1.4.4 follows that the saturation of its image consists of locally free $\mathcal{D} \boxtimes k$-modules. So $1 - u$ is injective or zero, but the former is impossible in view of $\deg(\mathcal{I}) < 0$. □

Corollary 1.4.7. *All stacks* Sht^m *are Deligne-Mumford stacks.* □

1.5 Properness

Contrary to the expectation the morphism $\mathrm{Sht}^m/a^{\mathbb{Z}} \to X'^{(m)} \times X'^{(m)}$ is not always proper, not even in the case $m = 1$. Here we only give a sufficient criterion for properness, but in appendix A we show that this is optimal in the case $d = 2$. In particular, for any $m \geq 1$ there are division algebras D for which the morphism π_I^m is not proper. In the present work we will (almost) exclusively be interested in the proper case.

Definition 1.5.1. The division algebra D is called sufficiently ramified with respect to m if for any set of md places of F the least common denominator of the local invariants of D at the remaining places equals d.

Proposition 1.5.2. *If D is sufficiently ramified with respect to m, then the morphism*

$$\pi_I^m \;:\; \mathrm{Sht}_I^m/a^{\mathbb{Z}} \longrightarrow (X' \setminus I)^{(m)} \times (X' \setminus I)^{(m)}$$

is proper, so in the case $I \neq \emptyset$ it is projective by Proposition 1.4.6.

Proof. Following a remark of L. Lafforgue, in the sufficiently ramified case a slight modification of the proof of [Laf97] IV.1, Theorem 1 is correct. By Proposition 1.3.8 we may assume $I = \emptyset$, and we have to prove the valuative criterion of properness for $\pi_\emptyset^m$ or equivalently for the morphism $\mathrm{Sht}^m \to X'^{(m)} \times X'^{(m)}$.

Let A be a complete discrete valuation ring with quotient field K, residue field k, and a uniformising element ϖ, and let A' be the local ring of $X \otimes A$ in the generic point of the special fibre. We must show that a given $\mathcal{D}$-shtuka $[\, \mathcal{E}_K \to \mathcal{E}'_K \leftarrow {}^\tau\mathcal{E}_K \,]$ over K with divisors $X_o = N(\mathcal{E}'_K/{}^\tau\mathcal{E}_K)$ and $X_\infty = N(\mathcal{E}'_K/\mathcal{E}_K)$ in $X'^{(m)}(A)$ can be extended to a $\mathcal{D}$-shtuka over A after a finite extension of A.

Let $V = \mathcal{E}_K \otimes K(X)$ be the generic fibre and let $\varphi = j^{-1}t : {}^\tau V \to V$.

Since vector bundles over the punctured spectrum of a two-dimensional regular local ring extend uniquely to the entire ring, the desired extension is given by an A'-lattice $M \subset V$ with $\varphi({}^\tau M) \subset M$. By [Laf98] 2, Lemma 3 (cf. also [Dri89]

Proposition 3.2) there is a maximal lattice M_0 with this property, and after a finite extension of A (which changes M_0) the induced map $\bar{\varphi} : {}^{\tau}M_0 \otimes k \to M_0 \otimes k$ is not nilpotent. Then the diagrams

$$[\, \mathcal{E}_K \xrightarrow{j_K} \mathcal{E}'_K \xleftarrow{t_K} \mathcal{E}''_K = {}^{\tau}\mathcal{E}_K \,]$$
$$[\, M_0 = M_0 = M_0 \xleftarrow{\varphi} {}^{\tau}M_0 \,]$$

induce a similar diagram of locally free $\mathcal{D} \boxtimes A$-modules of rank 1 on $X \times S$

$$[\, \mathcal{E} \xrightarrow{j} \mathcal{E}' \xleftarrow{t} \mathcal{E}'' \xleftarrow{t'} {}^{\tau}\mathcal{E} \,]$$

such that j and t are modifications with divisors X_∞ and X_o. It remains to show that t' or equivalently $\bar{\varphi}$ is an isomorphism.

Let $0 \neq N \subseteq M_0 \otimes k$ be the intersection of the images of all $\bar{\varphi}^r : {}^{\tau^r}M_0 \otimes k \to M_0 \otimes k$ and let $\mathcal{F} \subseteq \mathcal{E} \otimes k$, $\mathcal{F}' \subseteq \mathcal{E}' \otimes k$ be the maximal submodules with generic fibres N. We get a diagram of $\mathcal{D} \boxtimes k$-modules

$$\left[\, \mathcal{F} \longrightarrow \mathcal{F}' \longleftarrow {}^{\tau}\mathcal{F} \,\right]$$

with injective maps plus an injection $\mathcal{F}'/\mathcal{F} \subseteq \operatorname{Coker} j \otimes k$. The support of $\mathcal{F}'/{}^{\tau}\mathcal{F}$ lies over at most md different closed points of X because $\mathcal{F}'/\mathcal{F}$ and $\mathcal{F}'/{}^{\tau}\mathcal{F}$ have the same length. Therefore the hypothesis on D implies that there is a finite set of places $x_1 \ldots x_r$ at which the above diagram induces isomorphisms of the completions $\mathcal{F}_{x_i} \cong {}^{\tau}\mathcal{F}_{x_i}$ such that the least common denominator of the local invariants $\operatorname{inv}_{x_i}(D)$ equals d. Like in the proof of [Laf97] I.4, Proposition 7, this implies that the rank of $\mathcal{F}$ over $\mathcal{O}_{X\otimes k}$ is a multiple of d^2, i.e. $\mathcal{F} = \mathcal{E} \otimes k$ as $\mathcal{F} = 0$ is excluded. □

1.6 Partial Frobenii

The absolute Frobenius of Sht_I^m can be written as a product of two partial Frobenii between this stack and its variant with reversed arrows. Outside the diagonals these two stacks are canonically isomorphic.

Definition 1.6.1. For a given integer $m \geq 0$ let the fibre of the morphism of stacks

$$\alpha \;:\; \operatorname{Sht}_I^{-m} \longrightarrow \operatorname{Coh}_{\mathcal{D}}^m \times \operatorname{Coh}_{\mathcal{D}}^m$$

over K_o and $K_\infty \in \operatorname{Coh}_{\mathcal{D}}^m(S)$ be the groupoid of diagrams of locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank 1

$$\left[\, \mathcal{E} \xleftarrow{t} \mathcal{E}' \xrightarrow{j} {}^{\tau}\mathcal{E} \,\right]$$

with $\mathcal{E}/\mathcal{E}' = K_o$ and ${}^{\tau}\mathcal{E}/\mathcal{E}' = K_\infty$ plus a level-I-structure in the obvious sense.

All properties of Sht_I^m and their proofs carry over to Sht_I^{-m}: the morphism α is algebraic, locally of finite type, and smooth of relative dimension $2md$. Its composition with the norm, $\pi : \mathrm{Sht}_I^{-m}/a^{\mathbb{Z}} \to (X' \setminus I)^{(m)} \times (X' \setminus I)^{(m)}$, is of finite type. In the case $I \neq \emptyset$ it is representable quasiprojective, and in the sufficiently ramified case it is proper.

Proposition 1.6.2. *The symmetric product $X^{(m)}$ classifies divisors of degree m on X. Let $U \subseteq X^{(m)} \times X^{(m)}$ be the open subscheme where the two divisors are disjoint. Over U there is a canonical isomorphism $\mathrm{Sht}_I^m|_U \cong \mathrm{Sht}_I^{-m}|_U$ which is compatible with the morphisms α.*

Proof. Any sequence of two modifications $[\,\mathcal{E} \to \mathcal{E}' \leftarrow \mathcal{E}''\,]$ with disjoint divisors $X_o = N(\mathcal{E}'/\mathcal{E}'')$ and $X_\infty = N(\mathcal{E}'/\mathcal{E})$ can be extended uniquely to a cartesian and cocartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{E}} & \longrightarrow & \mathcal{E}'' \\ \downarrow & \square & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}' \end{array}$$

and vice versa. Any such diagram induces isomorphisms $\mathcal{E}'/\mathcal{E}'' \cong \mathcal{E}/\widetilde{\mathcal{E}}$ and $\mathcal{E}'/\mathcal{E} \cong \mathcal{E}''/\widetilde{\mathcal{E}}$. □

Definition 1.6.3. We define two partial Frobenius morphisms $\mathrm{Fr}_\infty : \mathrm{Sht}_I^m \to \mathrm{Sht}_I^{-m}$ and $\mathrm{Fr}_o : \mathrm{Sht}_I^{-m} \to \mathrm{Sht}_I^m$ by

$$\mathrm{Fr}_\infty \;:\; \big[\,\mathcal{E} \xrightarrow{j} \mathcal{E}' \xleftarrow{t} {}^\tau\mathcal{E}\,\big] \longmapsto \big[\,\mathcal{E}' \xleftarrow{t} {}^\tau\mathcal{E} \xrightarrow{{}^\tau j} {}^\tau\mathcal{E}'\,\big]$$

and

$$\mathrm{Fr}_o \;:\; \big[\,\mathcal{E} \xleftarrow{t} \mathcal{E}' \xrightarrow{j} {}^\tau\mathcal{E}\,\big] \longmapsto \big[\,\mathcal{E}' \xrightarrow{j} {}^\tau\mathcal{E} \xleftarrow{{}^\tau t} {}^\tau\mathcal{E}'\,\big].$$

With respect to α these partial Frobenii commute with the absolute Frobenius of the corresponding component $\mathrm{Coh}_{\mathcal{D}}$, i.e. there is a commutative diagram:

$$\begin{array}{ccccc} \mathrm{Sht}_I^m & \xrightarrow{\mathrm{Fr}_\infty} & \mathrm{Sht}_I^{-m} & \xrightarrow{\mathrm{Fr}_o} & \mathrm{Sht}_I^m \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ \mathrm{Coh}_{\mathcal{D}}^m \times \mathrm{Coh}_{\mathcal{D}}^m & \xrightarrow[\mathrm{id}\times\mathrm{Frob}_q]{} & \mathrm{Coh}_{\mathcal{D}}^m \times \mathrm{Coh}_{\mathcal{D}}^m & \xrightarrow[\mathrm{Frob}_q\times\mathrm{id}]{} & \mathrm{Coh}_{\mathcal{D}}^m \times \mathrm{Coh}_{\mathcal{D}}^m \end{array}$$

The compositions $\mathrm{Fr}_o \circ \mathrm{Fr}_\infty$ and $\mathrm{Fr}_\infty \circ \mathrm{Fr}_o$ are canonically isomorphic to the absolute Frobenius Frob_q of the respective stacks.

2 Chains of Modifications

The following variant of the stacks Sht_I^m does not present any new difficulties. We have tried to write down all relevant commuting diagrams, but compatibilities of the corresponding 2-isomorphisms have largely been ignored.

2.1 Definitions

First we extend Definition 1.2.3 of upper modifications to negative lengths.

Definition 2.1.1. For a given integer m let $\mathrm{Inj}_{\mathcal{D}}^{r,m}(S)$ be the groupoid of injective maps of locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank r

$$\begin{cases} \mathcal{E} \supseteq \mathcal{E}' & \text{if } m \geq 0 \\ \mathcal{E} \subseteq \mathcal{E}' & \text{if } m \leq 0 \end{cases}$$

with quotient $\mathcal{E}/\mathcal{E}' \in \mathrm{Coh}_{\mathcal{D}}^m(S)$ or $\mathcal{E}'/\mathcal{E} \in \mathrm{Coh}_{\mathcal{D}}^{-m}(S)$. These objects are called modifications of length m.

Definition 2.1.2. For a given sequence of integers $\underline{m} = (m_1 \dots m_r)$ with the sum zero let $\mathrm{Sht}_I^{\underline{m}}(S)$ be the groupoid of diagrams

$$\left[\, \mathcal{E} = \mathcal{E}_0 \leftrightharpoons \mathcal{E}_1 \leftrightharpoons \dots \leftrightharpoons \mathcal{E}_r = {}^\tau\mathcal{E} \,\right]$$

on $X \times S$ with $(\mathcal{E}_{i-1} \leftrightharpoons \mathcal{E}_i) \in \mathrm{Inj}_{\mathcal{D}}^{1,m_i}$ plus a level-I-structure in the obvious sense.

The connection to Definition 1.1.2 and Definition 1.6.1 is expressed by $\mathrm{Sht}_I^m = \mathrm{Sht}_I^{(-m,m)}$ for any integer m.

As before there is an action of $\mathrm{Pic}_I(X)$ on $\mathrm{Sht}_I^{\underline{m}}$, and we can form the quotient $\mathrm{Sht}_I^{\underline{m}}/a^{\mathbb{Z}}$. We consider the following commuting diagram in which β is defined by forgetting the level structure, while α is given by the quotients $\mathcal{E}_{i-1}/\mathcal{E}_i$ or $\mathcal{E}_i/\mathcal{E}_{i-1}$ depending on the sign of m_i.

$$\begin{array}{ccccc} \pi \;:\; \mathrm{Sht}^{\underline{m}}/a^{\mathbb{Z}} & \xrightarrow{\alpha} & \prod_{i=1}^r \mathrm{Coh}_{\mathcal{D}}^{|m_i|} & \xrightarrow{N^{\times r}} & \prod_{i=1}^r X'^{(|m_i|)} \\ \uparrow{\scriptstyle \beta} & & & & \uparrow \\ \pi_I \;:\; \mathrm{Sht}_I^{\underline{m}}/a^{\mathbb{Z}} & & \longrightarrow & & \prod_{i=1}^r (X' \setminus I)^{(|m_i|)} \end{array}$$

The following extensions of Propositions 1.3.5, 1.3.8, 1.4.5, 1.4.6, and 1.5.2 result from the obvious variants of their proofs.

Theorem 2.1.3. *The morphism α is algebraic, locally of finite type and smooth of relative dimension $d\sum|m_i|$. Outside I the morphism β is a $\mathcal{D}_I^*$-torsor.*

The stack $\mathrm{Sht}_I^{\underline{m}}/a^{\mathbb{Z}}$ is of finite type over $\mathbb{F}_q$, and in the case $I \neq \emptyset$ it is a quasiprojective scheme. Consequently all $\mathrm{Sht}^{\underline{m}}$ are Deligne-Mumford stacks.

If D is sufficiently ramified with respect to $\sum|m_i|/2$, then the morphism π_I is proper, thus projective in the case $I \neq \emptyset$. □

2.2 Permutations and partial Frobenius

If $\underline{m}'$ is a permutation of the sequence $\underline{m}$, then $\mathrm{Sht}_I^{\underline{m}}$ and $\mathrm{Sht}_I^{\underline{m}'}$ are isomorphic outside the diagonals. More precisely, for any $s \in \mathfrak{S}_r$ we denote the corresponding permutation of the sequence $\underline{m}$ by $\underline{m}\cdot s = (m_{s(1)}\dots m_{s(r)})$, and we denote the right permutation action by

$$i(s)\ :\ \prod_{i=1}^{r}\mathrm{Coh}_{\mathcal{D}}^{|m_i|} \cong \prod_{i=1}^{r}\mathrm{Coh}_{\mathcal{D}}^{|m_{s(i)}|}, \qquad \big(K_1\dots K_r\big)\longmapsto\big(K_{s(1)}\dots K_{s(r)}\big)\,.$$

Proposition 2.2.1. *Let $U \subseteq \prod X^{(|m_i|)}$ and $U' \subseteq \prod X^{(|m_{s(i)}|)}$ be the open substacks where the divisors are pairwise disjoint. Then there is a canonical isomorphism $j(s)$ which fits into the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{Sht}_I^{\underline{m}}|_U & \xrightarrow{j(s)} & \mathrm{Sht}_I^{\underline{m}\cdot s}|_{U'} \\ \downarrow{\scriptstyle \alpha_{\underline{m}}} & & \downarrow{\scriptstyle \alpha_{\underline{m}\cdot s}} \\ \prod_{i=1}^r \mathrm{Coh}_{\mathcal{D}}^{|m_i|} & \xrightarrow{i(s)} & \prod_{i=1}^r \mathrm{Coh}_{\mathcal{D}}^{|m_{s(i)}|} \end{array} \tag{2.2.1}$$

In particular we obtain an action of the stabiliser $\mathrm{Stab}(\underline{m}) \subseteq \mathfrak{S}_r$ on $\mathrm{Sht}_I^{\underline{m}}$ such that $\alpha_{\underline{m}}$ is equivariant.

Proof. Any sequence of two modifications $[\,\mathcal{E}\to\mathcal{E}'\leftarrow\mathcal{E}''\,]$, $[\,\mathcal{E}\leftarrow\widetilde{\mathcal{E}}\to\mathcal{E}''\,]$, or $[\,\widetilde{\mathcal{E}}\to\mathcal{E}\to\mathcal{E}'\,]$ with disjoint divisors can uniquely be extended to a cartesian and cocartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{E}} & \longrightarrow & \mathcal{E}'' \\ \downarrow & \square & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}' \end{array}$$

and any such diagram induces canonical isomorphisms $\mathcal{E}/\widetilde{\mathcal{E}} \cong \mathcal{E}'/\mathcal{E}''$ and $\mathcal{E}'/\mathcal{E} \cong \mathcal{E}''/\widetilde{\mathcal{E}}$. This implies the assertion for those s which exchange two adjacent letters, which generate $\mathfrak{S}_r$. □

Remark 2.2.2. It is sufficient that the divisors of those pairs of modifications are disjoint whose order is actually changed by the permutation s. Assume for example $\underline{m} = \underline{m}_1\underline{m}_2\underline{m}_3$ and $\underline{m} \cdot s = \underline{m}_1\underline{m}_3\underline{m}_2$. Then $\mathrm{Sht}_I^{\underline{m}}$ and $\mathrm{Sht}_I^{\underline{m}\cdot s}$ are isomorphic over the open subset $U \subseteq \prod X^{(|m_i|)}$ where any divisor with index in $\underline{m}_2$ is disjoint from any divisor with index in $\underline{m}_3$.

Definition 2.2.3. Let $\sigma \in \mathfrak{S}_r$ be the cyclic permutation such that $\underline{m} \cdot \sigma = (m_2 \ldots m_r, m_1)$. The partial Frobenius

$$\mathrm{Fr}_o \;:\; \mathrm{Sht}_I^{\underline{m}} \longrightarrow \mathrm{Sht}_I^{\underline{m}\cdot\sigma}$$

is defined by the assignment

$$\mathrm{Fr}_o \;:\; \left[\,\mathcal{E}_0 \leftrightharpoons \mathcal{E}_1 \ldots {}^{\tau}\mathcal{E}_0\,\right] \longmapsto \left[\,\mathcal{E}_1 \ldots {}^{\tau}\mathcal{E}_0 \leftrightharpoons {}^{\tau}\mathcal{E}_1\,\right]$$

with the obvious level structure.

The composition $(\mathrm{Fr}_o)^r$ is canonically isomorphic to the absolute Frobenius Frob_q of $\mathrm{Sht}_I^{\underline{m}}$, and there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Sht}_I^{\underline{m}} & \xrightarrow{\quad \mathrm{Fr}_o \quad} & \mathrm{Sht}_I^{\underline{m}\cdot\sigma} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ \prod_{i=1}^r \mathrm{Coh}_{\mathcal{D}}^{|m_i|} & \xrightarrow{i(\sigma)\circ\left(\mathrm{Frob}_q \times \mathrm{id}^{\times(r-1)}\right)} & \prod_{i=1}^r \mathrm{Coh}_{\mathcal{D}}^{|m_{\sigma(i)}|} \end{array} \tag{2.2.2}$$

Moreover the partial Frobenius is compatible with the permutations of $\underline{m}$ in the following sense: for any integer $0 \le k \le r$ and any permutation $s \in \mathfrak{S}_r$ which fixes the set $\{1 \ldots k\}$, outside appropriate diagonals (see Remark 2.2.2) there is the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Sht}_I^{\underline{m}} & \xrightarrow{(\mathrm{Fr}_o)^k} & \mathrm{Sht}_I^{\underline{m}\cdot\sigma^k} \\ {\scriptstyle j(s)}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle j(\sigma^{-k}s\sigma^k)} \\ \mathrm{Sht}_I^{\underline{m}\cdot s} & \xrightarrow{(\mathrm{Fr}_o)^k} & \mathrm{Sht}_I^{\underline{m}\cdot s\sigma^k} \end{array} \tag{2.2.3}$$

2.3 Collapsing maps

Let m_i and m_{i+1} be two neighbouring elements of the sequence $\underline{m}$ with the same sign, and let $\underline{\widetilde{m}}$ be the collapsed sequence $(m_1 \ldots m_{i-1}, m_i + m_{i+1}, m_{i+2} \ldots m_r)$. Forgetting $\mathcal{E}_i$ defines a morphism $\mathrm{Sht}_I^{\underline{m}} \to \mathrm{Sht}_I^{\underline{\widetilde{m}}}$.

The composition of such morphisms gives more general collapsing maps as far as this is permitted by the signs of the m_i. In order to fix a notation, let $r = r_1 + \ldots + r_\nu$ be a decomposition such that in any single block

$$(m_1 \ldots m_{r_1}),\ (m_{r_1+1} \ldots m_{r_1+r_2}),\ \ldots$$

all signs agree, and let $\underline{\widetilde{m}}$ be the corresponding collapsed sequence of length ν, i.e. $\widetilde{m}_j = m_{r_1+\ldots+r_{j-1}+1} + \ldots + m_{r_1+\ldots+r_j}$.

Definition 2.3.1. With this notation the collapsing map

$$p(\underline{r}) \ :\ \mathrm{Sht}_I^{\underline{m}} \longrightarrow \mathrm{Sht}_I^{\underline{\widetilde{m}}}$$

is given by forgetting all $\mathcal{E}_i$ except $\mathcal{E}_0$, $\mathcal{E}_{r_1}$, $\mathcal{E}_{r_1+r_2}$, etc.

Let $\widetilde{\mathrm{Coh}}_{\mathcal{D}}^{|\widetilde{m}_j|}$ be the stack of $K \in \mathrm{Coh}_{\mathcal{D}}^{|\widetilde{m}_j|}$ plus a filtration

$$K = K_0 \supseteq \ldots \supseteq K_{r_j} = 0 \quad \text{or} \quad 0 = K_0 \subseteq \ldots \subseteq K_{r_j} = K$$

depending on the sign of $\widetilde{m}_j$ such that the length of each quotient K_{i-1}/K_i or K_i/K_{i-1} is $|m_{r_1+\ldots+r_{j-1}+i}|$. Then there is a natural 2-commutative diagram

$$\begin{array}{ccc} \mathrm{Sht}_I^{\underline{m}} & \xrightarrow{p(\underline{r})} & \mathrm{Sht}_I^{\underline{\widetilde{m}}} \\ {\scriptstyle \widetilde{\alpha}_{\underline{m}}}\downarrow & \square & \downarrow{\scriptstyle \alpha_{\underline{\widetilde{m}}}} \\ \prod_j \widetilde{\mathrm{Coh}}_{\mathcal{D}}^{|\widetilde{m}_j|} & \xrightarrow[q(\underline{r})]{} & \prod_j \mathrm{Coh}_{\mathcal{D}}^{|\widetilde{m}_j|} \end{array} \tag{2.3.1}$$

in which $q(\underline{r})$ is given by forgetting the filtrations. The graduation of $\widetilde{\alpha}_{\underline{m}}$ is canonically isomorphic to $\alpha_{\underline{m}}$. From Lemma 1.2.6 follows that this diagram is 2-cartesian, so $p(\underline{r})$ is representable projective (the fibres of $q(\underline{r})$ are given by closed conditions in certain flag varieties).

Moreover $p(\underline{r})$ is compatible with permutations and with the partial Frobenius: assume that for some $s \in \mathfrak{S}_r$ the images of the chosen blocks of $\underline{m}$ remain connected in $\underline{m} \cdot s$, and denote by $\tilde{s} \in \mathfrak{S}_\nu$ their permutation. Then outside appropriate diagonals there is the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Sht}_I^{\underline{m}} & \xrightarrow{p(\underline{r})} & \mathrm{Sht}_I^{\underline{\widetilde{m}}} \\ {\scriptstyle j(s)}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle j(\tilde{s})} \\ \mathrm{Sht}_I^{\underline{m}\cdot s} & \xrightarrow{p(\underline{r}\cdot s)} & \mathrm{Sht}_I^{\underline{\widetilde{m}}\cdot\tilde{s}} \end{array} \tag{2.3.2}$$

Compatibility with the partial Frobenius is expressed by the commutative diagram drawn below.

$$\begin{array}{ccc} \mathrm{Sht}_I^{\underline{m}} & \xrightarrow{p(\underline{r})} & \mathrm{Sht}_I^{\underline{\widetilde{m}}} \\ {\scriptstyle (\mathrm{Fr}_o)^{r_1}}\downarrow & & \downarrow{\scriptstyle \mathrm{Fr}_o} \\ \mathrm{Sht}_I^{\underline{m}\cdot\sigma^{r_1}} & \xrightarrow{p(\underline{r}\cdot\tilde{\sigma})} & \mathrm{Sht}_I^{\underline{\widetilde{m}}\cdot\tilde{\sigma}} \end{array} \tag{2.3.3}$$

2.4 Hecke correspondences

The Hecke correspondences could be defined without leaving the context of finite level structures. Nevertheless we follow the presentations in [LRS] and [Laf97], which use infinite level structures, because this seems to clarify the construction.

Let $T \subset |X|$ be a fixed finite set of places. The projective limit

$$\mathrm{Sht}^{\underline{m},T}\big/a^{\mathbb{Z}} = \varprojlim_{I\cap T=\emptyset} \mathrm{Sht}_I^{\underline{m}}\big/a^{\mathbb{Z}}$$

is a quasicompact scheme over $\prod_i X_{(T)}'^{(|m_i|)}$ by Theorem 2.1.3. Its S-points are the set of isomorphism classes of $\mathcal{D}$-shtukas over S plus compatible isomorphisms

$$\iota_i \;:\; \mathcal{D}_{\mathbb{A}}^T \,\widehat{\boxtimes}\, \mathcal{O}_S \;\cong\; \mathcal{E}_i \,\widehat{\otimes}\, \mathcal{O}_{\mathbb{A}}^T$$

modulo the action of $a^{\mathbb{Z}}$. Here we use the notation $\mathcal{D}_{\mathbb{A}}^T \,\widehat{\boxtimes}\, \mathcal{O}_S = \varprojlim_I (\mathcal{D}_I \boxtimes \mathcal{O}_S)$ and similarly on the right hand side, the projective limit taken over $I \cap T = \emptyset$. There are a natural action of $\mathbb{A}^{T,*}$ and a natural right action of $(\mathcal{D}_{\mathbb{A}}^T)^*$ on this scheme: the former is defined via the homomorphism

$$\mathbb{A}^{T,*} \xrightarrow{b\mapsto \mathcal{O}(-b)} \mathrm{Pic}^T(X) = \varprojlim_{I\cap T=\emptyset} \mathrm{Pic}_I(X)$$

and the limit of the previously defined action of $\mathrm{Pic}_I(X)$ on $\mathrm{Sht}_I^{\underline{m}}/a^{\mathbb{Z}}$, while $g \in (\mathcal{D}_{\mathbb{A}}^T)^*$ acts by twisting the level structure $\iota_i \mapsto \iota_i \circ g$.

Over $\prod X_{(T)}'^{(|m_i|)}$ we have

$$\mathrm{Sht}^{\underline{m},T}\big/a^{\mathbb{Z}} K_I^T = \mathrm{Sht}_I^{\underline{m}}\big/a^{\mathbb{Z}}.$$

Lemma 2.4.1. *The actions of $(\mathcal{D}_{\mathbb{A}}^T)^*$ and of $\mathbb{A}^{T,*}$ on $\mathrm{Sht}^{\underline{m},T}/a^{\mathbb{Z}}$ can be extended to an action of $(D_{\mathbb{A}}^T)^*$ such that the natural morphism*

$$\alpha \;:\; \mathrm{Sht}^{\underline{m},T}/a^{\mathbb{Z}} \longrightarrow \prod \mathrm{Coh}_{\mathcal{D}}^{|m_i|}$$

is invariant, i.e. for any $g \in (D_{\mathbb{A}}^T)^*$ there are compatible 2-isomorphisms $u_g : \alpha \cong \alpha \circ g$.

Proof. It is sufficient to construct an extension of the action of $(\mathcal{D}_{\mathbb{A}}^T)^*$ to the monoid $\Gamma = (D_{\mathbb{A}}^T)^* \cap \mathcal{D}_{\mathbb{A}}^T$ whose restriction to the intersection $\Gamma \cap \mathbb{A}^{T,*}$ coincides with the given action of $\mathbb{A}^{T,*}$. So for $g \in \Gamma$ we have to define

$$\left[\,\mathcal{E}_0 \leftrightharpoons \ldots \leftrightharpoons \mathcal{E}_r = {}^\tau\mathcal{E}_0,\, \underline{\iota}\,\right] \cdot g \;=\; \left[\,\mathcal{E}'_0 \leftrightharpoons \ldots \leftrightharpoons \mathcal{E}'_r = {}^\tau\mathcal{E}'_0,\, \underline{\iota}'\,\right].$$

We define $\mathcal{E}'_i$ via the following cartesian diagram

$$\begin{array}{ccccc} \mathcal{E}_i & \longrightarrow & \mathcal{E}_i \,\widehat{\otimes}\, \mathcal{O}_{\mathbb{A}}^T & \xleftarrow[\iota_i]{\cong} & \mathcal{D}_{\mathbb{A}}^T \,\widehat{\boxtimes}\, \mathcal{O}_S \\ \uparrow & & \square & & \uparrow g \\ \mathcal{E}'_i & & \longrightarrow & & \mathcal{D}_{\mathbb{A}}^T \,\widehat{\boxtimes}\, \mathcal{O}_S \end{array} \tag{2.4.1}$$

and let its level structure ι'_i be the inverse of the isomorphism $\mathcal{E}'_i \,\widehat{\otimes}\, \mathcal{O}_{\mathbb{A}}^T \cong \mathcal{D}_{\mathbb{A}}^T \,\widehat{\boxtimes}\, \mathcal{O}_S$ induced by the lower arrow. Since the map $\mathcal{E}_i \to \operatorname{Coker} g$ is surjective, we have $\mathcal{E}_i/\mathcal{E}'_i = \operatorname{Coker} g$. This being flat over S, it follows that $\mathcal{E}'_i$ is locally free over $\mathcal{O}_{X\times S}$ (see the proof of Lemma 1.2.6).

That $\mathcal{E}'_i$ is locally free over $\mathcal{D} \boxtimes \mathcal{O}_S$ as well needs to be shown only over the geometric points of X by Lemma 1.2.5. Over geometric points in T the modules $\mathcal{E}_i$ and $\mathcal{E}'_i$ are isomorphic, and outside T the module $\mathcal{E}'_i$ admits arbitrary level structures.

From (2.4.1) we get isomorphisms $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{E}'_i/\mathcal{E}'_{i-1}$ or $\mathcal{E}_{i-1}/\mathcal{E}_i \cong \mathcal{E}'_{i-1}/\mathcal{E}'_i$ depending on the sign of m_i, which is the invariance of α. □

Definition 2.4.2. For given $g \in (D_{\mathbb{A}}^T)^*$ and a finite closed subscheme $I \subset X$ satisfying $I \cap T = \emptyset$ the associated Hecke correspondence is

$$(p, q) \;:\; \Gamma_I^{\underline{m}}(g) = \operatorname{Sht}^{\underline{m},T}/a^{\mathbb{Z}}(K_I^T \cap {}^gK_I^T) \xrightarrow{(\mathrm{id},g)} \operatorname{Sht}_I^{\underline{m}}/a^{\mathbb{Z}} \times \operatorname{Sht}_I^{\underline{m}}/a^{\mathbb{Z}}$$

plus the 2-isomorphism $u_g : \alpha p \cong \alpha q$. Restricted to $\prod X_{(T)}^{(|m_i|)}$ this is a finite étale correspondence over $\prod \operatorname{Coh}_{\mathcal{D}}^{|m_i|}$ in the sense of the following intermediate section.

Remark. The morphism $\Gamma_I^{\underline{m}}(g) \to (\operatorname{Sht}_I^{\underline{m}}/a^{\mathbb{Z}})^2$ is representable finite and unramified, but not in general an immersion. Aside from this technical point, Definition 2.4.2 is conform with [Laf97] I.4, while in [LRS] (7.5) the inverse g^{-1} is used instead of g. Since this is equivalent to the exchange of p and q, our action on cohomology (section 8.2) is conform with [LRS].

Finite étale correspondences

A finite étale correspondence of algebraic stacks over $\mathbb{F}_q$ is a diagram

$$X \xleftarrow{\;p\;} Z \xrightarrow{\;q\;} X'$$

with three algebraic stacks X, X', Z and two representable finite and étale morphisms p and q. For another algebraic stack Y along with two fixed morphisms $\alpha : X \to Y$ and $\alpha' : X' \to Y$ a finite étale correspondence over Y is a correspondence $X \leftarrow Z \to X'$ as before plus a 2-isomorphism $u : \alpha \circ p \cong \alpha' \circ q$. Any correspondence is a correspondence over $Y = \operatorname{Spec} \mathbb{F}_q$. We define an associative composition of correspondences over Y by

$$\big[\, X \xleftarrow{p} Z \xrightarrow{q} X' \,\big] \circ \big[\, X' \xleftarrow{p'} Z' \xrightarrow{q'} X'' \,\big] = \big[\, X \xleftarrow{pp_1} Z \times_{q,X',p'} Z' \xrightarrow{q'p_2} X'' \,\big]$$

with the obvious 2-isomorphism $\alpha p p_1 \cong \alpha q' p_2$.

Let $\operatorname{Corr}(X \,|\, Y)$ be the $\mathbb{Q}$-algebra generated by the pairs $[\, X \xleftarrow{p} Z \xrightarrow{q} X,\, u : \alpha p \cong \alpha q \,]$ with this product and with the following additional relations: for any representable finite étale morphism $r : Z' \to Z$ of constant degree n we demand

$$n \,\big[\, X \xleftarrow{p} Z \xrightarrow{q} X \,\big] = \big[\, X \xleftarrow{pr} Z' \xrightarrow{qr} X \,\big]\,.$$

A morphism $f : Y' \to Y$ induces a homomorphism of $\mathbb{Q}$-algebras

$$f^* \;:\; \operatorname{Corr}(X \,|\, Y) \longrightarrow \operatorname{Corr}(X \times_Y Y' \,|\, Y')$$

by base change, and a morphism $g : Y \to Y''$ induces a homomorphism

$$g_* \;:\; \operatorname{Corr}(X \,|\, Y) \longrightarrow \operatorname{Corr}(X \,|\, Y'')\,.$$

by prolonging α. For a representable finite étale morphism $h : X' \to X$ of constant degree n the assignment

$$h^{\#} \,\big[\, X \leftarrow Z \to X \,\big] = \frac{1}{n} \big[\, X' \to X \,\big] \circ \big[\, X \leftarrow Z \to X \,\big] \circ \big[\, X \leftarrow X' \,\big]$$

defines a homomorphism $h^{\#} : \operatorname{Corr}(X \,|\, Y) \to \operatorname{Corr}(X' \,|\, Y)$. The antiinvolution of $\operatorname{Corr}(X \,|\, Y)$ exchanging p and q will be called transposition.

Equivariant correspondences

A correspondence of two morphisms h and h' of algebraic stacks over $\mathbb{F}_q$ is a commutative diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & X_1' \\ \downarrow h & \square & \downarrow h'' & \square & \downarrow h' \\ X_2 & \xleftarrow[p_2]{} & Z_2 & \xrightarrow[q_2]{} & X_2' \end{array} \tag{2.4.2}$$

in which both rows are finite étale correspondences and both squares are 2-cartesian. Again we can define correspondences over a base $Y_1 \to Y_2$ and get for any 2-commutative diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_2 \end{array}$$

a $\mathbb{Q}$-algebra $\operatorname{Corr}(X_1 \to X_2 \,|\, Y_1 \to Y_2)$ equipped with morphisms to $\operatorname{Corr}(X_1 \,|\, Y_1)$ and to $\operatorname{Corr}(X_2 \,|\, Y_2)$. The above maps f^* and g_* extend to homomorphisms into $\operatorname{Corr}(X \times_Y Y' \to X \,|\, Y' \to Y)$ or $\operatorname{Corr}(X = X \,|\, Y \to Y'')$, respectively.

Notation. In the following, all stacks over $\prod X^{(|m_i|)}$ will be replaced by their restrictions to $\prod X'^{(|m_i|)}_{(T)}$, which will not be expressed by a change of notation.

Proposition 2.4.3. *The class of the correspondence* $[\Gamma^{\underline{m}}_I(g), u_g]$ *only depends on the double coset* $K^T_I g K^T_I$*. Thus the assignment*

$$\mathbb{1}_{K^T_I g K^T_I} \longmapsto \mu^T(K^T_I) \left[\Gamma^{\underline{m}}_I(g), u_g \right]$$

is well defined. It gives a homomorphism $h^{\underline{m}}_I$ *of* $\mathbb{Q}$*-algebras with the following compatibility for* $I \subseteq J$*:*

$$\begin{array}{ccc} h^{\underline{m}}_I \;:\; \mathcal{H}^T_I & \longrightarrow & \operatorname{Corr}\big(\operatorname{Sht}^{\underline{m}}_I/a^{\mathbb{Z}} \;\big|\; \prod \operatorname{Coh}^{|m_i|}_{\mathcal{D}}\big) \\ \downarrow & & \downarrow{\scriptstyle \beta^{\#}} \\ h^{\underline{m}}_J \;:\; \mathcal{H}^T_J & \longrightarrow & \operatorname{Corr}\big(\operatorname{Sht}^{\underline{m}}_J/a^{\mathbb{Z}} \;\big|\; \prod \operatorname{Coh}^{|m_i|}_{\mathcal{D}}\big) \end{array}$$

Here the left vertical arrow is the inclusion and $\beta : \operatorname{Sht}^{\underline{m}}_J \to \operatorname{Sht}^{\underline{m}}_I$ *is the reduction of level structure.*

Proof. Denoting $\mathcal{X} = \operatorname{Sht}^{\underline{m},T}/a^{\mathbb{Z}}$, the Hecke correspondences can be written as

$$\Gamma^{\underline{m}}_I(g) = \mathcal{X}\big/(K^T_I \cap {}^g K^T_I) \cong \mathcal{X} \times^{K^T_I} K^T_I g K^T_I \,/\, K^T_I \,,$$

where the isomorphism is given by the map $x \mapsto (x, g)$. Then the first assertion is clear. The following calculation implies that the diagram commutes (set $K = K^T_I$ and $K' = K^T_J$):

$$\mathcal{X}/K' \times_{\mathcal{X}/K} \big(\mathcal{X} \times^K KgK/K\big) \times_{\mathcal{X}/K} \mathcal{X}/K' = \mathcal{X} \times^{K'} KgK \,/\, K'$$
$$= \bigsqcup_{y \in K' \backslash KgK/K'} \mathcal{X} \times^{K'} K'yK' \,/\, K' = \bigsqcup_{y \in K' \backslash KgK/K'} \mathcal{X}\big/(K' \cap {}^y K')$$

We also use that the degree of the map $\mathcal{X}/K' \to \mathcal{X}/K$ equals $\mu^T(K)/\mu^T(K')$. The product of the correspondences $\Gamma_I^{\underline{m}}(g)$ and $\Gamma_I^{\underline{m}}(g')$ can be calculated as

$$\begin{aligned}
&\big(\mathcal{X} \times^K KgK \,/\, K\big) \times_{\mathcal{X}/K} \mathcal{X}/(K \cap {}^{g'}K) \\
&= \mathcal{X} \times^K KgK \,/\, (K \cap {}^{g'}K) = \mathcal{X} \times^K KgK \times^K Kg'K \,/\, K
\end{aligned}$$

and has an obvious finite étale map to

$$\mathcal{X} \times^K KgKg'K \,/\, K = \coprod_{g'' \in K\backslash KgKg'K/K} \mathcal{X}/(K \cap {}^{g''}K).$$

The degree of this map over the g''-component is the number of inverse images of g'' under the map $KgK \times^K Kg'K \to KgKg'K$, and this number equals $\mu^T(K)^{-1} \cdot \mathbb{1}_{KgK} * \mathbb{1}_{Kg'K}(g'')$. □

Lemma 2.4.4. *The Hecke correspondences are compatible with permutations of* $\underline{m}$*, with the partial Frobenius, and with the collapsing maps, i.e. if*

$$\begin{array}{ccc}
X_1 & \longrightarrow & X_2 \\
\big\downarrow & & \big\downarrow \\
Y_1 & \longrightarrow & Y_2
\end{array}$$

is one of the diagrams (2.2.1), (2.2.2)*, or* (2.3.1) *then the maps* $h_I^{\underline{m}}$ *can naturally be extended to homomorphisms into the algebra* $\operatorname{Corr}(X_1 \to X_2 \,|\, Y_1 \to Y_2)$.

Proof. Since the maps $j(s)$, Fr_o, $p(\underline{r})$ are equivariant with respect to the action in Lemma 2.4.1, we get diagrams of the form (2.4.2). In the case of $p(\underline{r})$ we also have to observe that the morphism $\widetilde{\alpha}_{\underline{m}}$ from (2.3.1) is invariant under this action. The relevant squares are 2-cartesian because for sufficiently large $J \supseteq I$ they can be written as a quotient of a map of K_I^T/K_J^T-torsors modulo a suitable subgroup. □

Remark 2.4.5. The homomorphisms $h_I^{\underline{m}}$ are also compatible with the change of T. More precisely, for $T' \subseteq T$ there is a natural inclusion $i_{T,T'} : \mathcal{H}_I^T \subseteq \mathcal{H}_I^{T'}$, which using the obvious decomposition $\mathcal{H}_I^{T'} = \mathcal{H}_I^T \otimes \mathcal{H}_{T\backslash T'}$ is given by $f^T \mapsto f^T \otimes 1$. Then $h_{I,T'}^{\underline{m}} \circ i_{T,T'}$ is the restriction of $h_{I,T}^{\underline{m}}$ to $\prod X'^{(|m_i|)}_{(T')}$.

3 $\mathcal{D}$-Shtukas with Base Points

In this section we consider moduli spaces of $\mathcal{D}$-shtukas with modifications of prescribed type in a fixed set of points. The constructions and results of the preceding sections carry over to this case without difficulties.

3.1 Definitions

The following definitions all depend on the $\mathcal{O}_X$-Algebra $\mathcal{D}$, which will sometimes be suppressed in the notation.

Definition 3.1.1. A modification in the point $z \in X'(S)$ is a triple $(\mathcal{E}, \mathcal{E}', i)$ with two locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules $\mathcal{E}$ and $\mathcal{E}'$ of rank 1 and an isomorphism of these outside the graph of z

$$i \;:\; \mathcal{E}\,\big|_{X \times S - \Gamma_z} \cong \mathcal{E}'\,\big|_{X \times S - \Gamma_z} .$$

We write $i : \mathcal{E} \doteq \mathcal{E}'$. Let $\mathrm{Mod}'_{\mathcal{D}}(S)$ be the groupoid of such modifications and let

$$\mathrm{Mod}'_{\mathcal{D}} \longrightarrow X'$$

be the map given by z.

In the case $S = \mathrm{Spec}\, k$ with a field k, the type of a modification $\mathrm{inv}(\mathcal{E} \doteq \mathcal{E}') \in P^+$ can be defined as follows. For a point $z \in X'(k)$ we denote by $\mathcal{O}_z$, F_z, $\mathcal{D}_z$, D_z the completions in z of the base changes of $\mathcal{O}_X$, F, $\mathcal{D}$, D to k. There is a natural isomorphism

$$\gamma \;:\; P^+ \cong \mathrm{GL}_d(\mathcal{O}_z)\big\backslash \mathrm{GL}_d(F_z)\big/ \mathrm{GL}_d(\mathcal{O}_z) \cong \mathcal{D}_z^*\backslash D_z^*/\mathcal{D}_z^*$$

where the first map is given by $\lambda \mapsto \lambda(\varpi_x)$ and the second map is induced by any isomorphism $\mathcal{D}_z \cong M_d(\mathcal{O}_z)$.

Definition 3.1.2. Let $\mathcal{E} \doteq \mathcal{E}'$ be a modification in $z \in X'(k)$. Any pair of generators of the $\mathcal{D}_z$-modules $e \in \mathcal{E}_z$ and $e' \in \mathcal{E}'_z$ determines a unique $g \in D_z^*$ such that $e' = eg$. We set

$$\mathrm{inv}(\mathcal{E} \doteq \mathcal{E}') \;=\; \gamma^{-1}(\mathcal{D}_z^* g \mathcal{D}_z^*)$$

which is independent of the choice of (e, e').

In order to give a precise definition for arbitrary base S, we fix a trivialisation of the Azumaya algebra $\mathcal{D}\,|_{X' \otimes \mathbb{F}_{q^n}}$ as in section 1.2. Using this, the restriction of a modification in $\mathrm{Mod}_{\mathcal{D}}(S)$ to $X' \times \mathrm{Spec}\, \mathbb{F}_{q^n} \times S$ can be considered as a modification $\widetilde{\mathcal{E}} \doteq \widetilde{\mathcal{E}}'$ of locally free $\mathcal{O}_{X' \times \mathbb{F}_{q^n} \times S}$-modules of rank d.

Definition 3.1.3. For $\lambda \in P^+$ let $\mathrm{Mod}_{\mathcal{D}}^{\leq \lambda} \subseteq \mathrm{Mod}'_{\mathcal{D}}$ be the closed substack where for any $1 \leq i \leq d$ the restriction of the modification to $X' \times \mathrm{Spec}\, \mathbb{F}_{q^n} \times S$ can be extended to a homomorphism

$$\Lambda^i \widetilde{\mathcal{E}} \longleftarrow \Lambda^i \widetilde{\mathcal{E}}'\big((\lambda^{(d)} + \ldots + \lambda^{(d-i+1)}) \cdot z\big)$$

which in addition is an isomorphism for $i = d$. We denote by $\mathrm{Mod}_{\mathcal{D}}^{\lambda} \subseteq \mathrm{Mod}_{\mathcal{D}}^{\leq \lambda}$ the open substack where these homomorphisms are surjective for all i.

That this really defines closed substacks may be checked over $\mathbb{F}_{q^n}$, and there this is clear. The partial order of the $\mathrm{Mod}_{\mathcal{D}}^{\leq\lambda}$ by inclusion coincides with the order in P^+ introduced on page xii, in particular $\mathrm{Mod}_{\mathcal{D}}^{\lambda} \subseteq \mathrm{Mod}_{\mathcal{D}}^{\leq\lambda}$ is the complement of the (finite) union of all $\mathrm{Mod}_{\mathcal{D}}^{\leq\lambda'}$ for $\lambda' < \lambda$.

For a field k, $\mathrm{Mod}_{\mathcal{D}}^{\lambda}(k) \subset \mathrm{Mod}'_{\mathcal{D}}(k)$ is the subset of modifications of type λ.

Remark. We do not claim that Definition 3.1.3 is 'correct' in the sense that $\mathrm{Mod}_{\mathcal{D}}^{\leq\lambda}$ is the schematic closure of $\mathrm{Mod}_{\mathcal{D}}^{\lambda}$. Nevertheless for the cofinal set of λ with $\lambda^{(2)} = \ldots = \lambda^{(d)}$ this holds. We ignore such questions in the sequel because they do not play a role for l-adic cohomology.

Definition 3.1.4. For any integer m we set

$$\mathrm{Mod}_{\mathcal{D}}^{m} = \bigcup_{\lambda \in P_m^+} \mathrm{Mod}_{\mathcal{D}}^{\leq\lambda}$$

and $\mathrm{Mod}_{\mathcal{D}} = \bigcup_m \mathrm{Mod}_{\mathcal{D}}^{m}$.

In other words, $\mathrm{Mod}_{\mathcal{D}}^{m} \subset \mathrm{Mod}'_{\mathcal{D}}$ is the closed substack where the determinant of i (defined using the reduced norm) extends to an isomorphism on $X \times S$

$$\det(i) \ : \ \det(\mathcal{E}) \ \cong \ \det(\mathcal{E}')(m \cdot z)\,.$$

Lemma 3.1.5. *The involution of* $\mathrm{Mod}_{\mathcal{D}}$ *which exchanges* $\mathcal{E}$ *and* $\mathcal{E}'$ *maps* $\mathrm{Mod}_{\mathcal{D}}^{\leq\lambda}$ *isomorphically to* $\mathrm{Mod}_{\mathcal{D}}^{\leq w_0\lambda^\vee}$ *with* $w_0\lambda^\vee = (-\lambda^{(d)} \ldots - \lambda^{(1)})$.

Proof. Over a field this is clear. Over a general base S the dual of the (potential) map in Definition 3.1.3 coincides with $\Lambda^{d-i}(\widetilde{\mathcal{E}}) \otimes ((-\lambda^{(1)} - \ldots - \lambda^{(d-i)}) \cdot z) \rightarrow \Lambda^{d-i}(\widetilde{\mathcal{E}}')$ up to multiplication with an invertible $\mathcal{O}_{X\times S}$-module. □

Definition 3.1.6. Let a sequence $\underline{\lambda} = (\lambda_1 \ldots \lambda_r)$ of dominant coweights for GL_d with $\sum \deg(\lambda_i) = 0$ be given. Let

$$\pi_{\underline{\lambda}} \ : \ \mathrm{Sht}^{\leq\underline{\lambda}} \longrightarrow (X')^r$$

be the morphism of stacks whose fibre over $x_1 \ldots x_r \in X'(S)$ is the groupoid of chains of modifications

$$\mathcal{E}_0 \doteq \mathcal{E}_1 \doteq \ldots \doteq \mathcal{E}_r \cong {}^\tau\mathcal{E}_0$$

on $X \times S$ such that any $(\mathcal{E}_{i-1} \doteq \mathcal{E}_i) \in \mathrm{Mod}_{\mathcal{D}}^{\leq\lambda_i}(S)$ is a modification in the point x_i. Here we use the notation ${}^\tau\mathcal{E}_0 = (\mathrm{id} \times \mathrm{Frob}_q)^*\mathcal{E}_0$. These objects will be called $\mathcal{D}$-shtukas (with base points) of type $\leq \underline{\lambda}$. We denote by

$$j_{\underline{\lambda}} \ : \ \mathrm{Sht}^{\underline{\lambda}} \subseteq \mathrm{Sht}^{\leq\underline{\lambda}}$$

the open substack where the type of the modifications equals $\underline{\lambda}$, which means $(\mathcal{E}_{i-1} \doteq \mathcal{E}_i) \in \mathrm{Mod}_{\mathcal{D}}^{\lambda_i}(S)$.

Definition 3.1.7. Let $I \subset X$ be a finite closed subscheme. A level-I-structure for a $\mathcal{D}$-shtuka as above with $x_i \in X' \setminus I$ for all i is a sequence of isomorphisms $\iota_i : \mathcal{D}_I \boxtimes \mathcal{O}_S \cong (\mathcal{E}_i)_I$ which together with the given modifications form the following commutative diagram.

$$\begin{array}{ccccccccc} \mathcal{E}_{0,I} & \overset{\sim}{\text{———}} & \mathcal{E}_{1,I} & \overset{\sim}{\text{——}} & \cdots & \overset{\sim}{\text{——}} & \mathcal{E}_{r,I} & \overset{\sim}{\text{———}} & {}^{\tau}\mathcal{E}_{0,I} \\ {\scriptstyle\iota_0}\uparrow{\scriptstyle\cong} & & {\scriptstyle\iota_1}\uparrow{\scriptstyle\cong} & & & & {\scriptstyle\cong}\uparrow{\scriptstyle\iota_r} & & {\scriptstyle\cong}\uparrow{\scriptstyle{}^{\tau}\iota_0} \\ \mathcal{D}_I \boxtimes \mathcal{O}_S & = & \mathcal{D}_I \boxtimes \mathcal{O}_S & = & \cdots & = & \mathcal{D}_I \boxtimes \mathcal{O}_S & = & \mathcal{D}_I \boxtimes \mathcal{O}_S \end{array}$$

We denote by

$$\pi_{I,\underline{\lambda}} \ : \ \mathrm{Sht}_I^{\leq\underline{\lambda}} \longrightarrow (X' \setminus I)^r$$

the stack of $\mathcal{D}$-shtukas of type $\leq \underline{\lambda}$ with a level-I-structure and by $\mathrm{Sht}_I^{\underline{\lambda}} \subseteq \mathrm{Sht}_I^{\leq\underline{\lambda}}$ the open substack where the type equals $\underline{\lambda}$.

As in section 1.1 the group $\mathrm{Pic}_I(X)$ acts on $\mathrm{Sht}_I^{\leq\underline{\lambda}}$, and the quotient $\mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}}$ can again be identified with a finite union of components of $\mathrm{Sht}_I^{\leq\underline{\lambda}}$. The following theorem summarises the main geometric properties of these stacks.

Theorem 3.1.8. *The morphism*

$$\pi_{I,\underline{\lambda}} \ : \ \mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}} \longrightarrow (X' \setminus I)^r$$

is always of finite type and is representable quasiprojective if $I \neq \emptyset$. If D is sufficiently ramified with respect to $\underline{\lambda}$ in the sense of Definition 3.3.5 below, then the morphism $\pi_{I,\underline{\lambda}}$ is proper. Its restriction to $\mathrm{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}}$ is always smooth of relative dimension $2\sum_i(\rho, \lambda_i) \in 2\mathbb{Z}$ where ρ denotes half the sum of the positive roots of GL_d. The morphism

$$\mathrm{Sht}_I^{\leq\underline{\lambda}} \longrightarrow \mathrm{Sht}^{\leq\underline{\lambda}}|_{(X'\setminus I)^r}$$

which is defined by forgetting the level-structure is a $\mathcal{D}_I^*$-torsor.

In section 3.3 these assertions will be deduced from the corresponding properties of the stacks $\mathrm{Sht}_I^{\underline{m}}/a^{\mathbb{Z}}$ by considering the $\mathrm{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}}$ as inverse images of strata in a stack of coherent sheaves. (The last assertion of the theorem is an immediate consequence of the isomorphism $\mathrm{Sht}^0_{\mathcal{D}_I} \cong \mathrm{Spec}\,\mathbb{F}_q/\mathcal{D}_I^*$ on page 8.)

Resolution of singularities

Let $\lambda_0 = (1, \ldots, 1) \in P_d^+$. Any dominant coweight $\lambda \in P^+$ can up to reordering uniquely be written as

$$\lambda = n\lambda_0 + \sum_{j=1}^{s} \mu_j$$

with minuscule μ_j, i.e. $\mu_j = (1, \ldots, 0)$ with $1 \leq \deg(\mu_j) \leq d-1$. For a given sequence $\underline{\lambda}$ with total degree zero we choose such representations $\lambda_i = n_i\lambda_0 + \sum_{j=1}^{s_i} \mu_{i,j}$ for all i and form the sequence

$$\underline{\widetilde{\lambda}} = \underline{\widetilde{\lambda}}_{(1)} \ldots \underline{\widetilde{\lambda}}_{(r)} \qquad \text{with} \qquad \underline{\widetilde{\lambda}}_{(i)} = (n_i\lambda_0, \mu_{i,1}, \ldots, \mu_{i,s_i}).$$

Let $\Delta_s : X \to X^s$ be the diagonal and let $\Delta = \Delta_{s_1+1} \times \ldots \times \Delta_{s_r+1}$.

Proposition 3.1.9. *Forgetting all $\mathcal{E}_i$ but $\mathcal{E}_0, \mathcal{E}_{s_1+1}, \mathcal{E}_{s_1+s_2+2}, \ldots$ defines a representable projective morphism*

$$q \;:\; \Delta^* \mathrm{Sht}_I^{\underline{\widetilde{\lambda}}}/a^{\mathbb{Z}} \longrightarrow \mathrm{Sht}_I^{\leq \underline{\lambda}}/a^{\mathbb{Z}}$$

which over the open substack $\mathrm{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}}$ is an isomorphism. With respect to the stratification by the $\mathrm{Sht}_I^{\underline{\lambda}'}/a^{\mathbb{Z}}$ for $\underline{\lambda}' \leq \underline{\lambda}$ the morphism q is semismall, more precisely the dimension of any irreducible component of a fibre equals half the codimension of the corresponding stratum in the base.

The proof can be found in section 3.3.

3.2 Stratification of $\mathrm{Coh}_{\mathcal{D}}$

The stack Coh_X^m classifies coherent $\mathcal{O}_X$-modules of length m, this is the case $\mathcal{D} = \mathcal{O}_X$ of Definition 1.1.1. Laumon [Lau87] defines the following stratification indexed by the set of partitions $\underline{m} = (m_1 \geq \ldots \geq m_r)$ of m. Let

$$X^{(\underline{m})} = X^{(m_1-m_2)} \times \ldots \times X^{(m_{r-1}-m_r)} \times X^{(m_r)}$$

and let $i_{\underline{m}} \;:\; X^{(\underline{m})} \;\to\; \mathrm{Coh}_X^m$ be the morphism which maps a sequence $\underline{D} \in X^{(\underline{m})}(S)$ to

$$\mathcal{O}_{\underline{D}} = \mathcal{O}_{D_1+\ldots+D_r} \oplus \mathcal{O}_{D_2+\ldots+D_r} \oplus \ldots \oplus \mathcal{O}_{D_r}\,.$$

The image of $i_{\underline{m}}$ is the $\underline{m}$-stratum $\mathrm{Coh}_X^{\underline{m}}$, whose sections over S will be called coherent $\mathcal{O}_{X\times S}$-modules of type $\underline{m}$. The situation can be explained more detailed as follows.

Proposition 3.2.1. *The assignment $\underline{D} \mapsto \mathcal{O}_{\underline{D}}$ defines a morphism*

$$\big[\, X^{(\underline{m})}/\mathcal{A}ut(\mathcal{O}_{\underline{D}})\,\big] \longrightarrow \mathrm{Coh}_X^m$$

which identifies the quotient on left hand side with a locally closed substack $\mathrm{Coh}_X^{\underline{m}} \subseteq \mathrm{Coh}_X^m$. In particular the restriction of the norm factors as

$$\mathrm{Coh}_X^{\underline{m}} \xrightarrow{N^{\underline{m}}} X^{(\underline{m})} \longrightarrow X^{(m)}$$

such that the first map $N^{\underline{m}}$ is smooth of relative dimension $\sum(1-2i)m_i$ and the second map is given by $\underline{D} \mapsto \sum iD_i$ (this is a finite morphism).

Proof. Smoothness of the map $[\,X^{(\underline{m})}/\mathcal{A}ut(\mathcal{O}_{\underline{D}})\,] \to X^{(\underline{m})}$ and its relative dimension follow from the fact that for $\underline{D} \in X^{(\underline{m})}(S)$ the sheaf of automorphisms $\mathcal{A}ut(\mathcal{O}_{\underline{D}}) \to S$ is the open subsheaf of invertible elements in $\mathcal{E}nd(\mathcal{O}_{\underline{D}}) \to S$. The latter is the direct sum of all

$$\mathcal{H}om(\mathcal{O}_{D_i+\ldots+D_r}, \mathcal{O}_{D_j+\ldots+D_r}) \longrightarrow S$$

which can be identified with the vector bundle $(p_2)_*\mathcal{O}_{D_j+\ldots+D_r}$ in case $i \leq j$ or $(p_2)_*\mathcal{O}_{D_i+\ldots+D_r}$ in case $j \leq i$.

For the first assertion of the proposition we have to show that there is a locally closed substack $\mathrm{Coh}_X^{\underline{m}} \subseteq \mathrm{Coh}_X^m$ over which $i_{\underline{m}}$ factors, such that any $K \in \mathrm{Coh}_X^{\underline{m}}(S)$ locally in S (it turns out for the Zariski topology) is of the form $\mathcal{O}_{\underline{D}}$.

Let $\mathcal{F}_i$ be the i-th Fitting ideal of $K \in \mathrm{Coh}_X^m(S)$. We define $\mathrm{Coh}_X^{\underline{m}}$ by the locally closed conditions that each $(p_2)_*(\mathcal{O}_{X\times S}/\mathcal{F}_i)$ is a locally free $\mathcal{O}_S$-module of rank $m_{i+1} + \ldots + m_r$. Certainly $i_{\underline{m}}$ factors over this substack, so assume some K satisfies these conditions.

We claim that the invertible ideal $(\mathcal{F}_0 : \mathcal{F}_1) \subseteq \mathcal{O}_{X\times S}$ annihilates K. Since $\mathcal{F}_{r-1}$ is an invertible $\mathcal{O}_{X\times S}$-module by assumption, locally in S there is an exact sequence $0 \to K' \to K \to K'' \to 0$ with a locally free $\mathcal{O}_{X\times S}/\mathcal{F}_{r-1}$-module K'' of rank r. The fitting ideals of K' are $\mathcal{F}'_i = (\mathcal{F}_i : \mathcal{F}_{r-1}^{r-i})$, so the claim follows inductively.

Let $E \subset X \times S$ be the divisor defined by $(\mathcal{F}_0 : \mathcal{F}_1)$. For any $s \in S$ there is an embedding $\mathcal{O}_E \otimes k(s) \subseteq K \otimes k(s)$ as a direct factor, which in some neighbourhood can be extended to an exact sequence $0 \to \mathcal{O}_E \to K \to K_1 \to 0$ with S-flat K_1. Using $\mathcal{E}xt^1_{\mathcal{O}_E}(K_1, \mathcal{O}_E) = 0$ this splits in a possibly smaller neighbourhood. This implies $\mathcal{F}_i(K_1) = \mathcal{F}_{i+1}$, and we can proceed inductively. □

The stratification can be carried over to $\mathrm{Coh}_{\mathcal{D}}^m$ for arbitrary $\mathcal{D}$:

Corollary 3.2.2. *For any $\underline{m}$ there is a unique locally closed substack $\mathrm{Coh}_{\mathcal{D}}^{\underline{m}} \subseteq \mathrm{Coh}_{\mathcal{D}}^m$ which over a finite extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$ coincides with $\mathrm{Coh}_{X'}^{\underline{m}}$ via any of the isomorphisms $j : \mathrm{Coh}_{\mathcal{D}}^m \otimes \mathbb{F}_{q^n} \cong \mathrm{Coh}_{X'}^m \otimes \mathbb{F}_{q^n}$ from* (1.2.1). *The restriction of the norm factors as*

$$\mathrm{Coh}_{\mathcal{D}}^{\underline{m}} \xrightarrow{N^{\underline{m}}} X'^{\underline{m}} \longrightarrow X'^{(m)}\,,$$

such that $N^{\underline{m}}$ is smooth of relative dimension $\sum(1-2i)m_i$ and the second map is given by $\underline{D} \mapsto \sum iD_i$.

Proof. The locally closed substack $j^{-1}(\mathrm{Coh}_{X'}^{\underline{m}} \otimes \mathbb{F}_{q^n})$ and the morphism $N^{\underline{m}} \circ j$ from there to $X'^{(m)}$ are defined over $\mathbb{F}_q$ because j is locally unique. □

Definition 3.2.3. The following fibred product is the stack of coherent sheaves of length m which are concentrated in one (variable) point.

$$\begin{array}{ccc} \mathrm{coh}^m_{\mathcal{D}} & \longrightarrow & X' \\ \downarrow & \Box & \downarrow{\scriptstyle m} \\ \mathrm{Coh}^m_{\mathcal{D}} & \xrightarrow{N} & X'^{(m)} \end{array}$$

Let $\underline{m} : X \to X^{(\underline{m})}$ be the map $x \mapsto ((m_1 - m_2) \cdot x, \ldots, m_s \cdot x)$. We define the stack of coherent sheaves of type $\underline{m}$ which are concentrated in one point via the 2-cartesian diagram below.

$$\begin{array}{ccc} \mathrm{coh}^{\underline{m}}_{\mathcal{D}} & \longrightarrow & X' \\ \downarrow & \Box & \downarrow{\scriptstyle \underline{m}} \\ \mathrm{Coh}^{\underline{m}}_{\mathcal{D}} & \xrightarrow{N^{\underline{m}}} & X'^{(\underline{m})} \end{array}$$

In the following sense this is the inverse image of the given stratification by the morphism $\mathrm{coh}^m_{\mathcal{D}} \to \mathrm{Coh}^m_{\mathcal{D}}$.

Proposition 3.2.4. *The natural morphism* $\mathrm{coh}^{\underline{m}}_{\mathcal{D}} \to \mathrm{Coh}^{\underline{m}}_{\mathcal{D}} \times_{X'^{(m)}} X'$ *identifies* $\mathrm{coh}^{\underline{m}}_{\mathcal{D}}$ *with the maximal reduced substack of the fibred product. The morphism* $\mathrm{coh}^{\underline{m}}_{\mathcal{D}} \to X'$ *is smooth of relative dimension* $\sum(1-2i)m_i$.

Proof. Using Corollary 3.2.2 the stack $\mathrm{coh}^{\underline{m}}_{\mathcal{D}}$ is smooth over X' and in particular reduced. Since the first map in the proposition is a base change of $(\underline{m}, \mathrm{id}) : X \to X^{(\underline{m})} \times_{X^{(m)}} X$, it is a closed immersion with a nilpotent ideal, which proves the first assertion. $\Box$

From the proof of Proposition 3.2.1 we get the following description of the strata in $\mathrm{coh}^{\underline{m}}_X$.

Lemma 3.2.5. *Assume that* $\mathcal{D} = \mathcal{O}_X$. *Let* $\mathcal{F}_i \subseteq \mathcal{O}_{X\times S}$ *be the* i*-th Fitting ideal of* $K \in \mathrm{coh}^{\underline{m}}_X(S)$ *and let* $x \in X(S)$ *be the base point of* K. *Then the locally closed substack* $\mathrm{coh}^{\underline{m}}_X \subseteq \mathrm{Coh}^m_{X,0}$ *is defined by the conditions*

$$\mathcal{F}_i = \mathcal{O}_{X\times S}\big((-m_{i+1} - \ldots - m_r) \cdot x\big)$$

for $0 \leq i \leq r$. $\Box$

For general $\mathcal{D}$ this gives a description of the strata in $\mathrm{coh}^{\underline{m}}_{\mathcal{D}}$ over some $\mathbb{F}_{q^n}$.

Definition 3.2.6. In the case $\mathcal{D} = \mathcal{O}_X$ let $\mathrm{coh}_X^{\leq \underline{m}} \subseteq \mathrm{coh}_X^{\underline{m}}$ be the closed substack where

$$\mathcal{F}_i \subseteq \mathcal{O}_{X\times S}\big((-m_{i+1} - \ldots - m_r) \cdot x\big)$$

for $0 \leq i \leq r$ (equality for $i = 0$). In the general case let $\mathrm{coh}_{\mathcal{D}}^{\leq \underline{m}} \subseteq \mathrm{coh}_{\mathcal{D}}^{\underline{m}}$ be the closed substack which over $\mathbb{F}_{q^n}$ coincides with $\mathrm{coh}_{X'}^{\leq \underline{m}}$ via one of the isomorphisms (1.2.1).

Remark. One might find it more natural to define $\mathrm{coh}_{\mathcal{D}}^{\leq \underline{m}}$ as the schematic closure of $\mathrm{coh}_{\mathcal{D}}^{\underline{m}}$ in $\mathrm{coh}_{\mathcal{D}}^{m}$. This makes no difference for the geometric points.

Resolution of singularities of the closures of the strata

The sum of two partitions $\underline{m} = (m_1 \geq \ldots \geq m_r)$ and $\underline{m}' = (m_1 \geq \ldots \geq m'_{r'})$ of the integers m and m' will be understood componentwise, i.e.

$$\underline{m} + \underline{m}' = (m_1 + m'_1 \geq m_2 + m'_2 \geq \ldots)$$

Here we have to fill up the shorter partition with zeros in the end. Using this notation, any $\underline{m}$ can up to reordering of the summands uniquely be written as

$$\underline{m} = \sum_{i=1}^{s} \underline{m}_i \tag{3.2.1}$$

such that all $\underline{m}_i$ have the form $(1 \ldots 1)$ of various lengths.

Definition 3.2.7. We fix a decomposition (3.2.1). Let $\widetilde{\mathrm{coh}}_{\mathcal{D}}^{\underline{m}}(S)$ be the groupoid of coherent $\mathcal{D} \boxtimes \mathcal{O}_S$-modules K plus a filtration

$$0 = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_s = K$$

with quotients $K_i/K_{i-1} \in \mathrm{coh}_{\mathcal{D}}^{\underline{m}_i}(S)$ all of which have the same base point in $X'(S)$.

Proposition 3.2.8. *We denote by $\Delta : X \to X^s$ the diagonal. The morphism*

$$\widetilde{\mathrm{coh}}_{\mathcal{D}}^{\underline{m}} \longrightarrow \Delta^* \prod_{i=1}^{s} \mathrm{coh}_{\mathcal{D}}^{\underline{m}_i}$$

given by the quotients K_i/K_{i-1} is of finite type and smooth of relative dimension zero. Consequently $\widetilde{\mathrm{coh}}_{\mathcal{D}}^{\underline{m}}$ is smooth over S with relative dimension $\sum (1 - 2i) m_i$.

Proof. By Proposition 3.2.4 the second assertion follows from the first, which can be proved inductively using Lemma 3.2.9 below. □

Lemma 3.2.9. *For given K_1 and $K_2 \in \mathrm{Coh}_{\mathcal{D}}(S)$ let $\mathcal{E}xt(K_1, K_2)$ be the stack over S which to $f : S' \to S$ assigns the groupoid of extensions $0 \to f^*K_2 \to K \to f^*K_1 \to 0$. Then $\mathcal{E}xt(K_1, K_2) \to S$ is of finite type and smooth of relative dimension zero.*

Proof. Locally in S we can chose a presentation $0 \to \mathcal{E}' \to \mathcal{E} \to K_1 \to 0$ with two locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules $\mathcal{E}$ and $\mathcal{E}'$, cf. the proof of Proposition 1.2.4. Then we have

$$\mathcal{E}xt(K_1, K_2) = \big[\, (p_2)_* \mathcal{H}om_{\mathcal{D}\boxtimes\mathcal{O}_S}(\mathcal{E}', K_2) \,\big/\, (p_2)_* \mathcal{H}om_{\mathcal{D}\boxtimes\mathcal{O}_S}(\mathcal{E}, K_2) \,\big]$$

with $p_2 : X \times S \to S$. □

Proposition 3.2.10. *Forgetting the filtration defines a representable projective morphism*

$$q \ : \ \widetilde{\mathrm{coh}}^{\underline{m}}_{\mathcal{D}} \longrightarrow \mathrm{coh}^{\leq \underline{m}}_{\mathcal{D}}$$

which is an isomorphism over the dense open substack $\mathrm{coh}^{\underline{m}}_{\mathcal{D}}$. For any $\underline{m}' \leq \underline{m}$ the fibres of q over $\mathrm{coh}^{\underline{m}'}_{\mathcal{D}}$ have pure dimension $\sum i(m'_i - m_i)$, i.e. the morphism q is semismall with respect to the chosen stratification.

Proof. The morphism q exists because the natural morphism $\widetilde{\mathrm{coh}}^{\underline{m}}_{\mathcal{D}} \to \mathrm{coh}^{m}_{\mathcal{D}}$ factors over $\mathrm{coh}^{\leq \underline{m}}_{\mathcal{D}}$ pointwise and $\widetilde{\mathrm{coh}}^{\underline{m}}_{\mathcal{D}}$ is reduced. Since the filtrations of a given $K \in \mathrm{coh}^{\leq \underline{m}}_{\mathcal{D}}(S)$ can be considered as sections of a flag variety with certain closed conditions, q is representable projective.

It is an elementary fact that q induces a bijection on geometric points over $\mathrm{coh}^{\underline{m}}_{\mathcal{D}}$. That there q is an isomorphism can be seen as follows. Locally in S any $K \in \mathrm{coh}^{\underline{m}}_{\mathcal{D}}(S)$ has standard form $\mathcal{O}_{\underline{D}}$ and thus admits a filtration of the given type, so we only have to prove its uniqueness. Inductively this can be reduced to the case $s = 2$, however allowing more general $\underline{m}_1$ and $\underline{m}_2$. Assume the lengths satisfy $\ell(\underline{m}_1) \geq \ell(\underline{m}_2)$. Then the maximal submodules in K_1 and in K on which the ideal $\mathcal{I}$ of the given embedding $S \subseteq X \times S$ acts trivially coincide, and we can proceed inductively with the quotients. In the case $\ell(\underline{m}_1) \leq \ell(\underline{m}_2)$ we argue similarly using the maximal quotients of K and K_2 on which $\mathcal{I}$ acts trivially.

To calculate the dimension of the fibres we can work over $\overline{\mathbb{F}}_q$ and may therefore assume that $\mathcal{D}$ is trivial. For a given $K \in \mathrm{coh}^{\underline{m}'}_X(\overline{\mathbb{F}}_q)$ let $W \subseteq K$ be the kernel of multiplication by the uniformising element ϖ. Then $n = \dim(W)$ is the length of $\underline{m}'$. Let $k = \deg(\underline{m}_1)$. The first step in the filtration is parametrised by the Grassmannian $\mathrm{Gr}^{n,k}$ of k-dimensional subspaces $V \subseteq W$, which carries a stratification according to the type of $K' = K/V$. Inductively it suffices to show that when passing from K to K', the number $\sum i(m'_i - m_i)$ decreases precisely by the dimension of the stratum.

For this we need a closer description of the strata. The images of K under multiplication by ϖ induce a filtration

$$W = W_n \supseteq \ldots \supseteq W_0 = 0$$

with dimensions $d = d_N \geq \ldots \geq d_0 = 0$. The strata are described by the numbers $k_i = \dim(V \cap W_i)$, which satisfy the conditions $k_i \leq d_i$ and $k_i - k_{i-1} \leq d_i - d_{i-1}$. The dimension of the stratum defined by $\underline{k}$ equals $\sum (d_i - k_{i-1})(k_i - k_{i-1})$. A direct calculation gives the same value for the decrease of $\sum i(m'_i - m_i)$.

By Proposition 3.2.4 the such calculated dimension of the fibres equals half the codimension of the strata in $\mathrm{coh}_{\mathcal{D}}^{\leq m}$. □

3.3 Proof of the geometric properties

Definition 3.3.1. A modification $\mathcal{E} \doteq \mathcal{E}'$ in the sense of Definition 3.1.1 is called definite if it can be extended to a map $\mathcal{E} \leftarrow \mathcal{E}'$ (positive) or $\mathcal{E} \rightarrow \mathcal{E}'$ (negative).

For any integer m let $\mathrm{Mod}_{\mathcal{D}}^{\mathrm{def},m} \subset \mathrm{Mod}_{\mathcal{D}}^{m}$ be the closed substack of definite modifications. This is connected with the stacks $\mathrm{Inj}_{\mathcal{D}}^{1,m}$ (Definition 2.1.1) by the following 2-cartesian diagram.

$$\begin{array}{ccc} \mathrm{Mod}_{\mathcal{D}}^{\mathrm{def},m} & \longrightarrow & X' \\ \downarrow & \square & \downarrow{\scriptstyle |m|} \\ \mathrm{Inj}_{\mathcal{D}}^{1,m} & \xrightarrow{N} & X'^{(|m|)} \end{array}$$

Thus the quotient $\mathcal{E}/\mathcal{E}'$ or $\mathcal{E}'/\mathcal{E}$ depending on the sign of m defines a morphism

$$q \ : \ \mathrm{Mod}_{\mathcal{D}}^{\mathrm{def},m} \longrightarrow \mathrm{coh}_{\mathcal{D}}^{|m|}$$

over X'. Lemma 1.3.2 implies

Lemma 3.3.2. *The two morphisms*

$$\mathrm{Mod}_{\mathcal{D}}^{\mathrm{def},m} \longrightarrow \mathrm{Vect}_{\mathcal{D}}^{1} \times \mathrm{coh}_{\mathcal{D}}^{|m|}$$

given by $(\mathcal{E}, q)$ and by $(\mathcal{E}', q)$ are representable quasiaffine of finite type and smooth of relative dimension dm. □

A modification over a field is definite if and only if its type lies in P^{++}. This holds over arbitrary base as well: any P_m^{++} has the unique maximal element

$$\lambda_m^{\max} = \begin{cases} (m, 0 \ldots 0) & \text{if } m \geq 0 \\ (0 \ldots 0, m) & \text{if } m \leq 0 \end{cases}$$

and we have $\mathrm{Mod}_{\mathcal{D}}^{\mathrm{def},m} = \mathrm{Mod}_{\mathcal{D}}^{\le\lambda_m^{\max}}$. For $m \ge 0$ this follows directly from Definition 3.1.3, and in the case $m \le 0$ we may apply Lemma 3.1.5. For any integer n the assignment $\mathcal{E} \mapsto \mathcal{E}(n \cdot x)$ defines an isomorphism

$$\mathrm{Mod}_{\mathcal{D}}^{\le\lambda} \cong \mathrm{Mod}_{\mathcal{D}}^{\le\lambda+n\lambda_0}$$

with $\lambda_0 = (1, \dots, 1) \in P_d^+$, so all stacks $\mathrm{Mod}_{\mathcal{D}}^{\le\lambda}$ can be embedded into some $\mathrm{Mod}_{\mathcal{D}}^{\mathrm{def},m}$. In particular they are of finite type over $\mathbb{F}_q$.

Lemma 3.3.3. *For any $\lambda \in P_m^{++}$ the locally closed substack $\mathrm{Mod}_{\mathcal{D}}^{\le\lambda} \subseteq \mathrm{Mod}_{\mathcal{D}}^{\mathrm{def},m}$ is the inverse image under q of $\mathrm{coh}_{\mathcal{D}}^{\le\lambda^+} \subseteq \mathrm{coh}_{\mathcal{D}}^{|m|}$ with*

$$\lambda^+ = \begin{cases} \lambda & \text{if } m \ge 0 \\ w_0\lambda^\vee & \text{if } m \le 0 \end{cases} \tag{3.3.1}$$

Proof. In the case $m \ge 0$ the assertion follows from a comparison of Definition 3.1.3 and Definition 3.2.6, and in the case $m \le 0$ we can apply Lemma 3.1.5. □

Let a sequence $\underline{\lambda}$ of total degree zero be given. In order to reduce the proof of the geometric properties of the stacks $\mathrm{Sht}_I^{\le\underline{\lambda}}$ to the definite case we chose integers $n_1 \dots n_r$ and set

$$\underline{\lambda}' = (-n_1\lambda_0\,, \lambda_1 + n_1\lambda_0\,, \dots, -n_r\lambda_0\,, \lambda_r + n_r\lambda_0) \tag{3.3.2}$$

with $\lambda_0 = (1, \dots, 1)$ as before. Let $\Delta : X^r \to X^{2r}$ be the map $(x_1 \dots x_r) \mapsto (x_1, x_1 \dots x_r, x_r)$. Then clearly

Lemma 3.3.4. *With the above notations, forgetting the odd $\mathcal{E}_i$ defines isomorphisms $\Delta^*\big(\mathrm{Sht}_I^{\le\underline{\lambda}'}\big) = \mathrm{Sht}_I^{\le\underline{\lambda}}$ and $\Delta^*\big(\mathrm{Sht}_I^{\underline{\lambda}'}\big) = \mathrm{Sht}_I^{\underline{\lambda}}$.* □

Definition 3.3.5. The division algebra D is called sufficiently ramified with respect to $\underline{\lambda}$ if one of the following conditions holds:

1. All λ_i are definite and D is sufficiently ramified with respect to $\sum |\deg(\lambda_i)|/2$ in the sense of Definition 1.5.1.

2. There is a $\underline{\lambda}'$ like in (3.3.2) satisfying condition 1.

Proof of Theorem 3.1.8. By Lemma 3.3.4 we may assume that all λ_i are definite. Let $m_i = \deg \lambda_i$ and define λ_i^+ by (3.3.1). In view of Lemma 3.3.3 there is the following commutative diagram with 2-cartesian left half.

$$\begin{array}{ccccccc} \pi_{I,\underline{\lambda}} : \mathrm{Sht}_I^{\le\underline{\lambda}}/a^{\mathbb{Z}} & \xrightarrow{\alpha'} & \prod \mathrm{coh}_{\mathcal{D}}^{\le\lambda_i^+} & \xrightarrow{N} & (X')^r \\ \downarrow{\scriptstyle i'} & \square & \downarrow{\scriptstyle i} & & \downarrow \\ \pi_{I,m} : \mathrm{Sht}_I^{m}/a^{\mathbb{Z}} & \xrightarrow[\alpha]{} & \prod \mathrm{Coh}_{\mathcal{D}}^{|m_i|} & \longrightarrow & \prod X'^{(|m_i|)} \end{array} \tag{3.3.3}$$

This allows to deduce all assertions from Theorem 2.1.3:

Since i is representable finite, the same holds for its base change i'. Thus the morphism $\pi_{I,\underline{\lambda}}$ is of finite type, is representable quasiprojective if $I \neq \emptyset$ and is proper in the sufficiently ramified case, because the same holds for $\pi_{I,\underline{m}}$.

Moreover the morphism α' is smooth of relative dimension $d\sum|m_i|$ because this is true for α. Thus the smoothness assertions follow from Proposition 3.2.4 and the elementary equation

$$d\,|m_i| + \sum_j (1-2j)\,\lambda_i^{+(j)} = 2(\rho,\lambda_i)\,.$$

Outside I the map $\mathrm{Sht}_I^{\underline{m}} \to \mathrm{Sht}^{\underline{m}}$ is a $\mathcal{D}_I^*$-torsor, so the same holds for its base change. □

Corollary 3.3.6. *If all λ_i are definite, then the morphism*

$$\mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}} \longrightarrow \prod \mathrm{coh}_{\mathcal{D}}^{\leq\lambda_i}$$

given by the quotients $\mathcal{E}_{i-1}/\mathcal{E}_i$ or $\mathcal{E}_i/\mathcal{E}_{i-1}$ depending on the sign of $\deg(\lambda_i)$ is smooth of relative dimension $d\sum|\deg\lambda_i|$. □

Proof of Proposition 3.1.9. The morphism q in Proposition 3.1.9 is the base change of a product of r morphisms q from Proposition 3.2.10. □

3.4 Partial Frobenii

Conforming to section 2.2 we denote the cyclic permutation of a sequence $\underline{\lambda}$ of length r by $\underline{\lambda}\cdot\sigma = (\lambda_2 \ldots \lambda_r, \lambda_1)$.

Definition 3.4.1. The partial Frobenius is the morphism

$$\mathrm{Fr}_o \;:\; \mathrm{Sht}_I^{\leq\underline{\lambda}} \longrightarrow \mathrm{Sht}_I^{\leq\underline{\lambda}\cdot\sigma}$$

given by $\mathrm{Fr}_o : [\,\mathcal{E}_0 \doteq \mathcal{E}_1 \doteq \ldots \doteq {}^{\tau}\mathcal{E}_0\,] \longmapsto [\,\mathcal{E}_1 \doteq \ldots \doteq {}^{\tau}\mathcal{E}_0 \doteq {}^{\tau}\mathcal{E}_1\,]$.

This morphism is compatible with the endomorphism $\mathrm{Frob}_q \times \mathrm{id}^{\times r-1}$ of the base $(X' \setminus I)^r$ followed by an appropriate permutation. The composition $(\mathrm{Fr}_o)^r$ is canonically isomorphic to the absolute Frobenius Frob_q, so Fr_o is a universal homeomorphism.

For any permutation $s \in \mathfrak{S}_r$ we denote by $\underline{\lambda}\cdot s$ the sequence $(\lambda_{s(1)} \ldots \lambda_{s(r)})$. We have the following variant of Proposition 2.2.1.

Lemma 3.4.2. *Let $U \subseteq (X' \setminus I)^r$ be the complement of all diagonals. Then there are canonical isomorphisms*

$$j(s) \ : \ \mathrm{Sht}_I^{\leq\lambda}\big|_U \cong \mathrm{Sht}_I^{\leq\lambda\cdot s}\big|_U$$

which are compatible with the corresponding permutations of the base. □

The isomorphisms $j(s)$ are compatible with the partial Frobenius in the following way: if $s \in \mathfrak{S}_r$ fixes the set $\{1\dots k\}$, then over the intersection $U \cap (\mathrm{id}^{\times k} \times \mathrm{Frob}_q^{\times(r-k)})(U)$ there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Sht}_I^{\leq\lambda} & \xrightarrow{(\mathrm{Fr}_o)^k} & \mathrm{Sht}_I^{\leq\lambda\cdot\sigma^k} \\ {\scriptstyle j(s)}\downarrow\cong & & \cong\downarrow{\scriptstyle j(\sigma^{-k}s\sigma^k)} \\ \mathrm{Sht}_I^{\leq\lambda\cdot s} & \xrightarrow{(\mathrm{Fr}_o)^k} & \mathrm{Sht}_I^{\leq\lambda\cdot s\sigma^k} \end{array} \tag{3.4.1}$$

Let $F_i : (X' \setminus I)^r \to (X' \setminus I)^r$ be the product of Frob_q in the i-th component and the identity in the remaining components. For any $i = 1 \dots r$ we chose a permutation $s_i \in \mathfrak{S}_r$ with $s_i(1) = i$.

Definition 3.4.3. The partial Frobenius $\mathrm{Fr}_i : \mathrm{Sht}_I^{\leq\lambda}|_{U\cap F_i^{-1}(U)} \to \mathrm{Sht}_I^{\leq\lambda}|_{F_i(U)\cap U}$ is the composition $\mathrm{Fr}_i = j(s_i\sigma)^{-1} \circ \mathrm{Fr}_o \circ j(s_i)$.

This morphism commutes with $F_i : U \cap F_i^{-1}(U) \to F_i(U) \cap U$ and is a universal homeomorphism. Using the case $k = 1$ of the commutative diagrams (3.4.1) we see that the definition does not depend on the choice of s_i, and we have $\mathrm{Fr}_i\, j(s) = j(s)\mathrm{Fr}_{s(i)}$ for any $s \in \mathfrak{S}_r$. The case $k = 2$ of (3.4.1) implies that the different Fr_i commute. Their product in any order (over an appropriate open subset of the base) is naturally isomorphic to the absolute Frobenius Frob_q.

Definition 3.4.4. For any sequence $\underline{a} \in (\mathbb{Z}_{\geq 0})^r$ let

$$F^{\underline{a}} = (\mathrm{Frob}_q)^{a_1} \times \ldots \times (\mathrm{Frob}_q)^{a_r} \ : \ (X' \setminus I)^r \longrightarrow (X' \setminus I)^r.$$

Let $U \subseteq (X' \setminus I)^r$ be the complement of all diagonals and let $\Lambda \subset X^r$ be the intersection of all $\mathrm{Frob}^{\underline{a}}(U)$. We denote by

$$\mathrm{Fr}^{\underline{a}} \ : \ \mathrm{Sht}_I^{\leq\lambda}|_\Lambda \to \mathrm{Sht}_I^{\leq\lambda}|_\Lambda$$

the product of $(\mathrm{Fr}_i)^{a_i}$ for $i = 1 \dots r$ (in any order).

3.5 Hecke correspondences

The construction of Hecke correspondences is completely analogous to section 2.4 and could in fact be formally reduced to that case.

For a fixed finite set $T \subset |X|$ the projective limit

$$\mathrm{Sht}^{\leq\lambda,T}/a^{\mathbb{Z}} = \varprojlim_{I\cap T=\emptyset} \mathrm{Sht}_I^{\leq\lambda}/a^{\mathbb{Z}}$$

is a quasiprojective scheme over $(X'_{(T)})^r$. It carries an action of $\mathbb{A}^{T,*}$ via the given homomorphism $\mathbb{A}^{T,*} \to \mathrm{Pic}^T(X)$ and a right action of $(\mathcal{D}_{\mathbb{A}}^T)^*$ by twisting the level structure. Like in Lemma 2.4.1 these actions can be extended to $(D_{\mathbb{A}}^T)^*$. Then for a finite closed subscheme $I \subset X$ with $I \cap T = \emptyset$,

$$\Gamma_I^{\leq\lambda}(g) = \mathrm{Sht}^{\leq\lambda,T}/a^{\mathbb{Z}}(K_I^T \cap {}^gK_I^T) \xrightarrow{(1,g)} \mathrm{Sht}_I^{\leq\lambda}/a^{\mathbb{Z}} \times \mathrm{Sht}_I^{\leq\lambda}/a^{\mathbb{Z}}$$

is a finite étale correspondence over $(X'_{(T)})^r$. The assignment

$$\mathbb{1}_{K_I^T g K_I^T} \longmapsto \mu^T(K_I^T)\cdot\big[\,\Gamma_I^{\leq\lambda}(g)\,\big]$$

defines a homomorphism of $\mathbb{Q}$-algebras h_I^{λ} which for $I \subseteq J$ fits into the following commutative diagram.

$$\begin{array}{ccc} h_I^{\lambda} \,:\, \mathcal{H}_I^T & \longrightarrow & \mathrm{Corr}\big(\,\mathrm{Sht}_I^{\leq\lambda}/a^{\mathbb{Z}} \,\big|\, (X'_{(T)})^r\,\big) \\ \big\downarrow & & \big\downarrow \beta^{\#} \\ h_J^{\lambda} \,:\, \mathcal{H}_J^T & \longrightarrow & \mathrm{Corr}\big(\,\mathrm{Sht}_J^{\leq\lambda}/a^{\mathbb{Z}} \,\big|\, (X'_{(T)})^r\,\big) \end{array} \tag{3.5.1}$$

Here $\beta : \mathrm{Sht}_J^{\leq\lambda}/a^{\mathbb{Z}} \to \mathrm{Sht}_I^{\leq\lambda}/a^{\mathbb{Z}}$ is given by reduction of the level structure. The isomorphisms from Lemma 3.3.4 are equivariant, i.e. the composition

$$\mathcal{H}_I^T \xrightarrow{h_I^{\lambda'}} \mathrm{Corr}\big(\,\mathrm{Sht}_I^{\leq\lambda'}/a^{\mathbb{Z}} \,\big|\, (X'_{(T)})^{2r}\,\big) \xrightarrow{\Delta^*} \mathrm{Corr}\big(\,\mathrm{Sht}_I^{\leq\lambda}/a^{\mathbb{Z}} \,\big|\, (X'_{(T)})^{r}\,\big)$$

equals h_I^{λ}. If all λ_i are definite, then the morphism i' from diagram (3.3.3) is equivariant as well. More precisely we have $h_I^{\lambda} = N_* i^* h_I^m$ with h_I^m as defined in Proposition 2.4.3. Similarly the partial Frobenius Fr_o and the isomorphism $j(s)$ from Lemma 3.4.2 are equivariant. In particular, denoting $\Lambda_{(T)} = \Lambda \cap (X'_{(T)})^r$, the homomorphism h_I^{λ} can be extended to

$$h_I^{\lambda} \,:\, \mathcal{H}_I^T \longrightarrow \mathrm{Corr}\big(\,\mathrm{Sht}_I^{\leq\lambda}/a^{\mathbb{Z}} \xrightarrow{\mathrm{Fr}_i} \mathrm{Sht}_I^{\leq\lambda}/a^{\mathbb{Z}} \,\big|\, \Lambda_{(T)}\,\big)\,.$$

3.6 Groupoids of fixed points

For a scheme Y of finite type over $\mathbb{F}_q$ and an integer $a \geq 1$ we denote by $Y(a) \subseteq Y$ the subscheme of fixed points of the endomorphism $(\mathrm{Frob}_q)^a$. This is the disjoint union over the closed points $y \in Y$ with $\deg(y) \mid a$ of the schemes $\mathrm{Spec}\, k(y)$.

Let $T \subset |X|$ be as before. For any sequence of positive integers $\underline{a} \in (\mathbb{Z}_{\geq 1})^r$ the intersection

$$\Lambda_{(T)}(\underline{a}) = \Lambda_{(T)} \cap \prod_i X(a_i)$$

is the disjoint union of all $\mathrm{Spec}\, k(x_1) \times \ldots \times \mathrm{Spec}\, k(x_r)$ for pairwise distinct closed points $x_i \in X' \cap T$ which satisfy $\deg(x_i) \mid a_i$.

Definition 3.6.1. We restrict all relevant stacks to $\Lambda_{(T)}$. Let a finite closed subscheme $I \subset X$ with $I \cap T = \emptyset$, a sequence $\underline{\lambda}$ of length r and total degree zero, an element $g \in (D_{\mathbb{A}}^T)^*$, and a sequence $\underline{a} \in (\mathbb{Z}_{\geq 1})^r$ be given. The associated groupoid of fixed points is the following fibred product:

$$\begin{array}{ccc} \mathrm{Fix}_I^{\underline{\lambda}}(g, \underline{a}) & \longrightarrow & \mathrm{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}} \\ \downarrow & \square & \downarrow {\scriptstyle (\mathrm{Fr}^{\underline{a}}, \mathrm{id})} \\ \Gamma_I^{\underline{\lambda}}(g) & \xrightarrow{(1,g)} & \mathrm{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}} \times \mathrm{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}} \end{array} \tag{3.6.1}$$

This is the stack of fixed points of the composition of the finite correspondence $(\mathrm{Fr}^{\underline{a}}, \mathrm{id})$ and the transposed of the correspondence $[\,\Gamma_I^{\underline{\lambda}}\,]$.

Lemma 3.6.2. *The natural morphism* $\mathrm{Fix}_I^{\underline{\lambda}}(g, \underline{a}) \to \Lambda_{(T)}(\underline{a})$ *is of finite type and étale.*

Proof. The stack $\mathrm{Fix}_I^{\underline{\lambda}}(g, \underline{a})$ is a Deligne-Mumford stack of finite type over $\mathbb{F}_q$ because the same holds for $\mathrm{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}}$ and the morphism $(1, g)$ is representable finite. Therefore it suffices to show that $\mathrm{Fix}_I^{\underline{\lambda}}(g, \underline{a})$ is smooth over $\mathbb{F}_q$ of dimension zero, which follows from Lemma 1.3.6. ☐

Definition 3.6.3. For a given $\underline{z} \in \Lambda_{(T)}(\underline{a})(\overline{\mathbb{F}}_q)$ let $\mathrm{Fix}_I^{\underline{\lambda}}(g, \underline{a})(\underline{z}) \subseteq \mathrm{Fix}_I^{\underline{\lambda}}(g, \underline{a})(\overline{\mathbb{F}}_q)$ be the inverse image of that point. We set

$$\#\mathrm{Fix}_I^{\underline{\lambda}}(g, \underline{a})(\underline{z}) = \sum_y \frac{1}{\#\,\mathrm{Aut}(y)} \tag{3.6.2}$$

with y running through a system of representatives of the isomorphism classes of the groupoid. This is a finite sum of rational numbers by Lemma 3.6.2.

Remark. In the case $I \neq \emptyset$ the groupoid of fixed points is a scheme, which means that in (3.6.2) all groups $\mathrm{Aut}(y)$ are trivial. For the computation of these sums this will however make no difference.

Part II
Counting Fixed Points

In the third part of this work we will study the cohomology of the varieties $\mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}}$. The starting point for this is the calculation of the cardinalities of the groupoids of fixed points which have been defined in section 3.6. Our approach is based on the presentations in [Laf97] and [Lau96].

The computation of the number of fixed points will be a consequence of their adelic description, which rests on the classification of the generic fibres (isogeny classes) of $\mathcal{D}$-shtukas over $\overline{\mathbb{F}}_q$. One important point is a transfer between conjugacy classes of generic automorphisms of $\mathcal{D}$-shtukas and conjugacy classes in D^*. At first, the computation results in a sum of adelic orbital integrals which are twisted at finitely many places. Using the fundamental lemma for GL_d (in those cases where it is known) the twisted integrals can be replaced by untwisted integrals. In view of the Selberg trace formula this means that the number of fixed points can be expressed as the automorphic trace of a certain Hecke function.

Prior to the actual computation we give a detailed description of the category of (D, φ)-spaces, which arise as generic fibres of $\mathcal{D}$-shtukas (section 6). In section 5 the statement of the fundamental lemma and some general notions are explained. In order to clarify the principle of the computation we first consider the case of $\mathcal{D}$-shtukas without modifications.

4 $\mathcal{D}$-Shtukas without Modifications

Let $\underline{\lambda} = \emptyset$ be the empty sequence. In view of Drinfeld's descent lemma 8.1.1, a $\mathcal{D}$-shtuka of type $\underline{\lambda}$ over $\overline{\mathbb{F}}_q$ is the same as a locally free $\mathcal{D}$-module of rank 1. Thus there is a canonical isomorphism

$$\mathrm{Sht}_I^{\emptyset}/a^{\mathbb{Z}}\,(\overline{\mathbb{F}}_q) \;\cong\; D^*\backslash D^*_{\mathbb{A}}/K_I a^{\mathbb{Z}} \tag{4.1}$$

such that the natural action of $\mathrm{Aut}(\overline{\mathbb{F}}_q \,|\, \mathbb{F}_q)$ on the left hand side corresponds to the trivial action on the right hand side. Since the stack $\mathrm{Sht}_I^{\emptyset}/a^{\mathbb{Z}}$ is of finite type and étale over $\mathbb{F}_q$, the groupoid of its geometric points is finite. Using (4.1) this translates to the quotient $D^*\backslash D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ being compact, which of course can also be proved directly (see [Weil] Lemma 3.1.1). In the case $I \neq \emptyset$ the stack $\mathrm{Sht}^{\emptyset}_I/a^{\mathbb{Z}}$ is a scheme, which means that the double quotient on the right hand side of (4.1) has trivial stabilisers.

Let $I \subset X$ and $g \in D^*_{\mathbb{A}}$ be given. The groupoid of fixed points $\mathrm{Fix}^{\emptyset}_I(g)$ is then defined by the right half of the following 2-cartesian diagram, while its left half is an auxiliary definition.

$$\begin{array}{ccccc}
\widetilde{\mathrm{Fix}}^{\emptyset}_I(g) & \longrightarrow & \mathrm{Fix}^{\emptyset}_I(g) & \longrightarrow & D^*\backslash D^*_{\mathbb{A}}/K_I a^{\mathbb{Z}} \\
\downarrow & \square & \downarrow & \square & \downarrow \Delta \\
D^*_{\mathbb{A}}/(K_I \cap {}^gK_I) & \longrightarrow & D^*\backslash D^*_{\mathbb{A}}/(K_I \cap {}^gK_I)a^{\mathbb{Z}} & \xrightarrow[(1,g)]{} & \big[D^*\backslash D^*_{\mathbb{A}}/K_I a^{\mathbb{Z}}\big]^2
\end{array}$$

Here the second component of the map denoted by $(1, g)$ is the map induced by right multiplication by g. We can identify the fibred product $\widetilde{\mathrm{Fix}}^{\emptyset}_I(g)$ with the subset of

$$D^*_{\mathbb{A}}\big/(K_I \cap {}^gK_I) \times D^* \times \mathbb{Z}$$

consisting of all elements $(\bar{y}, \delta, n)$ satisfying $ygK_I = \delta a^n yK_I$. Then the natural map identifies $\widetilde{\mathrm{Fix}}^{\emptyset}_I(g)$ with the subset of $D^*_{\mathbb{A}}/K_I \times D^* \times \mathbb{Z}$ defined by the condition $y^{-1}\delta a^n y \in K_I g K_I$. The action of $\delta' \in D^*$ is given by $\bar{y} \mapsto \delta'\bar{y}$ and $\delta \mapsto \delta'\delta\delta'^{-1}$. We denote by $D^*_\natural$ a system of representatives of the conjugacy classes in D^* and get

$$\begin{aligned}
\mathrm{Fix}^{\emptyset}_I(g) &= D^*\big\backslash \widetilde{\mathrm{Fix}}^{\emptyset}_I(g)\big/a^{\mathbb{Z}} \\
&= \coprod_{\substack{\delta \in D^*_\natural \\ n \in \mathbb{Z}}} D^*_\delta \big\backslash \big\{ y \in D^*_{\mathbb{A}} \,\big|\, y^{-1}\delta a^n y \in K_I g K_I \big\} \big/ K_I a^{\mathbb{Z}}
\end{aligned} \tag{4.2}$$

As $\mathrm{Fix}^{\emptyset}_I(g)$ is a finite groupoid, only finitely many of the double quotients on the right hand side are non-empty. In particular any compact set in $D^*_{\mathbb{A}}$ meets only finitely many conjugacy classes of elements of the form δa^n with $\delta \in D^*_\natural$ and $n \in \mathbb{Z}$. From (4.2) we get the following equation for the number of fixed points.

$$\#\mathrm{Fix}^{\emptyset}_I(g) = \sum_{\substack{\delta \in D^*_\natural \\ n \in \mathbb{Z}}} \int_{y \in D^*_\delta \backslash D^*_{\mathbb{A}}} \mathbb{1}_{K_I g K_I}(y^{-1}\delta a^n y)\, \mu(K_I)^{-1} \frac{d\mu}{d\mu_0} \tag{4.3}$$

This calculation can be interpreted as a proof of the Selberg trace formula. Let $\mathrm{Aut} = \mathcal{C}^\infty(D^*\backslash D^*_{\mathbb{A}}/a^{\mathbb{Z}})$ be the space of locally constant rational functions on the compact topological space $D^*\backslash D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ on which the group $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ acts by right translation and the Hecke algebra $\mathcal{H} = \mathcal{C}^\infty(D^*_\mathbb{A})$ by the corresponding convolution. The restricted action of $\mathcal{H}_I$ on the invariants

$$\mathrm{Aut}^{K_I} = \mathcal{C}\big(D^*\backslash D^*_\mathbb{A}/K_I a^\mathbb{Z}\big)$$

can be identified with the transpose of the action defined via the Hecke correspondences, which means $\mathrm{Tr}(\mathbb{1}_{K_I g K_I}, \mathrm{Aut}) = \mu(K_I)\cdot \#\mathrm{Fix}^\emptyset_I(g)$. Linear extension of (4.3) gives for all $f \in \mathcal{H}$ the Selberg trace formula

$$\mathrm{Tr}(f, \mathrm{Aut}) = \sum_{\substack{\delta\in D^*_\natural \\ n\in\mathbb{Z}}} \int_{y\in D^*_\delta\backslash D^*_\mathbb{A}} f(y^{-1}\delta a^n y)\,\frac{d\mu}{d\mu_0} \tag{4.4}$$

Here for any f almost all summands are the integral of the zero function.

A more direct direct proof of (4.4) is given in [Laf97] IV.4, Proposition 1.

5 Preliminaries on Orbital Integrals

A detailed reference for the following explanations is Chapter 4 in [Lau96], cf. also [ArCl], Chapter 1.

5.1 Norm for GL_d

Let $E\,|\,F$ be a cyclic field extension of degree r with fixed generator σ of the Galois group. We denote by $\mathrm{GL}_d(F)_\natural$ the set of conjugacy classes in $\mathrm{GL}_d(F)$ and by $\mathrm{GL}_d(E)^\sigma_\natural$ the set of σ-conjugacy classes in $\mathrm{GL}_d(E)$. Here σ-conjugation by an element $h \in \mathrm{GL}_d(E)$ is the map $g \mapsto h^{-1} g\,\sigma(h)$. There is a norm morphism

$$N_r \;:\; \mathrm{GL}_d(E) \longrightarrow \mathrm{GL}_d(E)\,, \quad g \longmapsto g\,\sigma(g)\dots\sigma^{r-1}(g)\,.$$

Lemma 5.1.1. *Any norm in* $\mathrm{GL}_d(E)$ *is conjugate to an element of* $\mathrm{GL}_d(F)$. *The thus well-defined map*

$$N_r \;:\; \mathrm{GL}_d(E)^\sigma_\natural \longrightarrow \mathrm{GL}_d(F)_\natural$$

is injective. For a given $\gamma \in \mathrm{GL}_d(E)$ *with* $N(\gamma) \in \mathrm{GL}_d(F)$ *the* σ*-centraliser* $(\mathrm{Res}_{E|F}\,\mathrm{GL}_d)^\sigma_\gamma$ *is an inner form of the centraliser* $(\mathrm{GL}_d)_{N(\gamma)}$.

Proof. See [ArCl] Lemma 1.1. and its proof or [Lau96], Proposition 4.4.2. □

An element $\delta \in \mathrm{GL}_d(F)$ is called elliptic if the algebra $F[\delta]$ is a field, and δ is called semisimple if $F[\delta]$ is a product of fields. In the latter case a decomposition $F[\delta] \cong \prod F_i$ with fields F_i induces a corresponding decomposition $F^d \cong \bigoplus V_i$ plus elliptic elements $\delta_i \in \mathrm{GL}(V_i)$ with $F_i = F[\delta_i]$.

Lemma 5.1.2. *Let F be a local field and let $\delta \in \mathrm{GL}_d(F)$ be semisimple. Then δ is a norm if and only if all $\det(\delta_i)$ are norms for $E \,|\, F$.*

Proof. Cf. [ArCl] Chapter 1, Lemma 4. Since any inverse image of δ under N_r commutes with δ, the element δ is a norm if and only if all δ_i are norms. By multiplicativity of the determinant it remains to show that an elliptic δ is a norm as soon as $\det(\delta)$ is a norm for $E \,|\, F$.

In the non-archimedean case we choose an extension F' of $F[\delta]$ with $[F' \!:\! F] = d$ plus an embedding $F' \subseteq M_d(F)$ over $F[\delta]$. Then we have $N_{F' \,|\, F}(\delta) = \det(\delta)$. Let E' be a factor of $E \otimes_F F'$, i.e. a common field extension of E and F' generated by these two. It suffices to show that δ is a norm for $E' \,|\, F'$. This holds because the map

$$N_{F' \,|\, F} \;:\; F'^{*} \big/ N_{E' \,|\, F'} E'^{*} \longrightarrow F^{*} \big/ N_{E \,|\, F} E^{*}$$

is isomorphic to the restriction $\mathrm{Gal}(E' \,|\, F') \to \mathrm{Gal}(E \,|\, F)$, which is injective.

In the archimedean case we can argue similarly. □

For $\gamma \in \mathrm{GL}_d(E)$ the σ-centraliser $(\mathrm{Res}_{E \,|\, F} \mathrm{GL}_d)_\gamma^\sigma$ is the multiplicative group of the F-algebra $M_d(E)_\gamma^\sigma$. If $N(\gamma) \in \mathrm{GL}_d(F)$ is semisimple, then from the following lemma we obtain a description of this algebra.

Lemma 5.1.3. *For a given elliptic element $\delta \in \mathrm{GL}_d(F)$ we write $F' = F[\delta]$ and $d' = d/[F' \!:\! F]$. Let $\alpha \in H^2(F, \mathbb{Z}) = \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})$ be the homomorphism defined by $\alpha(\sigma) = 1/r$.*

If $\delta = N_r(\gamma)$ is a norm, then $M_d(E)_\gamma^\sigma$ is a central simple F'-algebra of dimension d'^2 with class $-\mathrm{res}(\alpha) \cup \delta \in H^2(F', \mathbb{G}_m)$. Here δ is considered as an element of $H^0(F', \mathbb{G}_m)$.

Proof. The algebra $E \otimes F'$ can be decomposed into a product of fields $\prod_{i=1}^s E_i'$ such that $\sigma \otimes \mathrm{id}$ induces isomorphisms $E_i' \cong E_{i+1}'$. Each extension $E_i' \,|\, F'$ is cyclic of degree r/s with Galois group generated by $\sigma' = (\sigma \otimes \mathrm{id})^s$.

The given embedding $F' \subseteq M_d(E)$ gives rise to a decomposition $E^d \cong \bigoplus_{i=1}^s V_i$ corresponding to the decomposition of $E \otimes F'$. We define a $(\sigma \otimes \mathrm{id})$-linear endomorphism ψ of E^d via $\psi = \gamma \circ \sigma^{\times d}$. In view of $\psi^r = \delta$ and $\psi : V_i \cong V_{i+1}$ the algebra

$$M_d(E)_\gamma^\sigma = \mathrm{End}_{E \otimes F'}(E^d, \psi) = \mathrm{End}_{E_i'}(V_i, \psi^s)$$

can be identified with the centraliser of the cyclic F'-algebra $E_i'\{\tau\}/(\tau^{r/s} - \delta)$ in $\mathrm{End}_{F'}(V_i)$, which has the asserted class in H^2. Since the F'-dimension of V_i is $d'r/s$, we obtain the asserted dimension as well. □

Corollary 5.1.4. *Let F be a non-archimedean local field with valuation v. Then in the situation of Lemma 5.1.3 we have $\mathrm{inv}_{F'}(M_d(E)_\gamma^\sigma) = -v(\det \delta)/(rd')$.*

Proof. $\mathrm{inv}_{F'}(\mathrm{res}\, \alpha \cup \delta) = \mathrm{inv}_F(\alpha \cup \mathrm{cor}\, \delta) = v(N_{F' \,|\, F} \delta)/r$. □

5.2 Satake isomorphism

In the following let F be a non-archimedian local field with valuation v, with residue field $\mathbb{F}_q$, and with a fixed uniformising element ϖ. We write

- $G = \mathrm{GL}_d(F)$,
- $K = \mathrm{GL}_d(\mathcal{O}_F)$,
- $A = \mathbb{Q}[q^{1/2}, q^{-1/2}]$.

Let $\mathcal{H} = \mathcal{H}_G = \mathcal{C}_0(G//K, \mathbb{Q})$ be the Hecke algebra of K-biinvariant rational functions on G with compact support. Its multiplication is given by convolution with respect to the biinvariant measure μ of G normalised by $\mu(K) = 1$.

For a parabolic subgroup $\mathcal{P} \subseteq \mathrm{GL}_d$ defined over $\mathcal{O}_F$ with a Levi group $\mathcal{M}$ defined over $\mathcal{O}_F$ and with unipotent radical $\mathcal{N}$ we write $P = \mathcal{P}(K)$, $M = \mathcal{M}(K)$ and $N = \mathcal{N}(K)$. We denote by $\mathcal{H}_M = \mathcal{C}_0(M//K \cap M, \mathbb{Q})$ the corresponding Hecke algebra, by $\delta_P : M \to q^{\mathbb{Z}}$ the modulus character and by dn the measure on N normed by $dn(K \cap N) = 1$. Then there is an injective homomorphism of algebras

$$\mathcal{H}_G \otimes A \longrightarrow \mathcal{H}_M \otimes A, \quad f \longmapsto f^P$$

given by $f^P(m) = \delta_P^{1/2}(m) \cdot \int_N f(mn)dn$. This does not depend on $\mathcal{P}$ and is transitive in the obvious sense.

For a split maximal torus $\mathcal{T} \subseteq \mathrm{GL}_d$ defined over $\mathcal{O}_F$ and $T = \mathcal{T}(F)$ there is a canonical isomorphism $\mathcal{H}_T \cong \mathbb{Q}[X_*(\mathcal{T})]$. Moreover the choice of a Borel group $\mathcal{B}$ containing $\mathcal{T}$ determines an isomorphism $\mathcal{T} \cong \mathbb{G}_m^d$ and accordingly $\mathcal{H}_T \cong \mathbb{Q}[z_1, z_1^{-1} \ldots z_d, z_d^{-1}]$.

Using these notations the map $f \mapsto f^B$ induces an isomorphism (Satake isomorphism)

$$\mathcal{H}_G \otimes A \cong (\mathcal{H}_T \otimes A)^W = A[z_1, z_1^{-1} \ldots z_d, z_d^{-1}]^{\mathfrak{S}_d}, \quad \text{write } f \mapsto f^\vee,$$

which does not depend on T or B. Here W denotes the Weyl group of $T \subseteq G$. Similarly, for $T \subseteq M$ we get an isomorphism $\mathcal{H}_M \otimes A \cong (\mathcal{H}_T \otimes A)^{W_M}$. Using this, the map $f \mapsto f^P$ can be considered as the inclusion of subrings of $\mathcal{H}_T \otimes A$.

Let $E \mid F$ be the unramified extension of degree r and let

- $G_r = \mathrm{GL}_d(E)$,
- $K_r = \mathrm{GL}_d(\mathcal{O}_E)$,
- $\mathcal{H}_r = \mathcal{C}_0(G_r//K_r, \mathbb{Q})$.

Let $b_r : A[z_1 \ldots z_d^{-1}] \to A[z_1 \ldots z_d^{-1}]$ be the homomorphism defined by $z_i \mapsto z_i^r$. Via the Satake isomorphism this defines a homomorphism

$$b_r \ : \ \mathcal{H}_r \otimes A \longrightarrow \mathcal{H} \otimes A \,.$$

5.3 Fundamental lemma

We keep the notations of section 5.2. By [Lau96], Corollary 4.3.3 the conjugacy class of an element $\delta \in G$ is closed with respect to the ϖ-adic topology if and only if δ is semisimple, i.e. if the algebra $F[\delta]$ is a product of fields. In that case for any $f \in \mathcal{C}_0^\infty(G)$ the orbital integral with respect to a biinvariant measure ν on the centraliser G_δ

$$\mathrm{O}_\delta(f, d\nu) = \int_{G_\delta \backslash G} f(g^{-1}\delta g) \, \frac{d\mu}{d\nu}$$

is absolutely convergent.

Let $\sigma \in \mathrm{Gal}(E \,|\, F)$ be the Frobenius. A given semisimple element $\delta \in G$ determines a decomposition $F^d \cong \bigoplus V_i$ plus elliptic elements $\delta_i \in \mathrm{GL}(V_i)$. By Lemma 5.1.2 δ is a norm if and only if all $v(\det \delta_i)$ are multiples of r. In that case for any $\gamma \in G_r$ with $N_r(\gamma) = \delta$ the σ-conjugacy class of γ is ϖ-adically closed in G_r, because it is the inverse image under N_r of the conjugacy class of the semisimple element $\delta \in \mathrm{GL}_d(E)$. Thus for any $f \in \mathcal{C}_0^\infty(G_r)$ the twisted orbital integral with respect to a biinvariant measure ν on the σ-centraliser $(G_r)_\gamma^\sigma$

$$\mathrm{TO}_\gamma(f, d\nu) = \int_{(G_r)_\gamma^\sigma \backslash G_r} f(g^{-1}\gamma\sigma(g)) \, \frac{d\mu}{d\nu}$$

converges absolutely. Since $(G_r)_\gamma^\sigma$ is the group of rational points of an inner form of $(\mathrm{GL}_d)_\delta$ (Lemma 5.1.1), there is a canonical transfer of invariant measures between $(G_r)_\gamma^\sigma$ and G_δ.

Questions about orbital integrals of functions in $\mathcal{H}$ and twisted orbital integrals of functions in $\mathcal{H}_r$ can be reduced to the case of elliptic δ's as follows. For a given semisimple $\delta \in G$, after conjugation we may assume that the associated decomposition $F^d \cong \bigoplus V_i$ is defined over $\mathcal{O}_F$. Then the stabiliser $\mathcal{M} \subseteq \mathrm{GL}_d$ of this decomposition is the Levi group of a parabolic $\mathcal{P}$ defined over $\mathcal{O}_F$, and G_δ is contained in $\mathcal{M}(F)$. By [Lau96], Proposition 4.3.11 there is a constant $c(\delta) \in F^*$ such that for any $f \in \mathcal{H}$ the following equation holds.

$$\mathrm{O}_\delta(f, d\nu) = |c(\delta)|^{-1/2} \, \mathrm{O}_\delta^M(f^P, d\nu)$$

If in addition $\delta = N(\gamma)$ is a norm, then $(G_r)_\gamma^\sigma$ is contained in $\mathcal{M}(E)$, and by [Lau96], Proposition 4.4.9 for any $f \in \mathcal{H}_r$ we have

$$\mathrm{TO}_\gamma(f, d\nu) = |c(\delta)|^{-1/2} \, \mathrm{TO}_\gamma^M(f^P, d\nu)$$

with the same constant $c(\delta)$. Thus we may pass from G to M.

Lemma 5.3.1. *Let $\delta \in G$ be semisimple and let $f \in \mathcal{H}_r$. If $\mathrm{O}_\delta(b_r f, d\nu) \neq 0$ then δ is a norm. If in addition $f^\vee$ is homogeneous of degree k then we have $v(\det \delta) = kr$.*

Proof. Using the above reduction we can assume that δ is elliptic. Then it suffices to note that for $f \in \mathcal{H}$ the Satake transform $f^\vee$ is a homogeneous polynomial of degree $k \in \mathbb{Z}$ if and only if f is supported in the set of elements $g \in G$ with $v(\det g) = k$. For such f the integral $\mathrm{O}_\delta(f, d\nu)$ can be nonzero only if $v(\det \delta) = k$. □

We denote by $\mu^+ \in P_1^{++}$ and by $\mu^- \in P_{-1}^{++}$ the unique elements, see page xii. The following statement for all $f \in \mathcal{H}_r$ is called a fundamental lemma for GL_d. The corresponding statement in characteristic zero is known, cf. [ArCl] 4.5 and 3.13.

Theorem 5.3.2 (Drinfeld). *Let $\delta \in G$ be semisimple, $\delta = N(\gamma)$, and let $f \in \mathcal{H}_r$ be the characteristic function of one of the double cosets $K_r\mu^+(\varpi)K_r$ or $K_r\mu^-(\varpi)K_r$. Then we have*

$$\mathrm{O}_\delta(b_r(f), d\nu) = \varepsilon(\delta)\, \mathrm{TO}_\gamma(f, d\nu)$$

with $\varepsilon(\delta) = (-1)^{\mathrm{rk}_{F[\delta]}\, G_\delta - \mathrm{rk}_{F[\delta]}(G_r)^\sigma_\gamma}$.

Proof. The case μ^+ is [Lau96], Theorem 4.5.5. Using the involution $g \mapsto {}^t g^{-1}$ of GL_d this implies the case μ^- as well. □

6 Some Semilinear Algebra

We recall Drinfeld's results on (F, φ)-spaces including some proofs and generalise them to (D, φ)-spaces in the obvious way. Afterwards we explain a transfer of conjugacy classes which is needed for the computation of the groupoids of fixed points.

6.1 (F, φ)-spaces

Here we explain the results from [Dri88] on (F, φ)-spaces and their localisations, cf. also [LRS], Appendices A and B. As usual F is the function field of the given geometrically irreducible smooth projective curve X over $\mathbb{F}_q$.

Definition 6.1.1. We write $\bar{F} = F \otimes \bar{\mathbb{F}}_q$ and $\sigma_q = \mathrm{id} \otimes \mathrm{Frob}_q$. An (F, φ)-space (over $\bar{\mathbb{F}}_q$) is a finite dimensional vector space V over $\bar{F}$ plus a bijective σ_q-linear map $\varphi : V \to V$.

Let E be a finite dimensional commutative F-algebra and let $\Pi \in E^* \otimes \mathbb{Q}$. The functor $E \mapsto E^* \otimes \mathbb{Q}$ commutes with fibred products and in particular with intersections, so there is a unique minimal subalgebra $F[\Pi] \subseteq E$ such that $\Pi \in F[\Pi]^* \otimes \mathbb{Q}$.

Definition 6.1.2. An F-pair is a pair (E, Π) as above satisfying $E = F[\Pi]$. The pair is called indecomposable if E is a field.

Remark. The minimality of E implies $E = F \cdot E^p$, which means that E is an étale F-algebra. In the indecomposable case we also have $E^* \otimes \mathbb{Q} = \mathrm{Div}^0(E) \otimes \mathbb{Q}$ where $\mathrm{Div}^0(E)$ denotes the group of divisors of degree zero on the smooth projective curve with function field E.

The following construction of Drinfeld assigns to any (F, φ)-space an F-pair plus an embedding of E into the centre of $\mathrm{End}(V, \varphi)$. The given (F, φ)-space is defined over a finite field $\mathbb{F}_{q^n}$, i.e.

$$V = V_n \otimes_{\mathbb{F}_{q^n}} \overline{\mathbb{F}}_q \quad \text{and} \quad \varphi = \varphi_n \otimes \mathrm{Frob}_q \,.$$

Then $\pi_n = (\varphi_n)^n \otimes \mathrm{id}$ is a (linear) automorphism of (V, φ) which commutes with all endomorphisms of (V, φ) defined over $\mathbb{F}_{q^n}$. The subalgebra $F[\pi_n]$ of $\mathrm{End}(V_n, \varphi_n) \subseteq \mathrm{End}(V, \varphi)$ has finite dimension over F and we can define $\Pi = (\pi_n)^{1/n} \in F[\pi_n]^* \otimes \mathbb{Q}$ and $E = F[\Pi]$. This does not depend on the choice of n and of (V_n, φ_n).

Theorem 6.1.3 (Drinfeld). *The abelian category of (F, φ)-spaces is semisimple. The above construction defines a bijection between the set of isomorphism classes of simple (F, φ)-spaces and the set of isomorphism classes of indecomposable F-pairs.*

Let (V, φ) be a simple (F, φ)-space and let (E, Π) be the associated F-pair. Then $\Delta = \mathrm{End}(V, \varphi)$ is a finite dimensional central division algebra over E with local invariants

$$\mathrm{inv}_y(\Delta) = -\deg_y(\Pi) = -\deg(y) y(\Pi) \,.$$

Let $d(\Pi) = d(\Delta)$ be the least common denominator of all $\deg_y(\Pi)$. Then we have $\dim_{\overline{F}}(V) = [E\!:\!F]\, d(\Pi)$.

This is [Dri88], Proposition 2.1. The proof in *loc. cit.* also gives the following explicit description of the inverse map $(E, \Pi) \mapsto (V, \varphi)$.

For a given indecomposable F-pair (E, Π) the set of representations $\Pi = b^{1/n}$ with $b \in E^*$ is made into an inductive system using the order $(b, n) \leq (b^m, nm)$. Let $\mathcal{C}_{b,n}$ be the category of (F, φ)-spaces over $\mathbb{F}_{q^n}$ with $(F[\varphi^n], \varphi^n) \cong (E, b)$. This category is equivalent to the category of finite modules over the algebra

$$A = E \otimes \mathbb{F}_{q^n}\{\tau\}/(\tau^n - b)$$

which is a central simple E-algebra of dimension n^2 with local invariants $\mathrm{inv}_y(A) = \deg_y(\Pi)$. So the category $\mathcal{C}_{b,n}$ is semisimple and has a unique simple object

$V_{b,n}$ whose endomorphism ring is the division algebra equivalent to A^{op}. From $\dim_E(V_{b,n}) = n\, d(\Pi)$ we get $\dim_{F\otimes\mathbb{F}_{q^n}}(V_{b,n}) = [E:F]\, d(\Pi)$. Since the category of (F,φ)-spaces with associated F-pair (E,Π) is the direct limit of the categories $\mathcal{C}_{b,n}$, the desired simple (F,φ)-space is $(V,\varphi) = (V_{n,b}\,,\tau)\otimes_{\mathbb{F}_{q^n}} \overline{\mathbb{F}}_q$.

Dieudonné modules

For a closed point $x \in X$ we denote by F_x the completion of F at x. Let $\varpi_x \in F_x$ be a uniformising element.

Definition 6.1.4. We write $\bar{F}_x = F_x \,\widehat{\otimes}\, \overline{\mathbb{F}}_q$ and $\sigma_q = \mathrm{id}\,\widehat{\otimes}\,\mathrm{Frob}_q$. A Dieudonné-$F_x$-module (over $\overline{\mathbb{F}}_q$) is a finitely generated $\bar{F}_x$-module V plus a bijective σ_q-linear map $\varphi : V \to V$.

V is a free $\bar{F}_x$-module because $\bar{F}_x$ is a product of fields and σ_q permutes the factors transitively. In particular its rank $\mathrm{rk}_{\bar{F}_x}(V)$ is well defined. We fix an embedding $k(x) \subset \overline{\mathbb{F}}_q$ over $\mathbb{F}_q$ and write

$$F_{\bar{x}} = F_x \,\widehat{\otimes}_{k(x)}\, \overline{\mathbb{F}}_q\ , \qquad \sigma_x = \mathrm{id}\,\widehat{\otimes}(\mathrm{Frob}_q)^{\deg(x)}.$$

Then a Dieudonné-F_x-module (V,φ) is the same as a finite dimensional $F_{\bar{x}}$-module W plus a bijective σ_x-linear map $\psi : W \to W$. We call (W,ψ) the reduced representation of (V,φ).

Let $\bar{\mathcal{O}}_x = \mathcal{O}_x \,\widehat{\otimes}\, \overline{\mathbb{F}}_q$. A lattice in a free $\bar{F}_x$-module V of finite rank is a finitely generated $\bar{\mathcal{O}}_x$-submodule $M \subseteq V$ which generates V over $\bar{F}_x$ (this implies M is free over $\bar{\mathcal{O}}_x$). The degree of a Dieudonné-F_x-module is

$$\deg(V,\varphi) = \dim_{\overline{\mathbb{F}}_q}\big(M \,/\, M\cap\varphi M\big) - \dim_{\overline{\mathbb{F}}_q}\big(\varphi M \,/\, M\cap\varphi M\big)$$

which does not depend on the chosen lattice $M \subseteq V$ and could have been defined using $\mathcal{O}_{\bar{x}}$-lattices in W as well. Its slope is $\mu = \deg(V,\varphi)/\,\mathrm{rk}(V)$. A detailed proof of the following proposition can be found in [Lau96], Appendix B.

Proposition 6.1.5. *The abelian category of Dieudonné-F_x-modules is semisimple. For any $\mu \in \mathbb{Q}$ there is a unique simple object (V_μ,φ_μ) with slope μ. Its rank is the denominator of μ, while $\mathrm{End}(V_\mu,\varphi_\mu)$ is a finite dimensional central division algebra over F_x with local invariant $-\mu$. The natural functor from Dieudonné-F_x-modules to Dieudonné-$F_x[\varpi_x^{1/n}]$-modules respects isotypic objects and multiplies all slopes by n.*

Corollary 6.1.6. *Let (V,φ) be a simple Dieudonné-F_x-module and let Δ be its ring of endomorphisms. Then $\mathrm{rk}_{\bar{F}_x}(V) = d(\Delta)$.* □

In view of the proposition the isomorphism class of a Dieudonné module is given by the sequence $(\mu_1 \geq \ldots \geq \mu_n)$ of its slopes in which the multiplicity of μ is the rank of the μ-isotypic component. There is a lattice with $\varphi M \subseteq M$ (respectively $M \subseteq \varphi M$) if and only if all slopes are ≥ 0 (respectively ≤ 0).

A Dieudonné-F_x-module is called trivial if it is isotypic with slope zero. An equivalent condition is that there is a lattice with $\varphi M = M$. In the trivial case such lattices are the same as $\mathcal{O}_x$-lattices in the F_x-vector space $V^{\varphi=1} = W^{\psi=1}$.

Let (V, φ) be a Dieudonné-F_x-module and let (W, ψ) be its reduced representation. For a given automorphism b of (V, φ) or equivalently of (W, ψ) we form $(W, \psi \circ b)$, which is the reduced representation of a Dieudonné-F_x-module $(V_b\,, \varphi_b)$.

Lemma 6.1.7. *We assume that the subalgebra $E = F[b]$ of $\mathrm{End}(V, \varphi)$ is a field and denote by y its unique discrete valuation over x. Then the slopes of the b-twist $(V_b\,, \varphi_b)$ equal the slopes of (V, φ) plus $y(b)/e(y \,|\, x)$ (including multiplicities).*

Proof. It suffices to consider an isotypic Dieudonné module. Adjoining $\varpi_x^{1/n}$ we may assume that both its unique slope and $y(b)/e(y \,|\, x)$ are integers. Multiplying ψ and b with powers of ϖ_x we can further assume that (V, φ) is trivial and that $b \in \mathcal{O}_E^*$. Then an $\mathcal{O}_E$-lattice in the E-vector space $W^{\psi=1}$ corresponds to an $\mathcal{O}_{\bar{x}}$-lattice $M \subseteq W$ with $\psi b M = M$, which implies $(V_b\,, \varphi_b)$ is trivial as well. □

Let $r \geq 1$ be an integer and let $F_{x,r} \subset F_{\bar{x}}$ be the unramified extension of F_x of degree r. For any Dieudonné-F_x-module (V, φ) with reduced representation (W, ψ) the pair (W, ψ^r) determines a Dieudonné-$F_{x,r}$-module $(V^{(r)}, \varphi^{(r)})$.

Lemma 6.1.8. *The functor $(V, \varphi) \mapsto (V^{(r)}, \varphi^{(r)})$ multiplies all slopes with r, respecting their multiplicities.*

Proof. After adjunction of $\varpi_x^{1/n}$ all slopes of V are integers, in which case the assertion is obvious. □

Localisation

The localisation of an (F, φ)-space (V, φ) at $x \in |X|$ is the Dieudonné-F_x-module

$$(V_x, \varphi_x) = (V \mathbin{\widehat{\otimes}}_F F_x\,, \varphi \mathbin{\widehat{\otimes}} \mathrm{id})\,.$$

Let (E, Π) be the associated F-pair. The algebra $E_x = E \otimes_F F_x$ is the product of the completions E_y of E at the places $y \,|\, x$, so the given action of E on (V, φ) induces a decomposition

$$(V_x, \varphi_x) = \bigoplus_{y \,|\, x} (V_y, \varphi_y)\,.$$

The following description of the localisations is given in [Dri88].

Proposition 6.1.9. *Let (V,φ) be the simple (F,φ)-space with associated F-pair (E,Π). Then the Dieudonné-F_x-module (V_y,φ_y) in the above decomposition of (V_x,φ_x) is isotypic with slope*

$$\mu(y) = \frac{\deg_y(\Pi)}{[E_y : F_x]} = \deg(x)\,\frac{y(\Pi)}{e(x\,|\,y)}$$

and its rank is $[E_y : F_x]\, d(\Pi)$.

Proof. Cf. [LRS] Appendix B for a different proof. Since σ_q permutes the factors of the product of fields $E \otimes \overline{\mathbb{F}}_q$ transitively, V is free over $E \otimes \overline{\mathbb{F}}_q$ of rank $\dim_{\overline{F}}(V)/[E:F] = d(\Pi)$. This implies that V_y is free over $E_y \mathbin{\widehat{\otimes}} \overline{\mathbb{F}}_q$ of rank $d(\Pi)$ as well, which gives the asserted rank of V_y over $F_x \mathbin{\widehat{\otimes}} \overline{\mathbb{F}}_q$.

In order to compute the slopes we choose a sufficiently divisible integer $n = r\deg(x)$ such that (V,φ) is defined over $\mathbb{F}_{q^n}$ and such that there is a representation $\Pi = b^{1/n}$ with $b \in E^*$. From the construction of the simple (F,φ)-spaces we get an isomorphism between a multiple of the Dieudonné-$F_{x,r}$-module $(V_y^{(r)}, \varphi_y^{(r)})$ and the b-twist of a trivial Dieudonné module. Using Lemma 6.1.8 and Lemma 6.1.7 the equality of their slopes means

$$r\mu = \frac{y(b)}{e(y\,|\,x)} = \frac{n\, y(\Pi)}{e(y\,|\,x)}$$

for all slopes μ of (V_y,φ_y). Dividing by r gives the desired equation. □

6.2 (D,φ)-spaces

Using the following general observation the preceding results on (F,φ)-spaces and their localisations generalise directly to (D,φ)-spaces (Definition 6.2.2 below).

Let k be a field and let $\mathcal{C}$ be a k-linear semisimple abelian category in which all spaces of homomorphisms are finite dimensional. For a central simple k-algebra A we denote by $\mathcal{C}(A)$ the category of $W \in \mathcal{C}$ plus a homomorphism $A^{\mathrm{op}} \to \mathrm{End}_{\mathcal{C}}(W)$.

Lemma 6.2.1. *The abelian category $\mathcal{C}(A)$ is semisimple. The forgetful functor $\mathcal{C}(A) \to \mathcal{C}$ maps simple objects to isotypic objects, and the thus well defined map from the set of isomorphism classes of simple objects in $\mathcal{C}(A)$ into the set of isomorphism classes of simple objects in $\mathcal{C}$ is bijective.*

For two simple objects $W \in \mathcal{C}$ and $W' \in \mathcal{C}(A)$ with an isomorphism $W' \cong W^m$ in $\mathcal{C}$ we write $\Delta = \mathrm{End}_{\mathcal{C}}(W)$ and $\Delta' = \mathrm{End}_{\mathcal{C}(A)}(W')$. These are finite dimensional division algebras over k with canonically isomorphic centre K, and

Δ' is equivalent to the central simple K-algebra $\Delta \otimes_k A$. The multiplicity of W in W' is

$$m = \sqrt{\dim_k(A)}\, \frac{d(\Delta')}{d(\Delta)}\,. \tag{6.2.1}$$

Proof. As the A-action respects the decomposition into $\mathcal{C}$-isotypic components, we may assume that up to isomorphism $\mathcal{C}$ has a unique simple object W_0. Let $\Delta = \mathrm{End}_{\mathcal{C}}(W_0)$. Then $W \mapsto \mathrm{Hom}_{\mathcal{C}}(W_0, W)$ is an equivalence of categories $\mathcal{C} \cong$ Mod-Δ, which means that $\mathcal{C}(A)$ is equivalent to Mod-$(\Delta \otimes_k A)$. From this all assertions follow (the multiplicity can be computed as quotient of the K-dimensions). □

Remark. In the case $A = M_d(k)$ the categories $\mathcal{C}$ and $\mathcal{C}(A)$ are equivalent (Morita equivalence): a homomorphism $M_d(k) \to \mathrm{End}_{\mathcal{C}}(W)$ is the same as a decomposition $W = \widetilde{W}^d$.

Definition 6.2.2. A (D, φ)-space is a (F, φ)-space (V, φ) plus a homomorphism $i : D^{\mathrm{op}} \to \mathrm{End}(V, \varphi)$ over F. Let $\overline{D} = D \otimes \overline{\mathbb{F}}_q$. The rank

$$\mathrm{rk}_{\overline{D}}(V) := \frac{1}{d^2} \dim_{\overline{F}}(V)$$

is an integer if and only if V is a free $\overline{D}$-module.

Using Lemma 6.2.1 we get as an immediate consequence of Theorem 6.1.3

Corollary 6.2.3. *The abelian category of (D, φ)-spaces is semisimple, any simple object is isotypic as an (F, φ)-space, and for any simple (F, φ)-space there is a unique simple (D, φ)-space of this type.*

Let (V, φ, i) be a simple (D, φ)-space and let (E, Π) be the associated F-pair. Then $\Delta = \mathrm{End}(V, \varphi, i)$ is a finite dimensional central division algebra over E with local invariants

$$\mathrm{inv}_y(\Delta) = \mathrm{inv}_y(D \otimes_F E) - \deg_y(\Pi)\,.$$

The multiplicity of (V, φ) as an (F, φ)-space is $m = d\, d(\Delta)/d(\Pi)$, which implies $\mathrm{rk}_{\overline{D}}(V) = [E\!:\!F]\, d(\Delta)/d$. □

Dieudonné-D_x-modules

For $x \in |X|$ the algebras $D_x = D \otimes_F F_x$ and $\overline{D}_x = D_x \widehat{\otimes}\, \overline{\mathbb{F}}_q$ are Azumaya algebras of dimension d^2 over F_x or $\overline{F}_x$, respectively.

Definition 6.2.4. A Dieudonné-D_x-module (over $\overline{\mathbb{F}}_q$) is a Dieudonné-F_x-module (V, φ) plus a homomorphism $i : D_x^{\mathrm{op}} \to \mathrm{End}(V, \varphi)$ over F_x. Its rank

$$\mathrm{rk}_{\bar{D}_x}(V) := \frac{1}{d^2}\,\mathrm{rk}_{\bar{F}_x}(V)$$

is an integer if and only if V is a free $\bar{D}_x$-module.

As a consequence of Proposition 6.1.5 and Lemma 6.2.1 the category of Dieudonné-D_x-modules is semisimple, and the simple objects are classified by the slopes of the underlying Dieudonné-F_x-modules. The endomorphism algebra of the simple Dieudonné-D_x-module with slope μ is a central division algebra over F_x with the invariant $\mathrm{inv}_x(D) - \mu$.

Lemma 6.2.5. *Let (V, φ, i) be a simple Dieudonné-D_x-module and let Δ be its algebra of endomorphisms. Then $d \cdot \mathrm{rk}_{\bar{D}_x}(V) = d(\Delta)$.*

Proof. We denote by (V', φ') the simple Dieudonné-F_x-module with the same slope as (V, φ) and by Δ' its endomorphisms. Using equation (6.2.1) and Corollary 6.1.6 we calculate $d \cdot \mathrm{rk}_{\bar{D}_x}(V) = \mathrm{rk}_{\bar{F}_x}(V')\, d(\Delta)/d(\Delta') = d(\Delta)$. □

Remark 6.2.6 (Lattices in Dieudonné modules**).** We choose an embedding $k(x) \subset \overline{\mathbb{F}}_q$ over $\mathbb{F}_q$ and write $\bar{\mathcal{D}}_x = \mathcal{D}_x \,\widehat{\otimes}\, \overline{\mathbb{F}}_q$, $\mathcal{D}_{\bar{x}} = \mathcal{D}_x \,\widehat{\otimes}_{k(x)}\, \overline{\mathbb{F}}_q$. Let (V, φ, i) be a Dieudonné-D_x-module with reduced representation (W, ψ, i), i.e. $V = W \oplus \varphi W \oplus \ldots \oplus \varphi^{\deg(x)-1} W$ and $\psi = \varphi^{\deg(x)}$.

Any $\bar{\mathcal{D}}_x$-lattice $M \subseteq V$ (that is a $\bar{\mathcal{D}}_x$-stable $\bar{\mathcal{O}}_x$-lattice) admits a decomposition $M = M_0 \oplus \ldots \oplus M_{\deg(x)-1}$ with $M_i \subseteq \varphi^i W$. There is a natural bijection between $\bar{\mathcal{D}}_x$-lattices satisfying $M_i = \varphi M_{i-1}$ for $i = 1 \ldots \deg(x) - 1$ and $\mathcal{D}_{\bar{x}}$-lattices $M_0 \subseteq W$. Here M is free over $\bar{\mathcal{D}}_x$ if and only if M_0 is free over $\mathcal{D}_{\bar{x}}$.

In the case $x \in |X'|$ there is an isomorphism $\mathcal{D}_{\bar{x}} \cong M_d(\mathcal{O}_{\bar{x}})$, which implies that a $\mathcal{D}_{\bar{x}}$-lattice in W is free if and only if W is free over $D_{\bar{x}}$ if and only if V is free over $\bar{D}_x$.

In the case $x \notin |X'|$ the maximality of $\mathcal{D}_x$ implies that $\mathcal{D}_{\bar{x}}$ is a hereditary order ([CR], Theorem 26.12 and Corollary 26.30), but it is not maximal. A $\mathcal{D}_{\bar{x}}$-lattice $M_0 \subseteq W$ is free if and only if the multiplicities of all simple projective $\mathcal{D}_{\bar{x}}$-modules in M_0 are equal and W is free over $D_{\bar{x}}$. If (V, φ) is trivial, then φ-stable $\bar{\mathcal{D}}_x$-lattices in V are base extensions of $\mathcal{D}_x$-lattices in $V^{\varphi=1} = W^{\psi=1}$ and these are free as soon as W is free over $D_{\bar{x}}$.

Localisation

Let (V, φ, i) be a (D, φ)-space with associated F-pair (E, Π). Its localisation at $x \in X$ is naturally a Dieudonné-D_x-module (V_x, φ_x, i_x). The decomposition of

(V_x, φ_x) with respect to the places $y \mid x$ of E is respected by the D_x-action:

$$(V_x, \varphi_x, i_x) = \bigoplus_{y \mid x} (V_y, \varphi_y, i_y)$$

Proposition 6.2.7. *Let (V, φ, i) be the simple (D, φ)-space with associated F-pair (E, Π) and let Δ be its algebra of endomorphisms. Then the Dieudonné-D_x-module (V_y, φ_y, i_y) is isotypic with slope $\mu(y) = \deg_y(\Pi)/[E_y : F_x]$ and has rank $[E_y : F_x]\, d(\Delta)/d$ over $\bar{D}_x$.*

Proof. This follows from Proposition 6.1.9, using that as an (F, φ)-space (V, φ) is isotypic with multiplicity $d\, d(\Delta)/d(\Pi)$. □

Proposition 6.2.8. *Let (V, φ, i) be an isotypic (D, φ)-space with associated F-pair (E, Π). Then for any $x \in |X|$ the natural map*

$$\mathrm{End}\big(V, \varphi, i\big) \otimes_F F_x \longrightarrow \mathrm{End}_E\big(V_x, \varphi_x, i_x\big) = \prod_{y \mid x} \mathrm{End}_{E_y}\big(V_y, \varphi_y, i_y\big)$$

is an isomorphism.

Proof. The map in question exists because E lies in the centre of $\mathrm{End}(V, \varphi, i)$. We have to show that for a simple (V, φ, i) with $\Delta = \mathrm{End}(V, \varphi, i)$ all maps

$$\Delta \otimes_E E_y \longrightarrow \mathrm{End}_{E_y}\big(V_y, \varphi_y, i_y\big)$$

are bijective. They are injective because the left hand side is a (central) simple E_y-algebra. Thus it suffices to prove equality of the dimensions over E_y of both sides. Using Proposition 6.2.7 and Lemma 6.2.5 we calculate: left dimension $= d(\Delta)^2 = (d\, \mathrm{rk}_{\bar{D}_x}(V_y)/[E_y : F_x])^2 =$ right dimension. □

Definition 6.2.9. A (D, φ)-space (V, φ, i) is called trivial outside X' if its localisations at all $x \in X \setminus X'$ are trivial Dieudonné modules, i.e. if they have pure slope zero.

Denoting by (E, Π) the F-pair associated to (V, φ), an equivalent condition is $\deg_y(\Pi) = 0$ for all $y \mid x$ with $x \in X \setminus X'$. Thus for any simple (D, φ)-space which is trivial outside X', using the notations of Corollary 6.2.3 we have

$$d(\Delta) = \mathrm{lcm}\big(d(\Pi), d(D \otimes_F E)\big)\,. \tag{6.2.2}$$

Proposition 6.2.10. *For any simple (D, φ)-space which is trivial outside X' its multiplicity as an (F, φ)-space is a multiple of d and equals d if and only if $d(\Pi)\, [E_y : F_x]\, \mathrm{inv}_x(D) = 0$ for all $y \mid x$.*

Proof. By Corollary 6.2.3 the multiplicity in question is $d\,d(\Delta)/d(\Pi)$. In view of (6.2.2) this is a multiple of d with equality if and only if $d(D \otimes E) \mid d(\Pi)$. □

Proposition 6.2.11. *Any (D, φ)-space which is trivial outside X' is free over $\bar{D}$, i.e. its rank is an integer. The (D, φ)-spaces which are trivial outside X' of rank 1 are precisely the simple (D, φ)-spaces whose associated F-pair (E, Π) satisfies the following conditions.*

1. $\deg_y(\Pi) = 0$ *for all places y of E lying over $X \setminus X'$,*
2. $\dfrac{d}{[E:F]} \cdot \Pi$ *lies in* $\mathrm{Div}^0(E) \subset \mathrm{Div}^0(E) \otimes \mathbb{Q}$ *,*
3. *there is an embedding $E \subseteq D$ over F.*

In this case we have $d(\Delta) = d(D \otimes E) = d/[E:F]$.

Proof. By Corollary 6.2.3 the rank of a simple (D, φ)-space is $[E:F]\,d(\Delta)/d$. If the space is trivial outside X' then by (6.2.2) this is an integer multiple of $[E:F]\,d(D \otimes E)/d$ which is an integer by Lemma 6.2.12 below. The rank equals 1 if and only if $[E:F]\,d(D \otimes E) = d$ and $d(\Pi) \mid d(D \otimes E)$. Using Lemma 6.2.12 again, the former condition is equivalent to (3). If it holds then the latter condition is equivalent to (2). □

Lemma 6.2.12. *Let k be a field, let B be a central division algebra over k of dimension d^2 and let K be a finite extension of k. Then*

$$d \;\Big|\; [K:k]\, d(B \otimes K)$$

with equality if and only if there is a k-embedding $K \subseteq B$.

Proof. The unique simple $B \otimes K$-module M has dimension $d\,[K:k]\,d(B \otimes K)$ over k. Since M is a free B-module this implies $d^2 \mid d\,[K:k]\,d(B \otimes K)$, and equality means that as a B-module M is isomorphic to B. An action of $B \otimes K$ on B is the same as a homomorphism $K \to B$. □

6.3 Transfer of conjugacy classes

Conjugacy classes in D^*

We fix a finite subset $T \subset |X'|$, and for any $x \in T$ we choose an integer $r_x \geq 1$.

Definition 6.3.1. An element $\delta \in D^*$ is called $\underline{r}$-admissible if for any $x \in T$ its image $\delta_x \in D_x^* \cong \mathrm{GL}_d(F_x)$ is the norm of an element of $\mathrm{GL}_d(F_{x,r_x})$ and if $\sum_{x \in T} x(\det \delta)/r_x = 0$. We denote by $D^*_{\natural,\underline{r}}$ the set of $\underline{r}$-admissible conjugacy classes.

Lemma 6.3.2. *For $\delta \in D^*$ the algebra $F' = F[\delta]$ is a field. The element δ is $\underline{r}$-admissible if and only if for all places x' of F' with $x' \mid x \in T$ the expression*

$$\frac{d}{[F':F]}\, f(x' \mid x) x'(\delta)/r_x \tag{6.3.1}$$

is an integer and if the sum of these (finitely many) numbers is zero.

Proof. The first assertion is clear. For $x \in T$ the algebra $F'_x = F_x[\delta_x]$ is the product of the completions $F'_{x'}$ at the places $x' \mid x$, so $\delta_x \in \mathrm{GL}_d(F_x)$ is semisimple and we have a decomposition $F_x^d \cong \bigoplus_{x' \mid x} V_{x'}$ plus elliptic elements $\delta_{x'} \in \mathrm{GL}(V_{x'})$. By Lemma 5.1.2 δ_x is a norm if and only if all $x(\det \delta_{x'})$ are integer multiples of r_x. Since F_x^d is free over F'_x of rank $d/[F':F]$, the quotient $x(\det \delta_{x'})/r_x$ coincides with (6.3.1). □

Let $\mathcal{M}$ be the set of isomorphism classes of finite field extensions of F plus a fixed generator. By the theorem of Skolem-Noether the assignment $\delta \mapsto (F[\delta], \delta)$ defines an injection $D^*_{\natural, \underline{r}} \hookrightarrow \mathcal{M}$.

Conjugacy classes of automorphisms of (D, φ)-spaces

As before we fix a finite set $T \subset |X'|$ and integers $r_x \geq 1$ for all $x \in T$.

Definition 6.3.3. Let (V, φ, i) be a simple (D, φ)-space which is concentrated in T (that means its localisations outside T are trivial) and let $\Delta = \mathrm{End}(V, \varphi, i)$. An element $\delta \in \Delta^*$ is called $\underline{r}$-admissible if for any $x \in T$ there is a $\overline{\mathcal{D}}_x$-lattice $M \subset V_x$ satisfying $\delta M = \varphi_x^{r_x \deg(x)} M$. We denote by $\Delta^*_{\natural, \underline{r}}$ the set of $\underline{r}$-admissible conjugacy classes.

Lemma 6.3.4. *In the situation of Definition 6.3.3 let (E, Π) be the F-pair associated to (V, φ). An element $\delta \in \Delta^*$ is $\underline{r}$-admissible if and only if $E \subseteq F[\delta]$ and for all places x' of $F[\delta]$ and y of E with $x' \mid y \mid x \in T$ we have*

$$r_x \deg(x) \frac{y(\Pi)}{e(y \mid x)} = \frac{x'(\delta)}{e(x' \mid x)}\,. \tag{6.3.2}$$

Proof. Let $F' = F[\delta]$ and $E' = E[\delta]$ as subalgebras of Δ. These are finite field extensions of F with $E' = E \cdot F'$. The decompositions of the localisations $V_x = \bigoplus_{y \mid x} V_y$ are refined by $V_y = \bigoplus_{y' \mid y} V_{y'}$.

By definition δ is $\underline{r}$-admissible if and only if for any place y' of E' lying over some $x \in T$ the δ^{-1}-twist of the Dieudonné-F_{x,r_x}-module $(V_{y'}^{(r_x)}, \varphi_{y'}^{(r_x)})$ is trivial. We know that (V_y, φ_y) is isotypic with slope $\deg(x) y(\Pi)/e(y \mid x)$. Thus using Lemma 6.1.8 and Lemma 6.1.7, admissibility of δ is equivalent to equation (6.3.2) for all y' lying over T (each y' determines y and x').

Since (V, φ) is trivial outside T, these equations determine $\Pi \in E^* \otimes \mathbb{Q}$ uniquely and imply $\Pi \in F'^* \otimes \mathbb{Q}$, so $E \subseteq F'$ by minimality of E. □

Corollary 6.3.5. *The assignment $\delta \mapsto (F[\delta], \delta)$ defines an injective map*

$$\bigsqcup_{(V,\varphi,i)} \Delta^*_{\natural,\underline{r}} \longrightarrow \mathcal{M}$$

where (V, φ, i) runs through a system of representatives of the simple (D, φ)-spaces which are concentrated in T.

Proof. The images of different $\Delta^*_{\natural,\underline{r}}$ are disjoint because the F-pair (E, Π) can be reconstructed using the equations (6.3.2). These equations also imply that for fixed (V, φ, i) any isomorphism of two pairs $(F[\delta], \delta)$ with $\underline{r}$-admissible $\delta \in \Delta^*$ induces the identity on $E \subseteq F[\delta]$. Thus the theorem of Skolem-Noether can be applied. □

Transfer of conjugacy classes

The main result of this section is

Proposition 6.3.6. *In $\mathcal{M}$ the image of $D^*_{\natural,\underline{r}}$ coincides with the (disjoint) union of the images of the $\Delta^*_{\natural,\underline{r}}$ for all simple (D, φ)-spaces of rank 1 which are concentrated in T. In other words there is a natural bijection $D^*_{\natural,\underline{r}} = \bigsqcup \Delta^*_{\natural,\underline{r}}$.*

Proof. For each pair $(F', \delta) \in \mathcal{M}$ we define $\Pi' \in \mathrm{Div}(F') \otimes \mathbb{Q}$ by

$$\begin{aligned} r_x \deg(x)\, x'(\Pi') &= x'(\delta) && \text{for } x' \mid x \in T \\ x'(\Pi') &= 0 && \text{otherwise} \end{aligned} \tag{6.3.3}$$

In the case $\deg(\Pi') = 0$, that is $\Pi \in F'^* \otimes \mathbb{Q}$, we also set $E = F[\Pi']$ and $\Pi = \Pi' \in E^* \otimes \mathbb{Q}$. The classification of the simple (D, φ)-spaces of rank 1 in Proposition 6.2.11 and Lemma 6.3.4 imply that the union of the $\Delta^*_{\natural,\underline{r}} \subset \mathcal{M}$ is characterised by the following conditions on (F', δ).

(1) $\deg(\Pi') = 0$,

(2) there is an embedding $E \subseteq D$ over F,

(3) for all $y \mid x \in T$ the number $\dfrac{d}{[E : F]} \deg_y(\Pi)$ is an integer,

(4) there is an embedding $F' \subseteq \Delta$ over E.

On the other hand we can rewrite the first line of (6.3.3) as

$$\deg_{x'}(\Pi') = f(x' \,|\, x) x'(\delta) \big/ r_x$$

Thus by Lemma 6.3.2 the subset $D^*_{\natural,\underline{r}} \subset \mathcal{M}$ is given by the following conditions. (Here (2) and (4′) just say that there is an embedding $F' \subseteq D$ over F.)

(1) $\deg(\Pi') = 0$,

(2) there is an embedding $E \subseteq D$ over F,

(3′) for all $x' \,|\, x \in T$ the number $\dfrac{d}{[F' \!:\! F]} \deg_{x'}(\Pi')$ is an integer,

(4′) there is an embedding $F' \subseteq \mathrm{Cent}_D(E)$ over E.

We have the implication (3′)⇒(3) which follows from the equations

$$\sum_{x' \,|\, y} \deg_{x'}(\Pi') = [F' \!:\! E] \deg_y(\Pi).$$

Thus we may assume that (1) – (3) hold and have to show (4) ⇔ (3′ ∧ 4′). Then Proposition 6.2.11 implies $d(\Delta) = d(D \otimes E)$, i.e. Δ and the division algebra $\mathrm{Cent}_D(E)$, which is equivalent to $D \otimes E$, have the same E-dimension $(d/[E \!:\! F])^2$. By Lemma 6.2.12 there is an embedding $F' \subseteq \Delta$ over E if and only if for all places x' of F' we have

$$\frac{d}{[F' \!:\! F]} \cdot \mathrm{inv}_{x'}(\Delta \otimes_E F') = 0$$

in $\mathbb{Q}/\mathbb{Z}$. The local invariants of this algebra are

$$\mathrm{inv}_{x'}(\Delta \otimes_E F') = \begin{cases} \mathrm{inv}_{x'}(D \otimes F') & \text{for } x' \text{ over } X \setminus X' \\ -\deg_{x'}(\Pi) & \text{for } x' \text{ over } X' \end{cases}$$

which implies the equivalence in question. □

In the application of Proposition 6.3.6 it will be necessary to know that two corresponding elements $\delta \in \Delta^*$ and $\delta' \in D^*$ are locally conjugate in the following sense.

We denote by $\delta'^T \in D^T_{\mathbb{A}}$ and by $\delta'_x \in D_{x,r_x}$ for $x \in T$ the images of δ'.

Since $V \mathbin{\widehat{\otimes}} \mathbb{A}^T$ admits a φ-invariant basis there is an isomorphism

$$D^T_{\mathbb{A}} \cong \mathrm{End}(V \mathbin{\widehat{\otimes}} \mathbb{A}^T, \varphi \mathbin{\widehat{\otimes}} \mathrm{id}, i \mathbin{\widehat{\otimes}} \mathrm{id})$$

which is well defined up to inner automorphisms. Similarly, for $x \in T$ there are isomorphisms $\mathrm{End}(V_x^{\varphi_x^{r_x \deg(x)} = \delta}) \cong D_{x,r_x}$ which are well defined up to inner automorphisms. We denote by $\delta^T \in D^T_{\mathbb{A}}$ and by $\delta_x \in D_{x,r_x}$ the images of δ.

Proposition 6.3.7. *Suppose $\delta \in \Delta^*$ and $\delta' \in D^*$ are $\underline{r}$-admissible and correspond to each other under the bijection in Proposition 6.3.6. Then each of the pairs (δ^T, δ'^T) and (δ_x, δ'_x) for $x \in T$ consists of two conjugate elements.*

Proof. We fix the isomorphism $F[\delta] \cong F[\delta']$ given by $\delta \mapsto \delta'$.

It is clear that $D \otimes \overline{\mathbb{F}}_q$ is a free module over $F[\delta'] \otimes \overline{\mathbb{F}}_q$. Moreover V is free over $F[\delta] \otimes \overline{\mathbb{F}}_q$ because φ permutes transitively the components of V arising from a decomposition of $F[\delta] \otimes \overline{\mathbb{F}}_q$ into a product of fields. Using Lemma 6.3.8 below this implies that δ and δ' are conjugate in $\mathrm{End}(V) \cong D \otimes \overline{\mathbb{F}}_q$. Then δ^T and δ'^T are conjugate in $D^T_{\mathbb{A}} \widehat{\otimes} \overline{\mathbb{F}}_q$ and thus in $D^T_{\mathbb{A}}$ as well.

Using Lemma 6.3.8 again, for $x \in T$ we have to show that $\mathrm{End}(V_x^{\varphi_x^{r_x \deg(x)} = \delta})$ is a free module over $F[\delta] \otimes F_{x,r_x}$. An equivalent condition is that V_x is free over $F[\delta] \otimes \bar{F}_x$. This is the case because V is free over $F[\delta] \otimes \overline{\mathbb{F}}_q$. □

Lemma 6.3.8. *Let K be a field, let A be a central simple K-algebra, and let B be a finite dimensional semisimple K-algebra. Two morphisms of K-algebras $f : B \to A$ are conjugate by an element of A if and only if the structures of A as B-modules induced by the f's are isomorphic.* □

Proof. Let M_f by the $B \otimes A^{\mathrm{op}}$-module A on which B acts by left multiplication via f and A by right multiplication. Two f's are conjugate if and only if the associated modules M_f are isomorphic. However two $B \otimes A^{\mathrm{op}}$-modules are isomorphic as soon as they are isomorphic as B-modules. □

7 Computation of Fixed Points

In this section we calculate the cardinalities of the groupoids of fixed points $\#\mathrm{Fix}_I^{\lambda}(g, \underline{a})(\underline{z})$ in the general case. If $I \neq \emptyset$ they are in fact sets, but this will not be used.

For the notations we refer to section 3.6. The choice of $\underline{z} \in \Lambda_{(T)}(\overline{\mathbb{F}}_q)$ is equivalent to the choice of pairwise different closed points $x_1 \dots x_r \in T \cap X'$ plus embeddings $k(x_i) \subset \overline{\mathbb{F}}_q$ over $\mathbb{F}_q$. Since the groupoids of fixed points do not change when T is made smaller, we may assume the minimal case $T = \{x_1 \dots x_r\}$, in particular $T \subset |X'|$. For $x = x_i$ we write $\lambda_x = \lambda_i$ etc.

7.1 $\mathcal{D}$-shtukas over $\overline{\mathbb{F}}_q$

The generic fibre of a $\mathcal{D}$-shtuka $\mathcal{E}^\bullet = [\,\mathcal{E}_0 \doteq \dots \doteq \mathcal{E}_r \cong {}^{\tau}\mathcal{E}_0\,]$ over $\overline{\mathbb{F}}_q$ is the right $\bar{D}$-module $V = \mathcal{E}_0 \otimes_{\mathcal{O}_{\bar{X}}} \bar{F}$ plus the isomorphism $\varphi : {}^{\tau}V \cong V$ which is induced by the given modifications. This isomorphism can be viewed as a σ_q-linear bijective map $\varphi : V \to V$. If we denote by $i : D \to \mathrm{End}(V, \varphi)$ the given action, then the triple (V, φ, i) is a (D, φ)-space of rank 1 in the sense of Definition 6.2.2.

Definition 7.1.1. Let $\mathrm{Sht}_I^{\underline{\lambda}}(\underline{z}) \subseteq \mathrm{Sht}_I^{\underline{\lambda}}(\overline{\mathbb{F}}_q)$ be the inverse image of $\underline{z}$ under the characteristic morphism $\mathrm{Sht}_I^{\underline{\lambda}} \to (X' \setminus I)^r$ and let

$$\mathcal{X} = \mathrm{Sht}^{\underline{\lambda},T}(\underline{z}) = \varprojlim_{I \cap T = \emptyset} \mathrm{Sht}_I^{\underline{\lambda}}(\underline{z})\,.$$

For a given (D,φ)-space (V,φ,i) of rank 1 we denote by $\mathcal{Y}_{(V,\varphi,i)}$ the set of pairs $(\mathcal{E}^\bullet, j)$ with $\mathcal{E}^\bullet \in \mathcal{X}$ and an isomorphism j between the generic fibre of $\mathcal{E}^\bullet$ and (V,φ,i).

Let a pair $(\mathcal{E}^\bullet, j) \in \mathcal{Y}_{(V,\varphi,i)}$ be given. For any $x \in X$ the sequence $\mathcal{E}_0 \dots \mathcal{E}_r$ defines a sequence of $\overline{\mathcal{D}}_x$-lattices in the localisation $V_x = V \mathbin{\widehat{\otimes}}_F F_x$ with the following properties.

In the case $x \notin T$ all lattices are equal to one lattice M_x satisfying $\varphi_x M_x = M_x$. In particular (V,φ,i) is trivial outside T. Such a lattice M_x is the same as the $\mathcal{D}_x$-lattice $M_x^{\varphi_x=1}$ in $V_x^{\varphi_x=1}$. A level structure outside T for $\mathcal{E}^\bullet$ is equivalent to an isomorphism $\mathcal{D}_{\mathbb{A}}^T \cong \prod_{x\notin T} M_x^{\varphi_x=1}$, that is an isomorphism $y^T : D_{\mathbb{A}}^T \cong (V \mathbin{\widehat{\otimes}} \mathbb{A}^T)^{\varphi=1}$ respecting the given lattices.

In the case $x = x_i \in T$ the modules $\mathcal{E}_0 \dots \mathcal{E}_{i-1}$ determine the same lattice M_x' and $\mathcal{E}_i \dots \mathcal{E}_r$ determine the twisted lattice $\varphi_x M_x'$. Outside z_x these coincide, while their relative position at z_x is $\mathrm{inv}_{z_x}(M_x', \varphi_x M_x') = \lambda_x$. Let $W_x \subseteq V_x$ be the subspace where $k(x)$ acts via the embedding $k(x) \subset \overline{\mathbb{F}}_q$ given by z_x and let $\psi_x = \varphi_x^{\deg(x)}$. Then a $\overline{\mathcal{D}}_x$-lattice M_x' in V_x with the given properties is the same as a $\mathcal{D}_{\bar{x}}$-lattice M_x in W_x satisfying $\mathrm{inv}(M_x, \psi_x M_x) = \lambda_x$.

From this construction we see that if $\mathcal{Y}_{(V,\varphi,i)}$ is not empty, then the (D,φ)-space (V,ϖ,i) is trivial outside X' and thus simple by Proposition 6.2.11. In particular $\Delta = \mathrm{End}(V,\varphi,i)$ is a finite dimensional division algebra over F. The groupoid $\mathcal{X}$ is the disjoint union over the occurring (D,φ)-spaces of the quotients $\Delta^* \backslash \mathcal{Y}_{(V,\varphi,i)}$ where Δ^* acts by twisting j.

Definition 7.1.2. For a given (D,φ)-space (V,φ,i) of rank 1 which is concentrated in T we denote by $\mathcal{Y}^T$ the set of isomorphisms $y^T : D_{\mathbb{A}}^T \cong (V \mathbin{\widehat{\otimes}} \mathbb{A}^T)^{\varphi=1}$ and by $\mathcal{M}_T$ the set of families $M_T = (M_x)_{x\in T}$ of $\mathcal{D}_{\bar{x}}$-lattices in W_x such that $\mathrm{inv}(M_x, \psi_x M_x) = \lambda_x$.

Lemma 7.1.3. *The map $\mathcal{Y}_{(V,\varphi,i)} \to \mathcal{Y}^T \times \mathcal{M}_T$ defined by y^T and M_T constructed above is bijective.*

Proof. We only have to show that for given M_T and y^T the corresponding lattices $M_x' \subset V_x$ for $x \in T$ and $M_x \subset V_x$ for $x \notin T$ come from a $\mathcal{D}$-shtuka with generic fibre (V,φ,i).

Since any locally free $\mathcal{O}_{\overline{X}}$-module $\mathcal{E}$ with generic fibre V coincides with these lattices outside a finite set of points, there are unique locally free $\mathcal{O}_{\overline{X}}$-modules $\mathcal{E}_i$ with generic fibre V which give rise to the given lattices at all places.

These $\mathcal{E}_i$ are in fact $\bar{\mathcal{D}}$-modules and φ extends to an isomorphism ${}^{\tau}\mathcal{E}_0 \cong \mathcal{E}_r$, so it remains to see that the $\mathcal{E}_i$ are locally free over $\bar{\mathcal{D}}$. By Lemma 1.2.5 this is equivalent to the corresponding lattices in V_x being free for all $x \in X$. In the case $x \in X'$ all lattices in V_x are free over $\bar{\mathcal{D}}_x$ because this is a maximal order. In the case $x \notin T$ our lattices are φ_x-stable and thus free because $\mathcal{D}_x$ is a maximal order for all $x \in X$. In view of $T \subset |X'|$ this finishes the proof. □

Using the bijection of Lemma 7.1.3 the relevant actions on $\mathcal{Y}_{(V,\varphi,i)}$ take the following form.

- $\delta \in \Delta^*$ acts from the left by $M_x \mapsto \delta M_x$ and $y^T \mapsto \delta \circ y^T$
- $g \in (D_{\mathbb{A}}^T)^*$ acts from the right on $\mathcal{Y}^T$ by $y^T \mapsto y^T \circ g$
- the partial Frobenius $\mathrm{Fr}_i^{\deg(x_i)}$ acts on $\mathcal{M}_T$ by $M_{x_i} \mapsto \psi_{x_i} M_{x_i}$

7.2 Adelic description of the groupoids of fixed points

Now we assume $T \cap T_a = \emptyset$ and choose in addition to $\underline{\lambda}$ and $\underline{z}$ a finite closed subscheme $I \subset X$ with $I \cap T = \emptyset$, an element $g \in (D_{\mathbb{A}}^T)^*$, and a sequence of positive integers $\underline{a} = (a_1 \dots a_r)$ with $\deg(x_i) \,|\, a_i$. Hence $r_i = a_i / \deg(x_i)$ are positive integers as well. We still assume $T = \{x_1, \dots, x_r\}$.

In order to describe $\mathrm{Fix} = \mathrm{Fix}_I^{\underline{\lambda}}(g, \underline{a})(\underline{z})$ we define an auxiliary set $\widetilde{\mathrm{Fix}}_{(V,\varphi,i)}$ by the following 2-cartesian diagram. Its right half is a repetition of the definition of Fix.

$$\begin{array}{ccccc}
\widetilde{\mathrm{Fix}}_{(V,\varphi,i)} & \longrightarrow & \mathrm{Fix} & \longrightarrow & \mathcal{X}/K_I a^{\mathbb{Z}} \\
\downarrow & \Box & \downarrow & \Box & \downarrow{\scriptstyle (\mathrm{Fr}^{\underline{a}},\mathrm{id})} \\
\mathcal{Y}_{(V,\varphi,i)}/(K_I \cap {}^gK_I) & \longrightarrow & \mathcal{X}/(K_I \cap {}^gK_I)a^{\mathbb{Z}} & \xrightarrow{(1,g)} & \mathcal{X}/K_I a^{\mathbb{Z}} \times \mathcal{X}/K_I a^{\mathbb{Z}}
\end{array}$$

Then Fix is the disjoint union over the (D,φ)-spaces of rank 1 which are concentrated in T of the quotients $\Delta^* \backslash \widetilde{\mathrm{Fix}}_{(V,\varphi,i)}/a^{\mathbb{Z}}$.

Using Lemma 7.1.3 the definition of $\widetilde{\mathrm{Fix}} = \widetilde{\mathrm{Fix}}_{(V,\varphi,i)}$ can be expressed by the following 2-cartesian diagram.

$$\begin{array}{ccc}
\widetilde{\mathrm{Fix}} & \longrightarrow & \Delta^* \backslash (\mathcal{M}_T \times \mathcal{Y}^T/K_I)/a^{\mathbb{Z}} \\
\downarrow & \Box & \downarrow{\scriptstyle (\mathrm{Fr}^{\underline{a}},\mathrm{id})} \\
\mathcal{Y}^T/(K_I \cap {}^gK_I) \times \mathcal{M}_T & \xrightarrow{(1,g)} & \left[\Delta^* \backslash (\mathcal{Y}^T/K_I \times \mathcal{M}_T)/a^{\mathbb{Z}}\right]^2
\end{array}$$

Since K_I acts freely on $\mathcal{Y}^T$, this implies that Fix is the disjoint union over the (D,φ)-spaces of rank 1 concentrated in T of the quotients $\Delta^*\backslash\mathcal{Z}/a^{\mathbb{Z}}$ with

$$\mathcal{Z} \subseteq \mathcal{Y}^T/(K_I \cap {}^gK_I) \times \mathcal{M}_T \times \Delta^* \times \mathbb{Z} \ni (\bar{y}^T, M_T, \delta, n)$$

given by the conditions $y^T g K_I = \delta a^n y^T K_I$ and $\delta M_x = \psi_x^{r_x} M_x$ for $x \in T$. The induced action of Δ^* on the δ-component of $\mathcal{Z}$ is by conjugation.

Let $\Delta^*_{\natural,\underline{r}}$ be a system of representatives of the conjugacy classes in Δ^* for which the second condition can be fulfilled, i.e. the $\underline{r}$-admissible conjugacy classes in the sense of Definition 6.3.3. Then $\Delta^*\backslash\mathcal{Z}/a^{\mathbb{Z}}$ is the disjoint union over $\delta \in \Delta^*_{\natural,\underline{r}}$ and $n \in \mathbb{Z}$ of the groupoids

$$\Delta^*_\delta\backslash\mathcal{Z}_{\delta,n}/a^{\mathbb{Z}},$$

with $\mathcal{Z}_{\delta,n} \subseteq \mathcal{Y}^T/(K_I \cap {}^gK_I) \times \mathcal{M}_T$ given by the two conditions which define $\mathcal{Z}$.

We choose an element $y_0^T \in \mathcal{Y}^T$ (possible because (V,φ,i) is concentrated in T) and set $\delta_0^T = y_0^{T,-1}\delta a^n y_0^T \in (D_{\mathbb{A}}^T)^*$. Then we have

$$\begin{aligned}
&\left\{y^T \in \mathcal{Y}^T \,\middle|\, y^T g K_I = \delta a^n y^T K_I\right\} \big/ (K_I \cap {}^gK_I) \\
&= \left\{y^T \in \mathcal{Y}^T \,\middle|\, y^{T,-1}\delta a^n y^T \in K_I g K_I\right\} \big/ K_I \\
&\cong \left\{y^T \in (D_{\mathbb{A}}^T)^* \,\middle|\, y^{T,-1}\delta_0^T a^n y^T \in K_I g K_I\right\} \big/ K_I\,.
\end{aligned}$$

We denote by $\mathcal{Y}_T = \{y_x : D_{x,r_x} \cong W_x^{\psi_x^{r_x}=\delta} \text{ for } x \in T\}$ the set of trivialisations of the twisted Dieudonné modules, i.e. $\mathcal{Y}_T/\prod_{x\in T}\mathcal{D}^*_{x,r_x}$ is the set of lattices M_T satisfying $\psi_x^{r_x}M_x = \delta M_x$. We choose an element $y_{0,T} \in \mathcal{Y}_T$ (possible by the hypothesis on δ). Since δ and ψ_x commute, both of them fix $W_x^{\psi_x^{r_x}=\delta}$, which allows to define $\delta_{0,x} = y_{0,x}^{-1}\delta y_{0,x} \in D^*_{x,r_x}$ and $\psi_{0,x} = y_{0,x}^{-1}\psi_x y_{0,x} \in \mathrm{Aut}^{\sigma_x}(D_{x,r_x})$. We can write $\psi_{0,x} = \gamma_x \cdot \sigma_x$ with an element $\gamma_x \in D^*_{x,r_x}$ satisfying $N_{r_x}(\gamma_x) = \delta_{0,x}$. Then we have

$$\begin{aligned}
&\left\{M_T \in \mathcal{M}_T \,\middle|\, \delta M_x = \psi_x^{r_x}\right\} \\
&= \left\{y_T \in \mathcal{Y}_T \,\middle|\, y_x^{-1}\circ\psi_x\circ y_x \in \mathcal{D}^*_{x,r_x}\circ\lambda_x(\varpi_x)\sigma_x\circ\mathcal{D}^*_{x,r_x}\right\} \big/ \textstyle\prod_{x\in T}\mathcal{D}^*_{x,r_x} \\
&\cong \left\{y_T \in \textstyle\prod_{x\in T} D^*_{x,r_x} \,\middle|\, y_x^{-1}\gamma_x\sigma_x(y_x) \in \mathcal{D}_{x,r_x}\lambda_x(\varpi_x)\mathcal{D}_{x,r_x}\right\} \big/ \textstyle\prod_{x\in T}\mathcal{D}^*_{x,r_x}\,.
\end{aligned}$$

The chosen isomorphisms y_0^T and $y_{0,T}$ define an isomorphism

$$y_0 \;:\; D_{\mathbb{A}}\,\widehat{\otimes}\,\overline{\mathbb{F}}_q \cong V\,\widehat{\otimes}\,\mathbb{A}\,.$$

We can summarise the preceding considerations as follows.

Proposition 7.2.1. *For given $(\underline{\lambda}, I, g, \underline{a}, \underline{z})$ as above the groupoid $\mathrm{Fix}_I^{\underline{\lambda}}(g,\underline{a})(\underline{z})$ is bijective to the disjoint union over the (D,φ)-spaces (V,φ,i) of rank 1 which are* *concentrated in T, over $\delta \in \Delta^*_{\natural,\underline{r}}$ with the notation $\Delta = \mathrm{End}(V,\varphi,i)$, and over $n \in \mathbb{Z}$ of the groupoids*

$$\Delta^*_\delta \backslash \mathcal{Z}'_{\delta,n}/K_I a^{\mathbb{Z}} \times \prod_{x\in T} \mathcal{D}^*_{x,r_x}$$

where $\mathcal{Z}'_{\delta,n} \subseteq (D^T_{\mathbb{A}})^* \times \prod_{x\in T} D^*_{x,r_x}$ denotes the set of all (y^T, y_T) satisfying

$$\begin{aligned} (y^T)^{-1}\delta_0^T a^n y^T &\in K_I g K_I \\ y_x^{-1}\gamma_x\sigma_x(y_x) &\in \mathcal{D}^*_{x,r_x}\lambda_x(\varpi_x)\mathcal{D}^*_{x,r_x} \quad \textit{for } x \in T. \end{aligned} \tag{7.2.1}$$

The action of $\alpha \in \Delta^*_\delta$ on $\mathcal{Z}'_{\delta,n}$ is given by $y \mapsto y_0^{-1}\alpha y_0\, y$. □

7.3 Integral representation

Using the following general principle the adelic description of the groupoids of fixed points in Proposition 7.2.1 can be transformed into an integral representation of their cardinalities.

Lemma 7.3.1. *Let G be a locally profinite topological group, let $K \subseteq G$ be a compact open subgroup, and let μ be the left invariant measure on G normalised by $\mu(K) = 1$. Let $\Gamma \subseteq H \subseteq G$ be closed subgroups such that H is unimodular and Γ is discrete, and let $U \subseteq G$ be an open subset which is left-stable under H and right-stable under K. Then $\mathcal{G} = \Gamma\backslash U/K$ is a groupoid with finite automorphism groups having the cardinality*

$$\#\mathcal{G} = \frac{\mu}{\nu_0}(\Gamma\backslash U) = \frac{\nu}{\nu_0}(\Gamma\backslash H)\cdot\frac{\mu}{\nu}(H\backslash U)$$

Here ν is an arbitrary biinvariant measure on H and ν_0 is the counting measure on Γ. □

In the situation of Proposition 7.2.1 with fixed (V,φ,i), δ, n we choose

- $G = (D^T_{\mathbb{A}})^* \times \prod_{x\in T} D^*_{x,r_x}$
- $K = K^T_I \times \prod_{x\in T} \mathcal{D}^*_{x,r_x}$
- $H = y_0^{-1}\cdot(\Delta\otimes\mathbb{A})^*_\delta\cdot y_0$
- $\Gamma = y_0^{-1}\cdot\Delta^*_\delta a^{\mathbb{Z}}\cdot y_0$

and $U \subseteq G$ is defined by the conditions (7.2.1).

Lemma 7.3.2. *We have $H = (D^T_{\mathbb{A}})^*_{\delta_0^T} \times \prod_{x\in T}(D^*_{x,r_x})^{\sigma_x}_{\gamma_x}$.*

Proof. Let (E,Π) be the F-pair associated to (V,φ). Since δ is $\underline{r}$-admissible, Lemma 6.3.4 implies $E \subseteq F[\delta]$. Thus by Proposition 6.2.8 we have $(\Delta\otimes\mathbb{A})_\delta = \mathrm{End}(V \mathbin{\widehat{\otimes}} \mathbb{A}, \varphi \mathbin{\widehat{\otimes}} \mathrm{id}, i \mathbin{\widehat{\otimes}} \mathrm{id})_\delta$. Outside T this is the assertion, while for $x \in T$ we have to note in addition that $((D_{\bar{x}})^{\sigma_x}_{\gamma_x})_{\delta_{0,x}} = (D_{x,r_x})^{\sigma_x}_{\gamma_x}$. □

Let μ_x on D^*_{x,r_x} for $x \in T$ and μ^T on $(D^T_{\mathbb{A}})^*$ be the invariant measures satisfying $\mu_x(\mathcal{D}^*_{x,r_x}) = 1$ and $\mu^T(\mathcal{D}^T_{\mathbb{A}}) = 1$. We choose arbitrary invariant measures ν_x on $(\Delta \otimes F_x)^*_\delta \cong (D^*_{x,r_x})^{\sigma_x}_{\gamma_x}$ and ν^T on $(\Delta \otimes \mathbb{A}^T)^*_\delta \cong (D^T_{\mathbb{A}})^*_{\delta_0^T}$ and denote by ν their product measure on $(\Delta \otimes \mathbb{A})^*_\delta$. Proposition 7.2.1 then implies

$$\#\mathrm{Fix}^{\underline{\lambda}}_I(g, \underline{a})(\underline{z}) \cdot \mu^T(K^T_I) = \sum \frac{\nu}{\nu_0} \left(\Delta^*_\delta a^{\mathbb{Z}} \backslash (\Delta \otimes \mathbb{A})^*_\delta\right) \cdot \mathrm{O}^T_{\delta_0^T a^n}(\mathbb{1}_{K^T_I g K^T_I}, d\nu^T) \cdot \prod_{x \in T} \mathrm{TO}_{x,\gamma_x}(\mathbb{1}_{\mathcal{D}^*_{x,r_x} \lambda_x(\varpi_x) \mathcal{D}^*_{x,r_x}}, d\nu_x)$$

where the sum is over the (D, φ)-spaces of rank 1 which are concentrated in T, over $\delta \in \Delta^*_{\natural,\underline{r}}$, and over $n \in \mathbb{Z}$, using the notation

$$\mathrm{O}^T_{\delta_0^T a^n}(f^T, d\nu^T) = \int_{(D^T_{\mathbb{A}})^*_{\delta_0^T} \backslash (D^T_{\mathbb{A}})^*} f^T(y^{-1} \delta_0^T a^n y) \, \frac{d\mu^T}{d\nu^T} \, ,$$

$$\mathrm{TO}_{x,\gamma_x}(f_x, d\nu_x) = \int_{(D^*_{x,r_x})^{\sigma_x}_{\gamma_x} \backslash D^*_{x,r_x}} f_x(y^{-1} \gamma_x \sigma_x(y)) \, \frac{d\mu_x}{d\nu_x} \, .$$

Now the summation over the (D, φ)-spaces can be replaced as follows.

By Proposition 6.3.6 for any $\underline{r}$-admissible $\delta \in \Delta^*$ there is an embedding $F[\delta] \subseteq D$ over F, and this determines a bijection between the disjoint union of the occurring $\Delta^*_{\natural,\underline{r}}$ and $D^*_{\natural,\underline{r}}$ (Definition 6.3.1). Proposition 6.3.7 implies that the isomorphism y_0 can be chosen such that the images $\delta^T \in D^T_{\mathbb{A}}$ and $\delta_x \in D_{x,r_x}$ coincide with δ_0^T and $\delta_{0,x}$.

The algebraic groups D^*_δ and Δ^*_δ over $F[\delta]$ are two inner forms of $\mathrm{GL}_{d'}$ with $d' = d/[F[\delta]\!:\!F]$. As explained in [Lau96] (3.5) therefore the invariant measure ν on $(\Delta \otimes \mathbb{A})^*_\delta$ can be transferred to a measure on $(D \otimes \mathbb{A})^*_\delta$. More precisely, for $x \in T$ the measure ν_x is transferred from $(\Delta \otimes F_x)^*_\delta$ to $(D \otimes F_x)^*_\delta$ and this is multiplied by ν^T. The equality of Tamagawa numbers of inner forms of $\mathrm{GL}_{d'}$ then means (cf. [Laf97] III.6, Lemma 3)

$$\frac{\nu}{\nu_0} \left(\Delta^*_\delta a^{\mathbb{Z}} \backslash (\Delta \otimes \mathbb{A})^*_\delta\right) = \frac{\nu}{\nu_0} \left(D^*_\delta a^{\mathbb{Z}} \backslash (D \otimes \mathbb{A})^*_\delta\right) .$$

For $x \in T$ we write for short $\mathbb{1}_{\lambda_x} = \mathbb{1}_{\mathcal{D}^*_{x,r_x} \lambda_x(\varpi_x) \mathcal{D}^*_{x,r_x}}$. The conclusion of the above considerations can be stated as follows.

Theorem 7.3.3. *For given $(\underline{\lambda}, I, g, \underline{a}, \underline{z})$ as in the beginning of section 7.2 we have*

$$\#\mathrm{Fix}^{\underline{\lambda}}_I(g, \underline{a})(\underline{z}) \cdot \mu^T(K^T_I) = \sum_{\substack{\delta \in D^*_\natural \\ n \in \mathbb{Z}}} \frac{\nu}{\nu_0} \left(D^*_\delta a^{\mathbb{Z}} \backslash (D^*_{\mathbb{A}})_\delta\right) \cdot \mathrm{O}^T_{\delta a^n}(\mathbb{1}_{K^T_I g K^T_I}, d\nu^T) \cdot \prod_{x \in T} \mathrm{TO}_{x,\gamma_x}(\mathbb{1}_{\lambda_x}, d\nu_x) \quad (7.3.1)$$

□

We recall the notation $\mathrm{Aut} = \mathcal{C}^\infty(D^*\backslash D^*_{\mathbb{A}}/a^{\mathbb{Z}})$.

Corollary 7.3.4. *If for any $x \in T$ either $\lambda_x \in \{\mu^+, \mu^-\}$ or $r_x = 1$, then*

$$\#\mathrm{Fix}^{\lambda}_I(g, \underline{a})(\underline{z}) \cdot \mu^T(K^T_I) = \mathrm{Tr}\left(\mathbb{1}_{K_I g K_I} \cdot \prod_{x \in T} b_{r_x}(\mathbb{1}_{\lambda_x}), \mathrm{Aut} \otimes \mathbb{Q}(\sqrt{q})\right)$$

The fundamental lemma for GL_d would imply this equation also without the conditions on λ_x or r_x.

Proof. Under the given conditions the fundamental lemma is either trivial or holds by Theorem 5.3.2, i.e. the twisted orbital integrals in (7.3.1) can be replaced by $\varepsilon_x(\delta)\, \mathrm{O}_{x,\delta_x}(b_{r_x}(f^{\lambda_x}_{x,r_x}, d\nu_x)$. Then the summation over $D^*_{\natural,\underline{r}}$ can be extended to all conjugacy classes in D^*, because if for some $\delta \in D^*$ the new orbital integrals do not vanish for any $x \in T$, then δ is $\underline{r}$-admissible by Lemma 5.3.1 combined with the hypothesis $\sum \deg(\lambda_i) = 0$. Since the signs $\varepsilon_x(\delta)$ cancel (Lemma 7.3.5 below), the assertion follows from the Selberg trace formula (4.4). □

Lemma 7.3.5. *For any $\underline{r}$-admissible $\delta \in D^*$ we have $\prod_{x \in T} \varepsilon_x(\delta) = 1$.*

Proof. The decomposition $F[\delta] \otimes F_x \cong \prod F_i$ into a product of fields corresponds to a decomposition $(\mathrm{Res}_{F_{x,r_x}|F_x} \mathrm{GL}_d)^{\sigma_x}_{\gamma_x} \cong \prod G_i$. By Corollary 5.1.4 G_i is the multiplicative group of the central simple F_i-algebra of dimension d'^2 with local invariant $-x(\det \delta_i)/(r_x d')$. So modulo 2 we have

$$\mathrm{rk}_{F_i}(G_i) = \gcd(d', x(\det \delta_i)/r_x) \equiv \begin{cases} 1 & \text{if } d' \text{ is odd,} \\ x(\det \delta_i)/r_x & \text{if } d \text{ is even.} \end{cases}$$

For odd d' each single difference $\mathrm{rk}_{F_i}(\mathrm{GL}_{d'}) - \mathrm{rk}_{F_i}(G_i)$ is even, thus $\varepsilon_x(\delta) = 1$. For even d' we get $\varepsilon_x(\delta) = (-1)^{x(\det \delta)/r_x}$, and by Definition 6.3.1 of $\underline{r}$-admissibility the sum over $x \in T$ of the exponents is zero. □

Part III
Cohomology

8 Galois Representations in Cohomology

Initially, the cohomology of $\mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}}$ is a representation of the fundamental group of the product $(X' \setminus I)^r$. Using an argument of Drinfeld, the partial Frobenii extend this to an action of the product of the fundamental groups. The permutation action of the stabiliser $\mathrm{Stab}(\underline{\lambda}) \subseteq \mathfrak{S}_r$ on the base points induces an action on cohomology, and the Hecke algebra acts via the Hecke correspondences.

As usual, for an irreducible admissible representation π of $D_{\mathbb{A}}^*/a^{\mathbb{Z}}$ one can define the π-isotypic component of the cohomology, which is a finite dimensional virtual l-adic representation $H_{\underline{\lambda}}(\pi)$ of the semidirect product $(G_F)^r \rtimes \mathrm{Stab}(\underline{\lambda})$. Its ramification locus is contained in the union of the ramification loci of D and π.

8.1 Frobenius descent

Lemma 8.1.1 (Drinfeld). *Let X_0 be a projective scheme over $\mathbb{F}_q$ and let $k \supset \mathbb{F}_q$ be an algebraically closed field, $X = X_0 \otimes k$. Then the functor $\mathcal{F}_0 \mapsto \mathcal{F}_0 \otimes k$ induces an equivalence of categories*

$$\left\{ \begin{array}{c} \text{Coherent sheaves} \\ \mathcal{F}_0 \text{ on } X_0 \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Coherent sheaves } \mathcal{F} \text{ on } X \text{ plus} \\ \text{an isomorphism } \alpha : {}^{\tau}\mathcal{F} \cong \mathcal{F} \end{array} \right\}$$

with the notation ${}^{\tau}\mathcal{F} = (\mathrm{id} \otimes \mathrm{Frob}_q)^* \mathcal{F}$, preserving the natural tensor structure on both sides.

Proof. See [Dri87], Proposition 1.1, or [Laf97] I.3, Lemma 3 for a more detailed proof. □

Lemma 8.1.2. *Let X_0 be a smooth scheme of finite type over $\mathbb{F}_q$ and let $k \supset \mathbb{F}_q$ be an algebraically closed field. We write $X = X_0 \otimes k$ and denote by $F = \mathrm{Frob}_q \otimes \mathrm{id}$ its relative Frobenius. Then the functor $Y_0 \mapsto Y_0 \otimes k$ induces an equivalence of categories*

$$\left\{ \begin{array}{c} Y_0 \to X_0 \text{ finite} \\ \text{and étale} \end{array} \right\} \cong \left\{ \begin{array}{c} Y \to X \text{ finite, étale plus an} \\ \text{isomorphism } \beta : Y \cong F^* Y \end{array} \right\}$$

Proof. Cf. the proof of [Laf97] IV.2, Theorem 4. In view of the compatible isomorphisms $(\mathrm{Frob}_q)^*Y \cong Y$ for all étale $Y \to X$, giving β is equivalent to giving an isomorphism $\alpha : {}^\tau Y \cong Y$. Since the functor in Lemma 8.1.1 is fully faithful for arbitrary X_0, the same holds for the functor in the present lemma. Thus we need to show that (Y, α) descends to a Y_0. As locally existent Y_0's glue by uniqueness, X_0 may be assumed to be affine.

We choose an open embedding $X_0 \subseteq \widetilde{X}_0$ with projective $\widetilde{X}_0$. Let $L \,|\, K$ be the quotient rings of $Y \,|\, X$ and let $\widetilde{Y}$ be the normalisation of $\widetilde{X} = \widetilde{X}_0 \otimes k$ in L. Over X this coincides with Y because X_0 was assumed to be smooth. Since the underlying schemes do not change by ${}^\tau(\ )$, ${}^\tau\widetilde{Y}$ is the normalisation of ${}^\tau X$ in ${}^\tau L$. So α extends to an isomorphism ${}^\tau\widetilde{Y} \cong \widetilde{Y}$, and Lemma 8.1.1 can be applied to the coherent $\mathcal{O}_{\widetilde{X}}$-module $p_*\mathcal{O}_{\widetilde{Y}}$ and its algebra structure (writing $p : \widetilde{Y} \to \widetilde{X}$). □

We denote by $\pi(X)$ the fundamental groupoid of a scheme X. This is a profinite groupoid, i.e. all automorphism groups carry the structure of profinite groups such that conjugation by any isomorphism of two objects is continuous. Its objects are the geometric points of X and its morphisms are isomorphisms of the associated fibre functors on the category of étale coverings of X. The family of all fibre functors defines an equivalence between the category of étale coverings of X and the category of continuous representations of $\pi(X)$ in the category of finite sets.

The profinite completion of any topological groupoid has the same objects and the maximal continuous profinite quotients of the automorphism groups as automorphisms.

After these remarks the preceding lemma can be restated as follows.

Lemma 8.1.3. *In the situation of Lemma 8.1.2 F induces an equivalence $F : \pi(X) \to \pi(X)$, and there is a natural continuous functor*

$$\left[\, \pi(X) \,/\, F^{\mathbb{Z}} \,\right] \longrightarrow \pi(X_0) \tag{8.1.1}$$

which induces an equivalence of the profinite completions. □

Remark. In general, both the groupoid $\pi(X)$ and its quotient by $F^{\mathbb{Z}}$ depend on k. The above lemma therefore includes the assertion that this difference vanishes after profinite completion. In the case $k = \overline{\mathbb{F}}_q$ the functor (8.1.1) induces injective maps on the automorphism groups of the objects, but this is not true in general.

Theorem 8.1.4. *Let $X_1, \ldots, X_n$ be smooth schemes of finite type over $\mathbb{F}_q$ and let $\overline{X}_i = X_i \otimes \overline{\mathbb{F}}_q$. We write $\overline{X} = \overline{X}_1 \times \ldots \times \overline{X}_n$ and denote by $F_i : \overline{X} \to \overline{X}$ the product of $(\mathrm{Frob}_q \otimes \mathrm{id})$ on $\overline{X}_i$ and the identity on the remaining factors. These maps induce equivalences $F_i : \pi(\overline{X}) \to \pi(\overline{X})$, and the natural functor*

$$\left[\, \pi(\overline{X}) \,/\, F_1^{\mathbb{Z}} \times \ldots \times F_n^{\mathbb{Z}} \,\right] \longrightarrow \pi(X_1) \times \ldots \times \pi(X_n) \tag{8.1.2}$$

induces an equivalence of the profinite completions. In other words, a representation of the product of the $\pi(X_i)$ in the category of finite sets is the same as a finite étale covering $\overline{Y} \to \overline{X}$ plus compatible isomorphisms $\beta_i : \overline{Y} \cong F_i^* \overline{Y}$ for $1 \leq i \leq n$.

Proof. Cf. [Laf97] IV.2, Theorem 4 for the case of two curves. By Lemma 8.1.3 the right hand side of (8.1.2) is isomorphic to

$$\left[\, \pi(\overline{X}_1) \times \ldots \times \pi(\overline{X}_n) \,/\, F_1^{\mathbb{Z}} \times \ldots \times F_n^{\mathbb{Z}} \,\right]^{\wedge}.$$

The natural functor $\pi(\overline{X}) \to \pi(\overline{X}_1) \times \ldots \times \pi(\overline{X}_n)$ induces a bijection of the sets of connected components and surjections of the automorphism groups, which in general are not isomorphisms. Thus we must show that an étale covering $\overline{Y} \to \overline{X}$ plus compatible isomorphisms $\beta_i : \overline{Y} \cong F_i^* \overline{Y}$ for $1 \leq i \leq n$ as a representation of $\pi(\overline{X})$ factors over $\pi(\overline{X}_1) \times \ldots \times \pi(\overline{X}_n)$.

Inductively this can be reduced to the case $n = 2$.

Let then K be the quotient ring of $\overline{X}$ and let K_i be the quotient ring of $\overline{X}_i$ for $i = 1, 2$. These are finite products of fields. In the following commutative diagram of groupoids, aside from the rightmost arrow all functors induce bijections of the sets of connected components. The square is cocartesian because on the automorphism groups the horizontal maps are surjective and the vertical maps are injective with the same cokernel $\mathrm{Gal}(K_2^{\mathrm{alg}} \mid K_2)$.

$$\begin{array}{ccccccc}
 & & \pi(\overline{X}_1 \otimes K_2^{\mathrm{alg}}) & \twoheadrightarrow & \pi(\overline{X}_1) \times \pi(K_2^{\mathrm{alg}}) & \longrightarrow & \pi(X_1) \\
 & & \downarrow & \square & \downarrow & & \\
\pi(K) & \twoheadrightarrow & \pi(\overline{X}_1 \otimes K_2) & \twoheadrightarrow & \pi(\overline{X}_1) \times \pi(K_2) & &
\end{array}$$

We consider the given $\overline{Y} \to \overline{X}$ as a representation of $\pi(\overline{X}_1 \times \overline{X}_2)$ in the category of finite sets. By Lemma 8.1.2 its restriction to $\pi(\overline{X} \otimes K_2^{\mathrm{alg}})$ factors over $\pi(X_1)$ in virtue of the given isomorphism $\overline{Y} \cong F_1^* \overline{Y}$. So the restriction of $\overline{Y} \to \overline{X}$ to $\pi(K)$ factors over $\pi(\overline{X}_1) \times \pi(K_2)$. Similarly it factors over $\pi(K_1) \times \pi(\overline{X}_2)$ and therefore over $\pi(\overline{X}_1) \times \pi(\overline{X}_2)$ as well. □

Remark 8.1.5. The isomorphism in Theorem 8.1.4 is functorial in the following sense: for given maps $X_i' \to X_i$ such that X_i and X_i' satisfy the conditions of the theorem there is a commutative diagram

$$\begin{array}{ccc}
\left[\, \pi(\overline{X}') \,/\, F_1^{\mathbb{Z}} \times \ldots \times F_n^{\mathbb{Z}} \,\right] & \longrightarrow & \pi(X_1') \times \ldots \times \pi(X_n') \\
\downarrow & & \downarrow \\
\left[\, \pi(\overline{X}) \,/\, F_1^{\mathbb{Z}} \times \ldots \times F_n^{\mathbb{Z}} \,\right] & \longrightarrow & \pi(X_1) \times \ldots \times \pi(X_n)
\end{array} \qquad (8.1.3)$$

In the special case $X'_i = \operatorname{Spec} k(x_i)$ for closed points $x_i \in X_i$ with $\deg(x_i) = r_i$ the groupoid $\pi(\overline{X}')$ is discrete, and for any choice of geometric points $\bar{x}_i \in X'_i(\overline{\mathbb{F}}_q)$ the upper row of (8.1.3) induces an isomorphism

$$\underbrace{\widehat{\mathbb{Z}} \times \ldots \times \widehat{\mathbb{Z}}}_{n} \cong \pi_1(X'_1, \bar{x}_1) \times \ldots \times \pi_1(X'_n, \bar{x}_n)\,.$$

Here the i-th factor on the left hand side has the canonical generator $F_i^{r_i}$. The canonical isomorphism $\pi_1(X'_i, \bar{x}_i) \cong \operatorname{Aut}\big(\overline{\mathbb{F}}_q \,|\, k(x_i)\big)$ maps the image of $F_i^{r_i}$ to $(\operatorname{Frob}_q)^{r_i}$.

8.2 Construction of the representations

Let $I \subset X$ be a nonempty finite closed subscheme and let $\underline{\lambda} = (\lambda_1 \ldots \lambda_r)$ be a sequence of dominant coweights of total degree zero. By Theorem 3.1.8 the morphism

$$\pi_{I,\underline{\lambda}} \;:\; \operatorname{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}} \longrightarrow (X' \setminus I)^r$$

is representable quasiprojective. Its restriction to the open subset $\operatorname{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}}$ is smooth of relative dimension $\dim(\underline{\lambda}) = 2\sum(\rho, \lambda_i)$.

Assumption 8.2.1. The morphism $\pi_{I,\underline{\lambda}}$ and all $\pi_{I,\underline{\lambda}\cdot s}$ for $s \in \mathfrak{S}_r$ are projective. The holds for example if the division algebra D is sufficiently ramified with respect to $\underline{\lambda}$ in the sense of Definition 3.3.5. (In fact, projectivity of the different $\pi_{I,\underline{\lambda}\cdot s}$ are equivalent conditions, but we will not prove that here.)

Definition 8.2.2. Let $j_{\underline{\lambda}} : \operatorname{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}} \subseteq \operatorname{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}}$ be the natural embedding. We set

$$\begin{aligned} IC_{\underline{\lambda}} &= (j_{\underline{\lambda}})_{!*}\overline{\mathbb{Q}}_l \,\langle \dim(\underline{\lambda})\rangle \;, \\ H^n_{I,\underline{\lambda}} &= R^n(\pi_{I,\underline{\lambda}})_* IC_{\underline{\lambda}} \end{aligned}$$

with the abbreviation $\langle m\rangle = [m](m/2)$ for any $m \in \mathbb{Z}$.

Remark. Since $\dim(\underline{\lambda})$ is always even, the sheaves $IC_{\underline{\lambda}}$ and $H^n_{I,\underline{\lambda}}$ are defined over $\mathbb{Q}_l$, but this does not in general hold for the action of the partial Frobenii constructed below.

Lemma 8.2.3. *The a priori constructible l-adic sheaves $H^n_{I,\underline{\lambda}}$ on $(X' \setminus I)^r$ are smooth.*

Proof. In the case that all λ_i are minimal for the given order on P^+, the morphism $\pi_{I,\underline{\lambda}}$ is smooth and projective and the assertion follows. In the general case we use the semismall resolution of singularities q which has been defined in Proposition 3.1.9. On the smooth part $\operatorname{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}}$ we have the natural homomorphism

$\bar{\mathbb{Q}}_l \to Rq_*\bar{\mathbb{Q}}_l$, which by Gabber's decomposition theorem (cf. [KW], Theorem 10.6) extends to the inclusion of a direct factor $IC_{\underline{\lambda}} \subseteq Rq_*\bar{\mathbb{Q}}_l$. Thus every $H^n_{I,\underline{\lambda}}$ is a direct factor of a smooth l-adic sheaf. $\square$

Let $\mathrm{Stab}(\underline{\lambda}) \subseteq \mathfrak{S}_r$ be the subgroup of elements satisfying $\underline{\lambda} \cdot s = \underline{\lambda}$. For two finite closed subschemes $I \subseteq J$ of X the algebra $\mathbb{Q}[K_I/K_J]$ is naturally embedded into the Hecke algebra $\mathcal{H}_J$, i.e. any $\mathcal{H}_J$-module carries a natural action of K_I/K_J.

Proposition 8.2.4. *Using the partial Frobenii and the permutations of the base points, every $H^n_{I,\underline{\lambda}}$ naturally becomes a finite dimensional l-adic representation of the quotient groupoid*

$$\left[\, \pi(X' \setminus I)^r \,\big/\, \mathrm{Stab}(\underline{\lambda}) \,\right] \tag{8.2.1}$$

over $\bar{\mathbb{Q}}_l$ along with isomorphisms $p(s) : H^n_{I,\underline{\lambda}\cdot s} \cong \mathrm{Ad}(s)^* H^n_{I,\underline{\lambda}}$ for $s \in \mathfrak{S}_r$. Here $\mathfrak{S}_r$ acts on $\pi(X' \setminus I)^r$ from the left by permutation of the factors.

The transposed Hecke correspondences induce a left action of the Hecke algebra $\mathcal{H}_I$ on these representations which is compatible with the isomorphisms $p(s)$.

For $I \subseteq J$ the natural maps $H^n_{I,\underline{\lambda}} \to H^n_{J,\underline{\lambda}}$ are equivariant with respect to the inclusion $\mathcal{H}_I \subseteq \mathcal{H}_J$, the identity on $\mathrm{Stab}(\underline{\lambda})$, and the natural map $\pi(X' \setminus J) \to \pi(X' \setminus I)$, and they commute with the isomorphisms $p(s)$. They identify $H^n_{I,\underline{\lambda}}$ with the K_I/K_J-invariants in $H^n_{J,\underline{\lambda}}$.

Proof/Construction. (1) Let $U \subseteq (X' \setminus I)^r$ be the complement of all diagonals. For any permutation $s \in \mathfrak{S}_r$ we denote by $i(s) : (X' \setminus I)^r \to (X' \setminus I)^r$ the map $\underline{x} \mapsto \underline{x} \cdot s = (x_{s(1)} \dots x_{s(r)})$, which defines a right action of $\mathfrak{S}_r$. The isomorphisms $j(s)$ from Lemma 3.4.2 induce isomorphisms of smooth sheaves over U

$$p(s) \;:\; i(s)^* H^n_{I,\underline{\lambda}\cdot s} \;\cong\; H^n_{I,\underline{\lambda}}$$

which automatically extend to $(X' \setminus I)^r$.

(2) Let $F_i : (X' \setminus I)^r \to (X' \setminus I)^r$ be the product of Frob_q in the i-th component and the identity in the remaining components. In order to define an action of F_i on $H^n_{I,\underline{\lambda}}$ we need a cohomological correspondence for $IC_{\underline{\lambda}}$ over the partial Frobenius

$$\mathrm{Fr}_i \;:\; \mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}}\,\big|_{U \cap F_i^{-1}(U)} \longrightarrow \mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}}\,\big|_{F_i(U)\cap U}\,.$$

On $(X' \setminus I)^r$ we write

$$\bar{\mathbb{Q}}_l \left\langle \dim(\underline{\lambda}) \right\rangle = \bar{\mathbb{Q}}_l \left\langle 2(\rho, \lambda_1) \right\rangle \boxtimes \ldots \boxtimes \bar{\mathbb{Q}}_l \left\langle 2(\rho, \lambda_r) \right\rangle$$

and define $b_i^0 \;:\; F_i^* \,\bar{\mathbb{Q}}_l \left\langle \dim(\underline{\lambda}) \right\rangle \;\cong\; \bar{\mathbb{Q}}_l \left\langle \dim(\underline{\lambda}) \right\rangle$ as the product of the Frobenius correspondence in the i-th component and the identity in the remaining components. Using the equation $IC_{\underline{\lambda}} = \pi^*_{I,\underline{\lambda}} \bar{\mathbb{Q}}_l \left\langle \dim(\underline{\lambda}) \right\rangle$ on $\mathrm{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}}$, we can define the desired cohomological correspondence over $\mathrm{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}}\,|_{\,U\cap F_i^{-1}(U)}$ to be

$$b_i \;:\; \mathrm{Fr}_i^* IC_{\underline{\lambda}} \;=\; \pi_{I,\underline{\lambda}}^* F_i^* \,\overline{\mathbb{Q}}_l \,\langle \dim(\underline{\lambda})\rangle \overset{b_i^0}{\cong} \pi_{I,\underline{\lambda}}^* \,\overline{\mathbb{Q}}_l \,\langle \dim(\underline{\lambda})\rangle \;=\; IC_{\underline{\lambda}}$$

which admits a unique extension to $\mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}}\,|_{U\cap F_i^{-1}(U)}$ because Fr_i is a universal homeomorphism. This in turn induces an isomorphism over $U \cap F_i^{-1}(U)$:

$$c_i \;:\; F_i^* R(\pi_{I,\underline{\lambda}})_* IC_{\underline{\lambda}} \;=\; R(\pi_{I,\underline{\lambda}})_* \mathrm{Fr}_i^* IC_{\underline{\lambda}} \overset{b_i}{\cong} (R\pi_{I,\underline{\lambda}})_* IC_{\underline{\lambda}}$$

Since both sides are smooth sheaves, this again extends uniquely to an isomorphism $c_i : F_i^* H^n_{I,\underline{\lambda}} \cong H^n_{I,\underline{\lambda}}$ over $(X' \setminus I)^r$.

(3) The c_i commute pairwise and are permuted under conjugation with $p(s)$, which is expressed by the formulae $c_i \circ F_i^*(c_j) = c_j \circ F_j^*(c_i)$ and $p(s) \circ i(s)^*(c_i) = c_{s(i)} \circ F_{s(i)}^*(p(s))$. The product of all c_i in any order is the canonical isomorphism $(\mathrm{Frob}_q)^* H^n_{I,\underline{\lambda}} \cong H^n_{I,\underline{\lambda}}$.

At first, the smooth l-adic sheaves $H^n_{I,\underline{\lambda}}$ determine finite dimensional l-adic representations of the groupoid $\pi((X' \setminus I)^r)$. By Theorem 8.1.4 the isomorphisms $\beta_i = c_i^{-1}$ for $1 \leq i \leq r$ and $p(s)$ for $s \in \mathfrak{S}_r$ give rise to an action of the quotient (8.2.1) along with the asserted equivariant isomorphisms.[1]

(4) Let $\pi : X \to Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$ and let $j : U \subseteq X$ be the open immersion of the smooth locus. Then there is a natural homomorphism

$$\mathrm{Corr}(X\,|\,Y) \longrightarrow \mathrm{End}(R\pi_! j_{!*}\overline{\mathbb{Q}}_l)^{\mathrm{op}}\,,$$

cf. [Laf97] IV.2, Lemma 3. It maps a finite étale correspondence $[X \overset{p}{\longleftarrow} Z \overset{q}{\longrightarrow} X]$ over Y to the composition

$$R\pi_! j_{!*}\overline{\mathbb{Q}}_l \xrightarrow{\mathrm{Ad}} R\pi_! p_* p^* j_{!*}\overline{\mathbb{Q}}_l \cong R\pi_! q_* q^* j_{!*}\overline{\mathbb{Q}}_l \xrightarrow{\mathrm{Tr}} R\pi_! j_{!*}\overline{\mathbb{Q}}_l$$

where the middle isomorphism is induced by the equation $\pi p = \pi q$ and by the intermediate extension of the natural isomorphism $p^*\overline{\mathbb{Q}}_l \cong q^*\overline{\mathbb{Q}}_l$ from $p^{-1}U = q^{-1}U$ to Z. Let ι be the anti-involution of the algebra $\mathrm{Corr}(X\,|\,Y)$ which exchanges p and q. Using the homomorphisms $h_I^{\underline{\lambda}}$ defined in section 3.5 we obtain homomorphisms

$$\mathcal{H}_I^T \xrightarrow{\iota\circ h_I^{\underline{\lambda}}} \mathrm{Corr}\big(\mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}}\,\big|\,(X'_{(T)})^r\big)^{\mathrm{op}} \to \mathrm{End}\big(H^n_{I,\underline{\lambda}}\,\big|_{\,(X'_{(T)})^r}\big) = \mathrm{End}(H^n_{I,\underline{\lambda}})\,,$$

[1]For this it is necessary that the β_i are integral, meaning that they are defined on the level of l-adic systems. In [BBD] 3.3 it is said that the formalism of perverse sheaves, especially the definition of the intermediate extension, also works with integral l-adic coefficients.

compatible with the natural embeddings $\mathcal{H}_I^T \subseteq \mathcal{H}_I^{T'}$ for $T' \subseteq T$. These define the desired action of $\mathcal{H}_I$ on $H^n_{I,\underline{\lambda}}$. As the homomorphisms $h_I^{\underline{\lambda}}$ admit extensions to equivariant correspondences over Fr_i and over $j(s)$, this action commutes with the action of $[\pi(X' \setminus I)^r / \operatorname{Stab}(\underline{\lambda})]$ and with the isomorphisms $p(s)$.

(5) For $I \subseteq J$ let $\beta : \mathrm{Sht}_J^{\leq \underline{\lambda}}/a^{\mathbb{Z}} \to \mathrm{Sht}_I^{\leq \underline{\lambda}}/a^{\mathbb{Z}}$ be the reduction of the level structure. The compatibility of $h_I^{\underline{\lambda}}$ and $h_J^{\underline{\lambda}}$ expressed by the commutative diagram (3.5.1) implies that the natural map $\beta^* : H^n_{I,\underline{\lambda}} \to H^n_{J,\underline{\lambda}}$ is equivariant with respect to the inclusion $\mathcal{H}_I \subseteq \mathcal{H}_J$. It is clear that β^* is compatible with the remaining actions as well.

Since β is a torsor for the finite group $\operatorname{Ker}(\mathcal{D}_J^* \to \mathcal{D}_I^*)$ which acts by twisting the level structure, $H^n_{I,\underline{\lambda}}$ gets identified with the invariants of this group in $H^n_{J,\underline{\lambda}}$. By the construction of the Hecke correspondences the natural isomorphism $\operatorname{Ker}(\mathcal{D}_J^* \to \mathcal{D}_I^*) = K_I/K_J$ identifies this action with the action induced by the Hecke correspondences. □

We fix an algebraic closure F^{alg} of F and consider this as a geometric point $\bar{y} \in X(F^{\mathrm{alg}})$. There is a natural isomorphism

$$G_F = \operatorname{Aut}(F^{\mathrm{alg}} \,|\, F) \cong \varprojlim_I \pi_1(X' \setminus I,\, \bar{y}\,)\,.$$

We denote by $\mathcal{H} = \varinjlim_I \mathcal{H}_I$ the Hecke algebra of locally constant rational functions with compact support on the locally profinite group $D^*_{\mathbb{A}}$.

Corollary 8.2.5. *The group*

$$\big[\,(G_F)^r \rtimes \operatorname{Stab}(\underline{\lambda})\,\big] \times D^*_{\mathbb{A}}/a^{\mathbb{Z}}$$

acts on the direct limit $H^n_{\underline{\lambda}} = \varinjlim_I H^n_{I,\underline{\lambda}}$ such that $H^n_{\underline{\lambda}}$ is an l-adic representation of $(G_F)^r \rtimes \operatorname{Stab}(\underline{\lambda})$ over $\bar{\mathbb{Q}}_l$ and an admissible representation of $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$. For any $s \in \mathfrak{S}_r$ there are isomorphisms of these representations $H^n_{\underline{\lambda}\cdot s} \cong \operatorname{Ad}(s)^* H^n_{\underline{\lambda}}$.

Proof. We only have to show that $a^{\mathbb{Z}}$ acts trivially on $H^n_{\underline{\lambda}}$. By the construction of the Hecke correspondences the action of $\mathbb{A}^*$ respects the subspaces $H^n_{I,\underline{\lambda}}$, and on each of them it is given by the chosen homomorphism $\mathbb{A}^* \twoheadrightarrow \operatorname{Pic}_I(X)$ combined with the geometric action of $\operatorname{Pic}_I(X)$ on $\mathrm{Sht}_I^{\leq \underline{\lambda}}/a^{\mathbb{Z}}$. □

For a given irreducible, admissible (left) $\mathcal{H} \otimes \bar{\mathbb{Q}}_l$-module π the π-isotypic component of the semisimplification $(H^n_{\underline{\lambda}})^{\mathrm{ss}}$ can be defined as follows. The subspace of K_I-invariants π^{K_I} is either trivial or an irreducible $\mathcal{H}_I \otimes \bar{\mathbb{Q}}_l$-module which then determines π uniquely (cf. for example [Laf97] IV.3, Proposition 2). In this case we choose a maximal filtration of the finite dimensional $(G_F)^r \rtimes \operatorname{Stab}(\underline{\lambda}) \times \mathcal{H}_I \otimes \bar{\mathbb{Q}}_l$-module $(H^n_{\underline{\lambda}})^{K_I}$ and set

$$H^n_{\underline{\lambda}}(\pi) = (H^n_{\underline{\lambda}})^{K_I}(\pi^{K_I}) = \operatorname{Hom}_{\mathcal{H}_I \otimes \bar{\mathbb{Q}}_l}\big(\pi^{K_I}, \operatorname{gr}((H^n_{\underline{\lambda}})^{K_I})\big)\,. \qquad (8.2.2)$$

Up to isomorphism this is independent of the filtration and of I.

If π is an irreducible admissible right $\mathcal{H}\otimes\bar{\mathbb{Q}}_l$-module, we consider $\pi^\vee$ as a left $\mathcal{H}\otimes\bar{\mathbb{Q}}_l$-module and define

$$H^n_{\underline{\lambda}}(\pi) = H^n_{\underline{\lambda}}(\pi^\vee) = \operatorname{gr}((H^n_{\underline{\lambda}})^{K_I}) \otimes_{\mathcal{H}_I\otimes\bar{\mathbb{Q}}_l} \pi^{K_I} \,. \tag{8.2.3}$$

Corollary 8.2.6. *$H^n_{\underline{\lambda}}(\pi)$ is a semisimple, finite dimensional l-adic representation of $(G_F)^r \rtimes \operatorname{Stab}(\underline{\lambda})$ over $\bar{\mathbb{Q}}_l$. For all $s\in\mathfrak{S}_r$ there are isomorphisms*

$$H^n_{\underline{\lambda}\cdot s}(\pi) \cong \operatorname{Ad}(s)^* H^n_{\underline{\lambda}}(\pi)\,.$$

If the K_I-invariants of π are nonzero, then $H^n_{\underline{\lambda}}(\pi)$ is unramified over $X'\setminus I$, i.e. it is a representation of $\pi_1(X'\setminus I, \bar{y}\,)^r \rtimes \operatorname{Stab}(\underline{\lambda})$.

Proof. Since the irreducible representation π is defined over a finite extension of $\mathbb{Q}_l$ (cf. [Laf97] IV.3, Proposition 2), the same holds for $H^n_{\underline{\lambda}}(\pi)$. In view of $(H^n_{\underline{\lambda}})^{K_I} = H^n_{I,\underline{\lambda}}$ all assertions follow from Proposition 8.2.4. □

Remark. The restriction of $H^n_{\underline{\lambda}}(\pi)$ to $(G_F)^r$ is semisimple as well and could have been defined starting from $H^n_{I,\underline{\lambda}}$ considered as a representation of $\pi(X'\setminus I)^r\times\mathcal{H}_I\otimes\bar{\mathbb{Q}}_l$. This follows from the general observation that for a normal subgroup $H\lhd G$ any G-semisimple representation is automatically H-semisimple: the maximal H-semisimple subspace is G-stable and therefore has a complement, which then must be zero.

Definition 8.2.7. For any irreducible, admissible left or right $\mathcal{H}\otimes\bar{\mathbb{Q}}_l$-module π let

$$H_{\underline{\lambda}}(\pi) = \sum (-1)^n H^n_{\underline{\lambda}}(\pi)$$

as virtual l-adic representation of $(G_F)^r \rtimes \operatorname{Stab}(\underline{\lambda})$ over $\bar{\mathbb{Q}}_l$.

9 Consequences of Counting the Fixed Points

A description of $H_{\underline{\lambda}}(\pi)$ as virtual $(G_F)^r$-module follows directly from the computation of the fixed points in section 7 if the fundamental lemma for GL_d is at our disposal. Its use may be avoided by first proving that to any irreducible automorphic representation π there is an associated d-dimensional Galois representation $\sigma(\pi)$. The existence of a multiple of $\sigma(\pi)$ was proved by Lafforgue [Laf97]. However, a finer statement about the local L-functions of these Galois representations seems to depend on the fundamental lemma for the special coweight $\lambda=\mu^+ + \mu^-$.

Unless the contrary is stated explicitly, Assumption 8.2.1 persists.

9.1 First description of $H_{\underline{\lambda}}(\pi)$

First, we need some preliminary explanations.

An irreducible, admissible $\mathcal{H} \otimes \bar{\mathbb{Q}}_l$-module π can be written as a restricted tensor product $\pi \cong \bigotimes'_{x \in X} \pi_x$ with uniquely determined irreducible, admissible $\mathcal{C}_0^\infty(D_x^*) \otimes \bar{\mathbb{Q}}_l$-modules π_x. The module π is unramified (we use this term synonymously for 'spherical') at all but finitely many places $x \in X'$, which means $(\pi_x)^{\mathcal{D}_x^*}$ is a nontrivial irreducible representation of the algebra $\mathcal{C}_0(D_x^*//\mathcal{D}_x^*) \otimes \bar{\mathbb{Q}}_l$. This algebra is isomorphic to the (commutative) algebra $\bar{\mathbb{Q}}_l[z_1, z_1^{-1} \dots z_d, z_d^{-1}]^{\mathfrak{S}_d}$ via the Satake isomorphism $f \mapsto f^\vee$, in particular $(\pi_x)^{\mathcal{D}_x^*}$ is one-dimensional. The Satake parameter $\{z_1(\pi_x) \dots z_d(\pi_x)\} \in (\bar{\mathbb{Q}}_l^*)^d/\mathfrak{S}_d$ is characterised by the equations

$$\operatorname{Tr}(f, \pi_x) = f^\vee(z_1(\pi_x) \dots z_d(\pi_x)), \quad f \in \mathcal{C}_0^\infty(D_x^*//\mathcal{D}_x^*) \otimes \bar{\mathbb{Q}}_l \tag{9.1.1}$$

and determines π_x uniquely. The same information is carried by the local L-factor $L_x(\pi, T) = \prod_i (1 - z_i(\pi_x)T)^{-1}$.

The space $\operatorname{Aut} = \mathcal{C}^\infty(D^* \backslash D_{\mathbb{A}}^*/a^{\mathbb{Z}})$ of locally constant rational functions on the compact topological space $D^* \backslash D_{\mathbb{A}}^*/a^{\mathbb{Z}}$ is an admissible and semisimple right $\mathcal{H}$-module, cf. [Laf97] IV.4, Proposition 1. For any irreducible, admissible representation π of $D_{\mathbb{A}}^*/a^{\mathbb{Z}}$ over $\bar{\mathbb{Q}}_l$ let m_π be the (finite) multiplicity of π in $\operatorname{Aut} \otimes \bar{\mathbb{Q}}_l$. π is called automorphic if $m_\pi \geq 1$. One might expect $m_\pi = 1$ in that case.

Moreover one expects that for any irreducible automorphic representation π of $D_{\mathbb{A}}^*/a^{\mathbb{Z}}$ there exists a semisimple d-dimensional l-adic G_F-representation $\sigma = \sigma(\pi)$ over $\bar{\mathbb{Q}}_l$ which is uniquely determined by the following condition: if π is unramified at some place $x \in X'$, then so is σ, and we have

$$L_x(\sigma, T) = L_x(\pi, T). \tag{9.1.2}$$

This means that the set of eigenvalues of the geometric Frobenius τ_x on σ equals $\{z_1(\pi_x) \dots z_d(\pi_x)\}$. In Theorem 9.3.1 we will show the existence of such $\sigma(\pi)$ satisfying (9.1.2) outside some finite set of places.

The set of dominant coweights λ for GL_d can be naturally identified with the set of dominant weights for the dual group $\mathrm{GL}_d(\bar{\mathbb{Q}}_l)$. Let $\rho_\lambda : \mathrm{GL}_d(\bar{\mathbb{Q}}_l) \to \mathrm{GL}(V_\lambda)$ be the irreducible representation with highest weight λ. As a complete description of the virtual $(G_F)^r \rtimes \operatorname{Stab}(\underline{\lambda})$-module $H_{\underline{\lambda}}(\pi)$ one might expect

$$H_{\underline{\lambda}}(\pi) = m_\pi \cdot (\rho_{\lambda_1} \circ \sigma(\pi)) \boxtimes \dots \boxtimes (\rho_{\lambda_r} \circ \sigma(\pi)) \tag{9.1.3}$$

where $\operatorname{Stab}(\underline{\lambda})$ acts on the right hand side by permutation of the tensor factors.

The remainder of this section is concerned with the proof of the following partial result, which will later be improved by Theorem 9.3.3 and by Theorem 10.4.3 and its corollary.

Theorem 9.1.1. *Let π be an irreducible, admissible representation of $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ over $\bar{\mathbb{Q}}_l$ and let $T(\pi) \subset |X|$ be the finite set of places at which π is ramified. Suppose that all $\lambda_i \in \{\mu^+, \mu^-\}$ or that the fundamental lemma for GL_d holds. Then there is a finite set $T'(\pi) \subset |X|$ including $T(\pi)$ and $T(a)$ such that for any choice of closed points $x_1 \dots x_r \in X' \setminus T'(\pi)$ and for any integers $s_1 \dots s_r$ we have the following equation.*

$$\mathrm{Tr}\big(\tau_{x_1}^{s_1} \times \ldots \times \tau_{x_r}^{s_r}, H_{\underline{\lambda}}(\pi)\big) = m_\pi \cdot \prod_{i=1}^{r} f^\vee_{\lambda_i}\big(z_1(\pi_{x_i})^{s_i}, \ldots, z_d(\pi_{x_i})^{s_i}\big) \qquad (9.1.4)$$

Here $f^\vee_\lambda \in \bar{\mathbb{Q}}_l[z_1, z_1^{-1} \dots z_d, z_d^{-1}]^{\mathfrak{S}_d}$ denotes the character of the irreducible representation of $\mathrm{GL}_d(\bar{\mathbb{Q}}_l)$ with highest weight λ.

The case $\underline{\lambda} = (\mu^+, \mu^-)$ or (μ^-, μ^+) is [Laf97] IV.4, Theorem 9.

Proof. First we reduce the assertion to the case that the x_i are pairwise distinct. Let $\alpha_i : G_F \subseteq G_F^r$ be the embedding of the i-th component. Assuming (9.1.4) for pairwise distinct x_i we get for any $x \in X' \setminus T'(\pi)$ and $s \in \mathbb{Z}$

$$\mathrm{Tr}\left(\tau_x^s, \alpha_i^* H_{\underline{\lambda}}(\pi)\right) = m_\pi \cdot \prod_{j \neq i} \dim_{\bar{\mathbb{Q}}_l}(V_{\lambda_j}) \cdot f^\vee_{\lambda_i}\big(z_1(\pi_x)^s, \ldots, z_d(\pi_x)^s\big)$$

by setting $s_j = 0$ for $j \neq i$ and choosing any permitted x_j. Writing $M = m_\pi \cdot \prod_{i=1}^r \dim V_{\lambda_i}$ it follows that $M^{r-1} \cdot H_{\underline{\lambda}}(\pi) = \alpha_1^* H_{\underline{\lambda}}(\pi) \boxtimes \ldots \boxtimes \alpha_r^* H_{\underline{\lambda}}(\pi)$, because the traces of a dense set of G_F^r on both sides of this equation coincide. Since $M \neq 0$ this allows to compute the left hand side of (9.1.4) in the case that some x_i coincide as well.

Furthermore it is sufficient to consider exponents $s_i \geq 1$ because both sides of (9.1.4) are finite sums of terms $n\, \beta_1^{s_1} \dots \beta_r^{s_r}$ with $n \in \mathbb{Z}$ and $\beta_i \in \bar{\mathbb{Q}}_l^*$. Such sums are uniquely determined by their values for positive s_i.

Let $I \subset X$ be a finite closed subscheme such that π has nontrivial K_I-invariants, i.e. $H^n_{\underline{\lambda}}(\pi) = H^n_{I,\underline{\lambda}}(\pi^{K_I})$ for all n. Since Aut is an admissible $\mathcal{H}$-module, there are only finitely many isomorphism classes of irreducible $\mathcal{H}_I \otimes \bar{\mathbb{Q}}_l$-modules π_ν which occur in $\mathrm{Aut}^{K_I} \otimes \bar{\mathbb{Q}}_l$ or in one of the finitely many $H^n_{I,\underline{\lambda}}$. Hence there is a function $f \in \mathcal{H}_I \otimes \bar{\mathbb{Q}}_l$ satisfying $\mathrm{Tr}(f, \pi^{K_I}) = 1$ and $\mathrm{Tr}(f, \pi_\nu) = 0$ for all $\pi_\nu \ncong \pi^{K_I}$, cf. Bourbaki Algèbre, Ch. 8, §12, Proposition 3. This function is concentrated at finitely many places, which means there is a finite set $T'(\pi) \subset |X|$ such that $f \in \mathcal{H}_I^T \otimes \bar{\mathbb{Q}}_l$ for any finite $T \subset |X|$ disjoint from $T'(\pi)$. We enlarge $T'(\pi)$ by I and $T(a)$. Then Proposition 9.1.2 below for the chosen f and for pairwise distinct closed points $x_1 \dots x_r \in X' \setminus T'(\pi)$ implies

$$\mathrm{Tr}\big(\tau_{x_1}^{s_1} \times \ldots \times \tau_{x_r}^{s_r}, H_{\underline{\lambda}}(\pi)\big) = m_\pi \cdot \prod_{i=1}^{r} \mathrm{Tr}\big(b_{s_i}(f_{\lambda_i}), \pi_{x_i}\big)\,.$$

Using the characterisation of the Satake parameter (9.1.1) this is equal to the right hand side of (9.1.4). □

Proposition 9.1.2. *Suppose that all $\lambda_i \in \{\mu^+, \mu^-\}$ or that the fundamental lemma for GL_d holds. Let $x_1 \dots x_r \in X' \setminus T(a)$ be pairwise distinct closed points and $T = \{x_1 \dots x_r\}$, and let $I \subset X$ be a finite closed subscheme with $I \cap T = \emptyset$. Then for any positive integers $s_1 \dots s_r \geq 1$ and any $f^T \in \mathcal{H}_I^T$ the following equation holds.*

$$\mathrm{Tr}\big(\tau_{x_1}^{s_1} \times \ldots \times \tau_{x_r}^{s_r} \times f^T, H^*_{I,\underline{\lambda}}\big) = \mathrm{Tr}\Big(f^T \cdot \prod_{i=1}^{r} b_{s_i}(f_{\lambda_i}), \mathrm{Aut} \otimes \mathbb{Q}(\sqrt{q})\Big) \quad (9.1.5)$$

Here $f_\lambda \in \mathcal{C}_0(D_x^*//\mathcal{D}_x^*) \otimes \mathbb{Q}(\sqrt{q})$ denotes the Hecke function with Satake transform $f_\lambda^\vee$.

Proof. This assertion follows rather formally from the computation of fixed points (Corollary 7.3.4). We may assume $f^T = \mathbb{1}_{K_I g K_I}$ for some $g \in (D_{\mathbb{A}}^T)^*$, i.e. f^T acts by the $\mu^T(K_I^T)$-fold of the finite étale correspondence $\Gamma_I^{\leq\lambda}(g)^{\mathrm{op}}$. Let

$$p, q \;:\; \Gamma_I^{\leq\lambda}(g) \longrightarrow \mathrm{Sht}_I^{\leq\lambda}/a^{\mathbb{Z}}$$

be the given morphisms. We set $a_i = s_i \deg(x_i)$ for $i = 1 \dots r$ and choose geometric points $z_i \in X(\overline{\mathbb{F}}_q)$ over x_i, which will also be written as $\underline{z} \in X^r(\overline{\mathbb{F}}_q)$. By the functoriality of Theorem 8.1.4 (see Remark 8.1.5) the left hand side of (9.1.5) equals

$$LHS\,(9.1.5) = \mathrm{Tr}\big(c_1^{a_1} \times \ldots \times c_r^{a_r} \times f^T, (H^*_{I,\underline{\lambda}})_{\underline{z}}\big) \quad (9.1.6)$$

Here c_i has been defined in the proof of Proposition 8.2.4, and the trace more precisely means the trace of the composition:

$$(H^n_{I,\underline{\lambda}})_{\underline{z}} = ((F^{\underline{a}})^* H^n_{I,\underline{\lambda}})_{\underline{z}} \xrightarrow{c_1^{a_1} \times \ldots \times c_r^{a_r} \times f^T} (H^n_{I,\underline{\lambda}})_{\underline{z}}$$

Let $\mathrm{Sht}_I^{\leq\lambda}/a^{\mathbb{Z}}\,|_{\underline{z}} \to \mathrm{Spec}\,\overline{\mathbb{F}}_q$ be the fibre of $\pi_{I,\underline{\lambda}}$ in $\underline{z}$ and denote by $(IC_{\underline{\lambda}})_{\underline{z}}$ the restriction of $IC_{\underline{\lambda}}$ to this variety. Since $\underline{z}$ factors over $\Lambda_{(T)}$, the Hecke correspondence $\Gamma_I^{\leq\lambda}(g)$ can be restricted to the fibre in $\underline{z}$, and the partial Frobenii $\mathrm{Fr}_i^{\deg(x_i)}$ induce endomorphisms of this fibre. Via the base change isomorphism

$$(H^n_{I,\underline{\lambda}})_{\underline{z}} = H^n\big(\mathrm{Sht}_I^{\leq\lambda}/a^{\mathbb{Z}}\,|_{\underline{z}}, (IC_{\underline{\lambda}})_{\underline{z}}\big)$$

the actions of $c_i^{a_i}$ and of f^T coincide with the actions induced by the restrictions of the given cohomological correspondences.

As $\mathrm{Fr}^{\underline{a}}$ factors over the absolute Frobenius, the finite geometric correspondence

$$\big[\,(\mathrm{Fr}^{\underline{a}})_{\underline{z}}\,\big] \circ \big[\,\Gamma_I^{\leq\underline{\lambda}}(g)^{\mathrm{op}}_{\underline{z}}\,\big] = \\ \Big[\mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}}|_{\underline{z}} \xleftarrow{(\mathrm{Fr}^{\underline{a}})_{\underline{z}}} \mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}}|_{\underline{z}} \xleftarrow{q} \Gamma_I^{\leq\underline{\lambda}}(g)_{\underline{z}} \xrightarrow{p} \mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}}|_{\underline{z}}\Big]$$

is contracting in the sense that it satisfies the hypothesis of [Fu] Theorem 5.2.1. Consequently the local terms in the Lefschetz-Verdier formula coincide with the naive local terms and we get from (9.1.6):

$$LHS\,(9.1.5) = \mu^T(K_I^T)\cdot \sum_{y\in \mathrm{Fix}_I^{\leq\underline{\lambda}}(g,\underline{a})(\underline{z})} \mathrm{naive.loc}_y\big([\,\mathrm{Fr}^{\underline{a}}\,]\circ[\,\Gamma_I^{\leq\underline{\lambda}}(g)^{\mathrm{op}}\,], IC_{\underline{\lambda}}\big). \quad (9.1.7)$$

The naive local term at y is by definition the trace of the endomorphism of the fibre $(IC_{\underline{\lambda}})_y$ which is induced by the composition of the cohomological correspondences $b_i^{a_i} : (\mathrm{Fr}_i^{a_i})^* IC_{\underline{\lambda}} \cong IC_{\underline{\lambda}}$ and $b : p^* IC_{\underline{\lambda}} \cong q^* IC_{\underline{\lambda}}$.

In order to determine this trace we choose for each $i = 1\dots r$ an integer $n_i \geq 0$ such that in the sequence

$$\underline{\lambda}' = (-n_1\lambda_0\,, \lambda_1 + n_1\lambda_0\,, \dots, -n_r\lambda_0\,, \lambda_r + n_r\lambda_0)$$

all $\lambda'_{2i} = \lambda_i + n_i\lambda_0$ are positive, i.e. $\lambda'_{2i} \in P^{++}$ and $\deg(\lambda'_{2i}) \geq 0$. Then $\pi_{I,\underline{\lambda}}$ can be written as the composition

$$\mathrm{Sht}_I^{\leq\underline{\lambda}}/a^{\mathbb{Z}} \xrightarrow{\alpha} \prod_{i=1}^{r} \mathrm{coh}_{\mathcal{D}}^{\leq\lambda'_{2i}} \longrightarrow (X'\setminus I)^r$$

where α is given by the quotients $\mathcal{E}_{i-1}(n_i\cdot x_i)/\mathcal{E}_i$. This is a smooth morphism by Lemma 3.3.4, Corollary 3.3.6 and using that $\mathrm{coh}_{\mathcal{D}}^{\leq n_i\lambda_0}$ is smooth over X' (of negative dimension if $n_i > 0$).

Let $j'_i : \mathrm{coh}_{\mathcal{D}}^{\lambda'_{2i}} \subseteq \mathrm{coh}_{\mathcal{D}}^{\leq\lambda'_{2i}}$ be the natural embeddings and let $j' = \prod_{i=1}^r j'_i$. We set $\mathcal{F}_i = (j'_i)_{!*}\mathbb{Q}_l\,\langle 2(\rho,\lambda_i)\rangle$ and

$$\mathcal{F} = j_{!*}\mathbb{Q}_l\,\langle \dim(\underline{\lambda})\rangle = \mathcal{F}_1 \boxtimes \dots \boxtimes \mathcal{F}_r\,.$$

Let $b'_i : F_i^*\mathcal{F} \cong \mathcal{F}$ be the product of the Frobenius correspondence in the i-th component and the identity in the remaining components. Since α is smooth, there is a canonical isomorphism $IC_{\underline{\lambda}} = \alpha^*\mathcal{F}$. Using this, the cohomological correspondence b_i coincides with the composition:

$$\mathrm{Fr}_i^*\alpha^*\mathcal{F} \cong \alpha^* F_i^*\mathcal{F} \xrightarrow{\alpha^* b'_i} \alpha^*\mathcal{F}$$

Moreover the given 2-isomorphism $\alpha \circ p \cong \alpha \circ q$ induces an isomorphism $p^*\alpha^*\mathcal{F} \cong q^*\alpha^*\mathcal{F}$ which coincides with the isomorphism $p^*IC_{\underline{\lambda}} \cong q^*IC_{\underline{\lambda}}$ chosen before. Denoting $\alpha(y) = y' = (y'_1 \dots y'_r)$ this implies

$$\text{naive.loc}_y\big([\,\text{Fr}^{\underline{a}}\,] \circ [\,\Gamma_I^{\leq\underline{\lambda}}(g)^{\text{op}}\,], IC_{\underline{\lambda}}\big) = \text{naive.loc}_{y'}\big([\,F^{\underline{a}}\,], \mathcal{F}\big)$$
$$= \prod_{i=1}^{r} \text{naive.loc}_{y'_i}\big([\,\text{Frob}_q^{a_i}\,], \mathcal{F}_i\big)\,. \quad (9.1.8)$$

Here y'_i is an object of $\text{coh}_{\mathcal{D}}^{\leq\lambda'_{2i}}(\overline{\mathbb{F}}_q)$ over z_i plus an isomorphism $(\text{Frob}_q^{a_i})^* y'_i \cong y'_i$, which means that y'_i is defined over $\mathbb{F}_{q^{a_i}}$. The local term at y'_i depends only on the type of this coherent sheaf, which up to a shift by $n_i\lambda_0$ coincides with the type of the modification of the $\mathcal{D}$-shtuka y in z_i.

Inserting (9.1.8) into (9.1.7) we get that the left hand side of (9.1.5) equals the sum over the fixed points of a function which only depends on the type of the modifications. This function is determined by Lemma 9.1.3 below, and using Corollary 7.3.4 we get the right hand side of (9.1.5). □

Lemma 9.1.3. *For any $m \geq 0$ and $\lambda \in P_m^{++}$ let $j_\lambda : \text{coh}_{\mathcal{D}}^{\lambda} \subseteq \text{coh}_{\mathcal{D}}$ be the natural immersion and set $\mathcal{F}_\lambda = (j_\lambda)_{!*}\overline{\mathbb{Q}}_l\,\langle 2(\rho,\lambda)\rangle$. Let*

$$t \;:\; \text{coh}_{\mathcal{D}}(\mathbb{F}_q)/_{\sim} \longrightarrow P^{++} = \text{GL}_d\big(\mathbb{F}_q(\!(t)\!)\big)/\!/\,\text{GL}_d\big(\mathbb{F}_q[\![t]\!]\big)$$

be the map given by the type of the coherent sheaf. Then the function on $\text{coh}_{\mathcal{D}}(\mathbb{F}_q)/_{\sim}$ given by $y \mapsto \text{Tr}(\tau_y, \mathcal{F}_\lambda)$ equals $(-1)^{2(\rho,\lambda)} f_\lambda \circ t$ where f_λ is the Hecke function such that the Satake transform $f_\lambda^\vee$ is the character of the irreducible representation of $\text{GL}_d(\overline{\mathbb{Q}}_l)$ with highest weight λ.

Proof. This follows almost immediately from the corresponding statement about the affine Grassmannian of GL_d, which in a more sophisticated way can be expressed as an equivalence of the tensor category of perverse sheaves of weight zero on the affine Grassmannian and the tensor category of representations of the dual group $\text{GL}_d(\overline{\mathbb{Q}}_l)$, see [FGKV], Proposition 5.1.

More precisely, let $x \in X$ be a closed point and let $\text{Gr}(x)$ be the fibred product

$$\begin{array}{ccc} \text{Gr}(x) & \longrightarrow & \text{Spec}\,k(x) \\ \downarrow & \square & \downarrow \\ \text{Mod}_{\mathcal{D}} & \xrightarrow{(\mathcal{E},\,\text{can})} & \text{Vect}^1_{\mathcal{D}} \times X' \end{array}$$

with the right vertical map given by $\mathcal{D} \in \text{Vect}^1_{\mathcal{D}}(\mathbb{F}_q)$ and the natural immersion $\text{Spec}\,k(x) \to X'$. An isomorphism $\mathcal{D}_x \cong M_d(\mathcal{O}_x)$ identifies $\text{Gr}(x)$ with the maximal reduced closed subscheme of the affine Grassmannian.

The inverse image of the stack $\mathrm{Mod}^+_{\mathcal{D}} \subset \mathrm{Mod}_{\mathcal{D}}$ of positive modifications $\mathrm{Gr}^+(x) \subset \mathrm{Gr}(x)$ corresponds to the positive Grassmannian. The quotient $\mathcal{E}/\mathcal{E}'$ defines a morphism

$$\pi \; : \; \mathrm{Gr}^+(x) \longrightarrow \mathrm{coh}_{\mathcal{D}} \times_{X'} \mathrm{Spec}\, k(x) \; = \; \mathrm{coh}_{\mathcal{D}}(x)$$

which is smooth (for example by Lemma 3.3.2) and compatible with the stratifications of both sides indexed by the positive $\lambda \in P^{++}$. It remains to show that $\pi^*(\mathcal{F}_\lambda \,|_{\,\mathrm{coh}_{\mathcal{D}}(x)})$ is the intermediate extension of the sheaf $\overline{\mathbb{Q}}_l \,\langle 2(\rho, \lambda)\rangle$ on $\mathrm{Gr}^\lambda(x)$, or equivalently the corresponding assertion for $\mathcal{F}_\lambda \,|_{\,\mathrm{coh}_{\mathcal{D}}(x)}$. This follows from the fact that $\mathrm{coh}_{\mathcal{D}} \to X'$ is étale locally in X' isomorphic to the projection of a product onto the second factor (Lemma 10.1.5 below). □

Corollary 9.1.4. *If all $\lambda_i \in \{\mu^+, \mu^-\}$ or if the fundamental lemma for GL_d holds, then $H^n_{I,\underline{\lambda}}$ vanishes for odd n.*

Proof. Since $IC_{\underline{\lambda}}$ is pure of weight zero, the same holds for $R(\pi_{I,\underline{\lambda}})_* IC_{\underline{\lambda}}$, i.e. all eigenvalues of the geometric Frobenius τ on the smooth l-adic sheaf $H^n_{I,\underline{\lambda}}$ have absolute value $q^{n/2}$.

Let $x_1 \ldots x_r \in X'$ be any pairwise distinct closed points which do not meet I or $T(a)$, let N be a common multiple of the numbers $\deg(x_i)$, and let $s_i = N/\deg(x_i)$. Choosing for f^T the unit element of $\mathcal{H}^T_I$, Proposition 9.1.2 yields for

$$\mathrm{Tr}\big(\tau^{rN}, H^*_{I,\underline{\lambda}}\big) = \mathrm{Tr}\big(\tau^{rs_1}_{x_1} \times \ldots \times \tau^{rs_r}_{x_r} \times f^T, H^*_{I,\underline{\lambda}}\big)$$

a presentation as a finite sum $\sum n_i \alpha_i^r$ with positive integers n_i. By purity a non-trivial $H^n_{I,\underline{\lambda}}$ for an odd n would be reflected in this sum by a negative coefficient. □

9.2 Remarks on the non-proper case

For sufficiently large $\underline{\lambda}$ with respect to D the morphism $\pi_{I,\underline{\lambda}}$ will in general not be proper. Nevertheless in the case $I \neq \emptyset$ it is representable quasiprojective, and Definition 8.2.2 may be replaced by

$${}^0H^n_{I,\underline{\lambda}} = R^n(\pi_{I,\underline{\lambda}})_! IC_{\underline{\lambda}}$$

with $IC_{\underline{\lambda}}$ as before. These l-adic sheaves are smooth in some neighbourhood of the generic point $U \subseteq (X' \setminus I)^r$, but this might be small.

Lemma 9.2.1. *The largest open subset $U' \subseteq (X' \setminus I)^r$ to which all ${}^0H^n_{I,\underline{\lambda}}\,|_{\,U}$ can be extended as smooth l-adic sheaves has the form $U' = U_1 \times \ldots \times U_r$ with open sets $U_i \subseteq X$.*

Proof. U' is the complement of a divisor and is stable under the Frobenii of the components $F_i : X^r \to X^r$ thanks to the partial Frobenii. Hence it suffices to show that any closed subset $Y \subseteq X^r$ of pure codimension 1 which is stable under all F_i is the union of sets of the form $p_i^{-1}(x)$ for closed points $x \in X$, where $p_i : X^r \to X$ denotes the i-th projection.

First we consider the case $r = 2$. To an irreducible divisor in X^2 which is mapped surjectively onto both components one can assign a degree in $\mathbb{Q}_{>0}$ such that transforming the divisor by F_1 multiplies its degree by q. Therefore these divisors have infinite order under the action of F_1 and cannot be part of a F_1-stable $Y \subset X^2$ of codimension 1.

In the general case let $q_i : X^r \to X^{r-1}$ by the projection outside i and let $Y_i \subseteq Y$ be the maximal subset such that $q_i^{-1}(q_i Y_i) \subseteq Y$. Then Y_i is closed in Y, and the case $r = 2$ implies that Y is covered by Y_1 and Y_2. This remains true when the Y_i are replaced by their maximal closed subsets of pure codimension 1 in X^r. Since then $\overline{Y}_i = q_i Y_i \subset X^{r-1}$ for $i = 1, 2$ satisfy the hypotheses of the Lemma, we can proceed by induction. □

Similarly U' is stable under the action of $\mathrm{Stab}(\underline{\lambda})$, i.e. $\lambda_i = \lambda_j$ implies $U_i = U_j$. Let $H^n_{I,\underline{\lambda}}$ be the smooth l-adic sheaf on U' which over U coincides with ${}^0H^n_{I,\underline{\lambda}}$. Like in the proper case this becomes a representation of

$$\big(\pi(U_1) \times \ldots \times \pi(U_r)\big) \rtimes \mathrm{Stab}(\underline{\lambda}) \times \mathcal{H}_I^{\emptyset}.$$

Again, in the direct limit over I for any irreducible admissible representation π of $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ over $\overline{\mathbb{Q}}_l$ we get l-adic representations $H^n_{\underline{\lambda}}(\pi)$ of $(G_F)^r \rtimes \mathrm{Stab}(\underline{\lambda})$, but their ramification locus is not a priori bounded by the ramification loci of π and D as in the proper case (Corollary 8.2.6).

Theorem 9.2.2. *The statement of Theorem 9.1.1 about the trace of $\tau_{x_1}^{s_1} \times \ldots \times \tau_{x_r}^{s_r}$ on $H_{\underline{\lambda}}(\pi)$ holds literally in the non-proper case (with a possibly larger set $T'(\pi)$.)*

Proof. Like in the proof of Theorem 9.1.1, the proof can be reduced to the assertion that the calculation of the traces on $H^*_{I,\underline{\lambda}}$ stated in Proposition 9.1.2 holds under the additional condition $\mathrm{Spec}\, k(x_1) \times \ldots \times \mathrm{Spec}\, k(x_r) \subset U$ and for sufficiently large exponents $s_1, \ldots, s_r \gg 1$. This assertion is proved in precisely the same way as before except that in the Lefschetz-Verdier formula there might be contributions from infinity. These vanish for sufficiently large exponents by [Fu] Corollary 5.4.5 ('Deligne Conjecture'). □

In the non-proper case we cannot expect the analogue of Corollary 9.1.4 (purity). Therefore it is not clear whether the virtual representation $H_{\underline{\lambda}}(\pi)$ can be written as a sum of irreducible representations with positive coefficients.

9.3 Second description of $H_{\underline{\lambda}}(\pi)$

Theorem 9.3.1. *Suppose the morphism* $\mathrm{Sht}^{(\mu^+,\mu^-)}/a^{\mathbb{Z}} \to X' \times X'$ *is proper, for example when* D *is sufficiently ramified with respect to* 1.

Then for any irreducible automorphic representation π *of* $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ *over* $\bar{\mathbb{Q}}_l$ *there exists a* d*-dimensional* l*-adic* G_F*-representation* $\sigma = \sigma(\pi)$ *over* $\bar{\mathbb{Q}}_l$ *which is uniquely determined by the following property: if for some closed* $x \in X'$ *the local component* π_x *is unramified, then* σ *is unramified at* x*, and for all but finitely many such* x *we have* $L_x(\pi, T) = L_x(\sigma, T)$.

Proof. By Lafforgue's case of Theorem 9.1.1 at least a multiple of the desired representation σ exists, i.e. we can write σ as a linear combination of irreducible representations with positive rational coefficients (positive by Corollary 9.1.4).

Let $N(\sigma)$ be the denominator of σ, that is the least positive integer such that $N(\sigma) \cdot \sigma$ is a true representation. For any integer $m \geq 1$ we consider the sequence $\underline{\lambda}(m)$ of length $2m$

$$\underline{\lambda}(m) = (\mu^+)^m (\mu^-)^m = (\mu^+ \ldots \mu^+, \mu^- \ldots \mu^-) .$$

Theorem 9.2.2 then implies the equations $H_{\underline{\lambda}(m)} = m_\pi \cdot \sigma^{\boxtimes m} \boxtimes \sigma^{\vee \boxtimes m}$ of virtual representations of $(G_F)^{2m}$. Consequently all powers $N(\sigma)^{2m}$ divide the fixed number m_π, which implies $N(\sigma) = 1$. $\square$

Remark. For given D only finitely many of the stacks $\mathrm{Sht}^{\leq \underline{\lambda}(m)}/a^{\mathbb{Z}}$ are proper over $(X')^{2m}$. If $\mathrm{Sht}^{(\mu^+,\mu^-)}/a^{\mathbb{Z}} \to X' \times X'$ is not proper, the above proof only gives the existence of σ as virtual representation with integer coefficients.

Remark 9.3.2. In [Lau97] it is conjectured that there is an irreducible l-adic representation $\sigma' = \sigma'(\pi)$ with dimension d' dividing d, which is pure of weight $1 - d/d'$ such that

$$\sigma(\pi) = \sigma' \oplus \sigma'(-1) \oplus \ldots \oplus \sigma'(1 - d/d') .$$

In particular $\sigma(\pi)$ does not contain any irreducible representation with multiplicity greater than 1.

Theorem 9.3.3. *Suppose* $\mathrm{Sht}^{(\mu^+,\mu^-)}/a^{\mathbb{Z}}$ *is proper over* $X' \times X'$ *and* $\mathrm{Sht}^{\leq \underline{\lambda}}/a^{\mathbb{Z}}$ *is proper over* $(X')^r$*. Then there is an isomorphism of virtual* $(G_F)^r$*-modules*

$$H_{\underline{\lambda}}(\pi) \cong m_\pi \cdot (\rho_{\lambda_1} \circ \sigma(\pi)) \boxtimes \ldots \boxtimes (\rho_{\lambda_r} \circ \sigma(\pi)) . \tag{9.3.1}$$

For odd n *we have* $H^n_{\underline{\lambda}}(\pi) = 0$.

Proof. To prove equation (9.3.1) we have to show that the traces of a dense subset of $(G_F)^r$ coincide on both sides. One class of such subsets is given by the products $\tau_{x_1} \times \ldots \times \tau_{x_r}$ where $x_1 \ldots x_r$ are pairwise distinct closed points in X' which do not meet a fixed finite set $T'(\pi)$. Like in the proof of Theorem 9.1.1 the equality of traces on these elements can be reduced to the case $s_i = 1$ of Proposition 9.1.2, and (9.3.1) is proved.

This in turn implies the equality of traces (9.1.4) outside a finite set of places. The vanishing statement for odd n then follows from purity of $H^n_{I,\underline{\lambda}}$ as in the proof of Corollary 9.1.4. □

Theorem 9.3.4. *Assume that* $\mathrm{Sht}^{(\mu^+,\mu^-)}/a^{\mathbb{Z}}$ *is proper over* $X' \times X'$ *and let* $I \subset X$ *be fixed. Then the following three conditions are equivalent.*

(1) Let $\underline{\lambda}$ *be a sequence of dominant coweights for which* $\mathrm{Sht}^{\leq\underline{\lambda}}/a^{\mathbb{Z}}$ *is proper over* $(X')^r$*. Then for any pairwise distinct closed points* $x_1 \ldots x_r \in X' \setminus (I \cup T(a))$ *and* $T = \{x_1 \ldots x_r\}$*, any* $f \in \mathcal{H}_I^T$*, and any integers* $s_1 \ldots s_r \geq 1$ *equation* (9.1.5) *holds.*

(2) The same condition for $\underline{\lambda} = (\mu^+ + \mu^-)$ *with* $\mu^+ + \mu^- = (1, 0, \ldots, 0, -1)$*.*

(3) For any irreducible automorphic representation π *of* $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ *with nontrivial* K_I*-invariants and for any closed point* $x \in X' \setminus (I \cup T(a))$ *we have* $L_x(\pi, T) = L_x(\sigma(\pi), T)$*.*

Remark. The hypothesis of the theorem implies that $\mathrm{Sht}^{\leq(\mu^+ + \mu^-)}/a^{\mathbb{Z}}$ is proper over X' because there is a surjective map

$$\Delta^* \mathrm{Sht}^{\leq(\mu^+,\mu^-)}\big/a^{\mathbb{Z}} \longrightarrow \mathrm{Sht}^{\leq(\mu^+ + \mu^-)}\big/a^{\mathbb{Z}}.$$

(This is the resolution of singularities of Proposition 3.1.9.)

Corollary 9.3.5. *Suppose the fundamental lemma for* GL_d *holds for the characteristic function of the double coset of* $\varpi^{\mu^+ + \mu^-}$*. Then in the situation of Theorem 9.3.1 we have* $L_x(\pi, T) = L_x(\sigma, T)$ *for any* $x \in X'$ *such that* π_x *is unramified.*

Proof. In view of Proposition 9.1.2 this is clear if $x \notin T(a)$. However, for any irreducible automorphic representation π of $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ we can find a new $a' \in \mathbb{A}^*$ of degree 1 such that $T(a')$ avoids a given finite set of places and π is a representation of $D^*_{\mathbb{A}}/a'^{\mathbb{Z}}$ as well.

In fact, via class field theory the central character $F^* \backslash \mathbb{A}^*/a^{\mathbb{Z}} \rightarrow \bar{\mathbb{Q}}_l^*$ of π corresponds to a cyclic extension $E \mid F$ which contains no constant extension. Let T be the set of places of F which decompose completely in E, then any a' with $T(a') \subset T$ acts trivially on π. Thus it suffices to show that for any finite set of places S the greatest common divisor of $\deg(x)$ for $x \in T \setminus S$ is 1. Otherwise there is a nontrivial extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$ such that any $x \in T \setminus S$ decomposes completely in $F \otimes \mathbb{F}_{q^n}$ as well. This means that $T \setminus S$ is contained in the analogue set T' for the field $E \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, which contradicts Dirichlet densities. $\square$

Proof of Theorem 9.3.4. For any $\underline{\lambda}$ as in (1), Theorem 9.3.3 implies the equation of virtual $\mathcal{H}_I^T \times \pi(X' \setminus I)^r$-modules

$$H_{I,\underline{\lambda}} = \bigoplus_{\pi} m_\pi \cdot \pi^{K_I} \boxtimes (\rho_{\lambda_1} \circ \sigma(\pi)) \boxtimes \ldots \boxtimes (\rho_{\lambda_r} \circ \sigma(\pi))$$

with π running through the irreducible automorphic representations of $D_{\mathbb{A}}^*/a^{\mathbb{Z}}$ having nontrivial K_I-invariants.

Assuming $L_{x_i}(\sigma(\pi), T) = L_{x_i}(\pi, T)$ for all i and for all occurring π this implies equation (9.1.5), so we have (3)⇒(1)⇒(2), and it remains to show (2)⇒(3).

For a fixed closed point $x \in X' \setminus (I \cup T(a))$ let $\pi_1 \ldots \pi_k$ be a maximal family of irreducible automorphic representations with nontrivial K_I-invariants which differ only in the x-component. In particular they all have the same associated Galois representation $\sigma(\pi_1) \cong \ldots \cong \sigma(\pi_k) = \sigma$. There is a function $f \in \mathcal{H}_I^T$ satisfying $\mathrm{Tr}(f, \pi_\nu) = 1$ for all ν and $\mathrm{Tr}(f, \pi) = 0$ for any other π.

By Proposition 9.1.2, equation (9.1.5) holds in the case $\underline{\lambda} = (\mu^+, \mu^-)$. Using the chosen f, for the set $\{z_1(\sigma) \ldots z_d(\sigma)\}$ of the eigenvalues of τ_x on σ we get

$$\sum_\nu m_{\pi_\nu} \cdot \big\{z_1(\sigma) \ldots z_d(\sigma)\big\} = \bigsqcup_\nu m_{\pi_\nu} \cdot \big\{z_1(\pi_{\nu,x}) \ldots z_d(\pi_{\nu,x})\big\} . \qquad (9.3.2)$$

Similarly condition (2) gives the additional equation

$$\sum_\nu m_{\pi_\nu} \cdot \big\{z_i(\sigma)/z_j(\sigma)\big\}_{i\neq j} = \bigsqcup_\nu m_{\pi_\nu} \cdot \big\{z_i(\pi_{\nu,x})/z_j(\pi_{\nu,x})\big\}_{i\neq j} . \qquad (9.3.3)$$

Now (9.3.2) implies that the multiplicity of 1 on the left hand side of (9.3.3) is $\leq$ the multiplicity of 1 on the right hand side with equality if and only if the sets $\{z_i(\pi_{\nu,x})\}_i$ are equal for all ν. This means (2)⇒(3). $\square$

10 Decomposition under the Symmetric Groups

If the sequence $\underline{\lambda}$ consists only of repetitions of μ^+ and μ^-, then the isotypical components of $H^n_{I,\underline{\lambda}}$ with respect to the irreducible representations of the finite group $\mathrm{Stab}(\underline{\lambda})$ can be recognised as certain $H^n_{I,\underline{\lambda}'}$ for different $\underline{\lambda}'$. This follows rather formally from the sheaf theoretic version of the Springer correspondence for gl_n, which we recall first.

10.1 Generalities on $\mathrm{Coh}_{\mathcal{D}}$

In the sequel X may be any smooth curve over $\mathbb{F}_q$. The stack Coh_X^m classifies coherent $\mathcal{O}_X$-modules of length m and is equipped with a natural morphism norm:

$$N \ : \ \mathrm{Coh}_X^m \longrightarrow X^{(m)}$$

We denote by $N_0 : \mathrm{coh}_X^m \to X$ its base change by the diagonal $m : X \to X^{(m)}$, and for a closed point $x \in X$ let

$$\mathrm{coh}^m(x) \longrightarrow \mathrm{Spec}\, k(x)$$

be the base change of N_0 by the immersion $\mathrm{Spec}\, k(x) \to X$. This means $\mathrm{coh}^m(x)$ classifies coherent $\mathcal{O}_X$-modules with support in x. The notation is justified because this stack only depends on the complete local ring $\mathcal{O}_x$, which only depends on $k(x)$.

Example 10.1.1. In the case $X = \mathbb{A}^1 = \mathrm{Spec}\, \mathbb{F}_q[t]$ a coherent $\mathcal{O}_X$-module of length m is the same as an m-dimensional vector space over $\mathbb{F}_q$ with a linear action of the variable t, so

$$\mathrm{Coh}_{\mathbb{A}^1}^m = \mathrm{gl}_m \big/ \mathrm{GL}_m \ . \tag{10.1.1}$$

Under this isomorphism the norm can be identified with the map $\mathrm{gl}_m / \mathrm{GL}_m \to \mathbb{A}^m$ given by the coefficients of the characteristic polynomial. Let $\mathcal{N} \subseteq \mathrm{gl}_m$ be the closed subscheme where all coefficients of the characteristic polynomial vanish. Then (10.1.1) induces an isomorphism $\mathrm{coh}^m(0) \cong \mathcal{N} / \mathrm{GL}_m$. Using translations in $\mathbb{A}^1$ this extends to

$$\mathrm{coh}_{\mathbb{A}^1}^m \cong \mathrm{coh}^m(0) \times \mathbb{A}^1 \ .$$

Lemma 10.1.2. *Let $f : X \to Y$ be a dominant morphism of smooth curves over $\mathbb{F}_q$. Then the morphism $f_* : \mathrm{Coh}_X^m \to \mathrm{Coh}_Y^m$ is representable affine of finite type.*

Proof. If f is an open immersion, the same is true for f_*. Thus we have to show that for finite f and for $K \in \mathrm{Coh}_Y^m(S)$ the fibre of the morphism f_* in K is a relatively affine scheme over S of finite type. Let $N(K) \subseteq Y \times S$ be the norm of K and let $Z = N(K) \times_Y X$.

$$Z \xrightarrow{\ \psi\ } N(K) \xrightarrow{\ \pi\ } S$$

Here both morphisms are finite and flat. Since K is an $\mathcal{O}_{N(K)}$-module, the structure of an $\mathcal{O}_{Y\times S}$-module for K can be considered as a section of the vector bundle

$$\mathcal{H}om_{\mathcal{O}_S}\big(\pi_*\psi_*\mathcal{O}_Z \otimes_{\mathcal{O}_S} \pi_* K \ , \ \pi_* K\big) \longrightarrow S$$

satisfying certain closed conditions (compatible with $S' \to S$). $\square$

Lemma 10.1.3. *Let $f : X \to Y$ be an étale morphism of smooth curves over $\mathbb{F}_q$. Then the morphism $f_* : \mathrm{Coh}_X^m \to \mathrm{Coh}_Y^m$ is étale in precisely those points $K \in \mathrm{Coh}_Y^m(\overline{\mathbb{F}}_q)$ for which no two points of $N(K)$ have the same image under f.*

Moreover the following diagram is 2-cartesian, in particular the upper morphism f_* is étale.

$$\begin{array}{ccc} \mathrm{coh}_X & \xrightarrow{f_*} & \mathrm{coh}_Y \\ \big\downarrow & \square & \big\downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

Proof. We need to show that f_* is formally étale and that the S-valued points of the diagram are 2-cartesian. For both assertions it is sufficient to consider the category of S which are finite over $\overline{\mathbb{F}}_q$, in which case they follow from the fact that f induces isomorphisms of the complete local rings of $X_{\overline{\mathbb{F}}_q}$ and $Y_{\overline{\mathbb{F}}_q}$.

(If two points of $N(K)$ coincide, then the dimension of $\mathcal{A}ut(f_*K)$ is greater than the dimension of $\mathcal{A}ut(K)$, which means the fibre of f_* has positive dimension in K.) □

Corollary 10.1.4. *Locally in X there is an isomorphism $\mathrm{coh}_X^m \cong \mathrm{coh}^m(0) \times X$ such that the morphism N_0 is identified with the second projection. In particular the stack coh_X^m is reduced.*

Proof. Locally in X there is an étale morphism to $\mathbb{A}^1$, which proves the first assertion. The scheme $\mathcal{N}$ is reduced because it is generically smooth and a complete intersection (it is defined by m equations), i.e. it has no embedded components. □

Let now $\mathcal{D}$ be an Azumaya algebra of rank d^2 on X. As in Definition 1.1.1 we can define the stack $\mathrm{Coh}_{\mathcal{D}}^m$ of coherent $\mathcal{D}$-modules of length m, which is equipped with a morphism norm:

$$N \ : \ \mathrm{Coh}_{\mathcal{D}}^m \longrightarrow X^{(m)}$$

As before, the base change of N by the diagonal is denoted by $N_0 : \mathrm{coh}_{\mathcal{D}}^m \to X$, and the restriction of N_0 to a closed point $x \in X$ is $\mathrm{coh}_{\mathcal{D}}^m(x) \to \mathrm{Spec}\, k(x)$. Since an isomorphism $\mathcal{D}_x \cong M_d(\mathcal{O}_X)$ induces an isomorphism $\mathrm{coh}_{\mathcal{D}}^m(x) \cong \mathrm{coh}^m(x)$, this stack is independent of $\mathcal{D}$. Moreover a trivialisation of $\mathcal{D}$ over $\mathbb{F}_{q^n}$ induces an isomorphism $\mathrm{Coh}_{\mathcal{D}}^m \otimes \mathbb{F}_{q^n} \cong \mathrm{Coh}_X^m \otimes \mathbb{F}_{q^n}$ (cf. section 1.2). So Corollary 10.1.4 implies:

Corollary 10.1.5. *Locally in X there is an isomorphism*

$$\mathrm{coh}_{\mathcal{D}}^m \otimes \mathbb{F}_{q^n} \cong (\mathrm{coh}^m(0) \times X) \otimes \mathbb{F}_{q^n}$$

such that the morphism N_0 is identified with the second projection. In particular the stack $\mathrm{coh}_{\mathcal{D}}^m$ is reduced. □

Definition 10.1.6. Let $\widetilde{\mathrm{Coh}}^m_{\mathcal{D}}(S)$ be the groupoid of $K \in \mathrm{Coh}^m_{\mathcal{D}}(S)$ equipped with a filtration

$$K = K^m \supset \ldots \supset K^1 \supset K^0 = 0$$

such that all K^i/K^{i-1} lie in $\mathrm{Coh}^1_{\mathcal{D}}(S)$.

In the following commutative diagram the morphism N^m is given by the quotients K^i/K^{i-1}, and p is given by forgetting the filtration.

$$\begin{array}{ccc} \widetilde{\mathrm{Coh}}^m_{\mathcal{D}} & \xrightarrow{N^m} & X^m \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathrm{Coh}^m_{\mathcal{D}} & \xrightarrow{N} & X^{(m)} \end{array} \tag{10.1.2}$$

We denote by $X^{(m)}_{\mathrm{rss}} \subseteq X^{(m)}$ the complement of all diagonals.

Proposition 10.1.7. *The morphism p is representable projective. Over $X^{(m)}_{\mathrm{rss}}$ diagram* (10.1.2) *is 2-cartesian, in particular there p is an $\mathfrak{S}_m$-torsor.*

Proof. The fibre of p in $K \in \mathrm{Coh}^m_{\mathcal{D}}(S)$ can be viewed as a subsheaf of a flag variety of $(p_2)_*K$ which is defined by closed conditions (here $p_2 : X \times S \to S$ denotes the second projection). Thus p is representable projective.

Over $X^{(m)}_{\mathrm{rss}}$ any filtration of K splits uniquely, and a decomposition $N(K) = N_1 + \ldots + N_m$ with effective divisors N_i is equivalent to a decomposition $K = K_1 \oplus \ldots \oplus K_m$ with $N(K_i) = N_i$. The group $\mathfrak{S}_m$ acts in both cases by permutation of the constituents. $\square$

Proposition 10.1.8. *The morphism $N^m : \widetilde{\mathrm{Coh}}^m_{\mathcal{D}} \to X^m$ is smooth of relative dimension $-m$.*

Proof. First assume $m = 1$. Over $\overline{\mathbb{F}}_q$, using a trivialisation of $\mathcal{D}$ the norm can be identified with the smooth morphism $\overline{X}/\mathbb{G}_m \to \overline{X}$. The general case follows inductively using Lemma 3.2.9. $\square$

Let $\widetilde{\mathrm{coh}}^m_{\mathcal{D}} \to X$ be the base change of N^m by the diagonal $X \to X^m$. The morphism

$$q \;:\; \widetilde{\mathrm{coh}}^m_{\mathcal{D}} \longrightarrow \mathrm{coh}^m_{\mathcal{D}}$$

given by forgetting the filtration is the resolution of singularities of Definition 3.2.7 in the special case of the generic stratum. Up to a closed immersion with a nilpotent ideal it coincides with the base change of p. Its fibre in the closed point $x \in X$, which we write as $q(x) : \widetilde{\mathrm{coh}}^m_{\mathcal{D}}(x) \to \mathrm{coh}^m_{\mathcal{D}}(x)$, can be identified with the Springer resolution $\widetilde{\mathcal{N}}/\,\mathrm{GL}_m \to \mathcal{N}/\,\mathrm{GL}_m$ over $k(x)$.

In section 3.2 we defined locally closed substacks $\mathrm{Coh}^{\underline{m}}_{\mathcal{D}} \subseteq \mathrm{Coh}^m_{\mathcal{D}}$ and their inverse images $\mathrm{coh}^{\underline{m}}_{\mathcal{D}} \subseteq \mathrm{coh}^m_{\mathcal{D}}$. By Proposition 3.2.4 the latter are smooth over X of relative dimension $-m - d(\underline{m})$, using the notation

$$d(\underline{m}) = 2\sum(i-1)m_i\,. \tag{10.1.3}$$

Hence the codimension of $\mathrm{coh}^{\underline{m}}_{\mathcal{D}} \subseteq \mathrm{coh}^m_{\mathcal{D}}$ is $d(\underline{m})$.

Proposition 10.1.9. *The morphism p is small with respect to the given stratification. The morphisms q and $q(x)$ are semismall and all strata are relevant (this means the dimension of the fibre equals half the dimension of the stratum).*

Proof. Each stratum $\mathrm{Coh}^{\underline{m}}_{\mathcal{D}}$ has constant dimension in all points (Proposition 3.2.1) and only the generic stratum is mapped surjectively to $X^{(m)}$ by the norm. So we need to show that all fibres of p over $\mathrm{Coh}^{\underline{m}}_{\mathcal{D}}$ have dimension $d(\underline{m})/2$.

For $K = K_1 \oplus K_2$ in $\mathrm{Coh}^{\underline{m}}_{\mathcal{D}}(\overline{\mathbb{F}}_q)$ such that the K_i have disjoint supports, the number $d(\underline{m})$ is additive. Since the natural map from the maximal filtrations of K to the pairs of maximal filtrations of the K_i is finite and surjective, it suffices to consider the fibres of q. Their dimension has been computed in Proposition 3.2.10. $\square$

10.2 Springer correspondence for $\mathrm{Coh}_{\mathcal{D}}$

First we recall the classical parametrisation of the irreducible representations of the symmetric groups and their connection with the irreducible representations of the general linear groups.

Let $m \geq 0$ be an integer and let V be a finite dimensional vector space over $\overline{\mathbb{Q}}_l$. On $V^{\otimes m}$ there is the diagonal action of $\mathrm{GL}(V)$ and the action of $\mathfrak{S}_m$ by permutation of the tensor factors. For any finite dimensional representation χ of $\mathfrak{S}_m$ over $\overline{\mathbb{Q}}_l$ we denote by $V(\chi)$ the $\mathrm{GL}(V)$-representation $\mathrm{Hom}_{\mathfrak{S}_m}(\chi, V^{\otimes m})$.

For any partition $\underline{m} = (m_1 \geq \ldots \geq m_r)$ of the integer m we write $\mathfrak{S}_{\underline{m}} = \mathfrak{S}_{m_1} \times \ldots \times \mathfrak{S}_{m_r} \subseteq \mathfrak{S}_m$ and let $I_{\underline{m}} = \mathrm{Ind}_{\mathfrak{S}_{\underline{m}}}^{\mathfrak{S}_m} \mathbb{1}$, where $\mathbb{1}$ denotes the trivial one-dimensional representation. There is a natural bijection $\underline{m} \mapsto \lambda_{\underline{m}}$ between the partitions of m of length $\leq \dim(V)$ and the positive dominant weights of degree m for the group $\mathrm{GL}(V)$, given by $\lambda^{(i)}_{\underline{m}} = m_i$.

For any dominant weight λ we denote by V_λ the irreducible representation of $\mathrm{GL}(V)$ with highest weight λ.

Lemma 10.2.1. *There is a unique parametrisation $\underline{m} \mapsto \chi_{\underline{m}}$ of the irreducible representations of $\mathfrak{S}_m$ by the partitions of m such that $I_{\underline{m}} \equiv \chi_{\underline{m}}$ modulo all representations $\chi_{\underline{m}'}$ with $\underline{m}' > \underline{m}$. If the length of $\underline{m}$ is at most $\dim(V)$, we have $V(\chi_{\underline{m}}) \cong V_{\lambda_{\underline{m}}}$, and $V(\chi_{\underline{m}}) = 0$ otherwise.*

This is completely classical. A refinement of Lemma 10.2.1 is that if the length of $\underline{m}$ is at most $\dim(V)$, then the multiplicity of $\chi_{\underline{m}}$ in $I_{\underline{m}'}$ equals the dimension of the $\lambda_{\underline{m}'}$-eigenspace in $V_{\lambda_{\underline{m}}}$. A Reference for these assertions is [FH], Lemma 4.24, Corollary 4.39, Theorem 6.3, Proposition 15.47, and Formula (A.19).

We denote by $\mathrm{Coh}^m_{\mathcal{D},\mathrm{rss}}$ the inverse image of $X^{(m)}_{\mathrm{rss}}$ under the norm and by $j : \mathrm{Coh}^m_{\mathcal{D},\mathrm{rss}} \subseteq \mathrm{Coh}^m_{\mathcal{D}}$ the natural embedding. Writing $p' : \widetilde{\mathrm{Coh}}^m_{\mathcal{D},\mathrm{rss}} \to \mathrm{Coh}^m_{\mathcal{D},\mathrm{rss}}$ for the restriction of p, Proposition 10.1.9 implies

$$Rp_*\overline{\mathbb{Q}}_l = j_{!*}Rp'_*\overline{\mathbb{Q}}_l \,.$$

Therefore the natural $\mathfrak{S}_m$-action on $Rp'_*\overline{\mathbb{Q}}_l$ (Proposition 10.1.7) extends uniquely to an action on $Rp_*\overline{\mathbb{Q}}_l$, which in turn induces an action on $Rq_*\overline{\mathbb{Q}}_l$ by pullback. Since $\mathrm{Coh}^m_{\mathcal{D}}$ is smooth, the equation $(Rp'_*\overline{\mathbb{Q}}_l)^{\mathfrak{S}_m} = \overline{\mathbb{Q}}_l$ gives $(Rp_*\overline{\mathbb{Q}}_l)^{\mathfrak{S}_m} = \overline{\mathbb{Q}}_l$ by intermediate extension. This implies

$$(Rq_*\overline{\mathbb{Q}}_l)^{\mathfrak{S}_m} = \overline{\mathbb{Q}}_l \,.$$

Lemma 10.2.2. *Let $f : X \to Y$ be an étale morphism of smooth curves over $\mathbb{F}_q$ and let $\varphi_0 = f_* : \mathrm{coh}^m_X \to \mathrm{coh}^m_Y$. Then the natural homomorphism*

$$c \;:\; \varphi_0^* R(q_Y)_*\overline{\mathbb{Q}}_l \longrightarrow R(q_X)_*\overline{\mathbb{Q}}_l$$

is a $\mathfrak{S}_m$-equivariant isomorphism.

Proof. Let $U \subseteq \mathrm{Coh}^m_X$ be the open subset where $\varphi = f_* : \mathrm{Coh}^m_X \to \mathrm{Coh}^m_Y$ is étale. By Lemma 10.1.3 we have $U' = \varphi^{-1}(\mathrm{Coh}^m_{Y,\mathrm{rss}}) \subseteq U$. Over U' the natural map

$$b \;:\; \varphi^* R(p_Y)_*\overline{\mathbb{Q}}_l \longrightarrow R(p_X)_*\overline{\mathbb{Q}}_l$$

is an equivariant isomorphism, because it comes from a morphism of $\mathfrak{S}_m$-torsors. This property extends to U by intermediate extension. Now c is the base change of b by the natural morphism $\mathrm{coh}^m_X \to \mathrm{Coh}^m_X$, which factors over U, and the assertion follows by pullback. □

For any irreducible representation χ of $\mathfrak{S}_m$ over $\overline{\mathbb{Q}}_l$ the χ-isotypic component $(Rq_*\overline{\mathbb{Q}}_l)(\chi)$ is defined in such a way that $Rq_*\overline{\mathbb{Q}}_l$ is canonically isomorphic to the direct sum of all $(Rq_*\overline{\mathbb{Q}}_l)(\chi)\otimes\chi$. Let $j_{\underline{m}} : \mathrm{coh}^{\underline{m}}_{\mathcal{D}} \subseteq \mathrm{coh}^m_{\mathcal{D}}$ be the natural embedding and let

$$\mathcal{F}_{\underline{m}} = (j_{\underline{m}})_{!*}\overline{\mathbb{Q}}_l \left\langle -d(\underline{m})\right\rangle \,.$$

The sheaf theoretic version of the Springer correspondence for gl_m in this context assumes the following form.

Theorem 10.2.3. *For any $\underline{m}$ there is an isomorphism*

$$(Rq_*\overline{\mathbb{Q}}_l)(\chi_{\underline{m}}) \cong \mathcal{F}_{\underline{m}} \,.$$

Proof. It has been remarked in [Lau87] that the corresponding statement for $q(0) : \widetilde{\mathrm{coh}}^m(0) \to \mathrm{coh}^m(0)$ is precisely the Springer correspondence for gl_m, which is explained in [KW], Chapter VI.

Using the isomorphisms $\mathrm{coh}^m_{\mathbb{A}^1} \cong \mathrm{coh}^m(0) \times \mathbb{A}^1$ and $\widetilde{\mathrm{coh}}^m_{\mathbb{A}^1} \cong \widetilde{\mathrm{coh}}^m(0) \times \mathbb{A}^1$ we obtain the assertion of the theorem in the case $X = \mathbb{A}^1$ with trivial $\mathcal{D}$. Since any X admits locally an étale map to $\mathbb{A}^1$, Lemma 10.2.2 implies the assertion for arbitrary X with trivial $\mathcal{D}$ (that $(Rq_*\overline{\mathbb{Q}}_l)(\chi_{\underline{m}})$ is the intermediate extension of a smooth sheaf by $j_{\underline{m}}$ is a local statement, and the isomorphism class of this smooth sheaf can be detected locally as well).

For the proof of the general case let X be geometrically irreducible. Over $\overline{\mathbb{F}}_q$ there exists the desired isomorphism because there $\mathcal{D}$ can be trivialised. This means that

$$\mathcal{G} = (Rq_*\overline{\mathbb{Q}}_l)(\chi_{\underline{m}}) \langle d(\underline{m}) \rangle \Big|_{\mathrm{coh}^{\underline{m}}_{\mathcal{D}}}$$

is a smooth one-dimensional l-adic sheaf which is trivial over $\overline{\mathbb{F}}_q$, and we have to show that $\mathcal{G}$ is trivial itself. Since $\mathrm{coh}^{\underline{m}}_{\mathcal{D}}$ is geometrically irreducible, $\mathcal{G}$ can be considered as a one-dimensional l-adic representation of $\mathrm{Gal}(\overline{\mathbb{F}}_q \,|\, \mathbb{F}_q)$.

The isomorphisms $\mathrm{coh}^m_{\mathcal{D}}(x) \cong \mathrm{coh}^m(x)$ and $\widetilde{\mathrm{coh}}^m_{\mathcal{D}}(x) \cong \widetilde{\mathrm{coh}}^m(x)$ imply the assertion of the theorem for the morphisms $q(x)$. Because each $\mathrm{coh}^{\underline{m}}_{\mathcal{D}}(x)$ is geometrically irreducible over $\mathrm{Spec}\, k(x)$, this means that the restrictions of the representation $\mathcal{G}$ to $\mathrm{Gal}(\overline{\mathbb{F}}_q \,|\, k(x))$ are trivial. These subgroups generate $\mathrm{Gal}(\overline{\mathbb{F}}_q \,|\, \mathbb{F}_q)$. □

Remark. The construction of the Springer correspondence as explained in Chapter VI of [KW] only shows that the statement of the theorem holds for some parametrisation of the irreducible representations of $\mathfrak{S}_m$ by the set of partitions $\underline{m}$.

We sketch a proof that this parametrisation coincides with the one we fixed before: in view of the characterisation of our parametrisation in Lemma 10.2.1, we have to show that every $(Rq_*\overline{\mathbb{Q}}_l)^{\mathfrak{S}_{\underline{m}}}$ is the direct sum of some $\mathcal{F}_{\underline{m}'}$ with $\underline{m}' \geq \underline{m}$. This is equivalent to the condition

$$\underline{m}' \not\geq \underline{m} \implies \mathcal{H}^{d(\underline{m}')}(Rq_*\overline{\mathbb{Q}}_l)^{\mathfrak{S}_{\underline{m}}} \Big|_{\mathrm{coh}^{\underline{m}'}} = 0.$$

To prove this condition, we factor q as $\widetilde{\mathrm{coh}}^m \xrightarrow{q'} \widetilde{\mathrm{coh}}' \xrightarrow{q''} \mathrm{coh}^m$ where $\widetilde{\mathrm{coh}}'$ parametrises partial filtrations of type $\underline{m}$, i.e. with quotients of lengths $m_1 \ldots m_r$. Then there is a natural action of $\mathfrak{S}_{\underline{m}}$ on $Rq'_*\overline{\mathbb{Q}}_l$ with invariants $\overline{\mathbb{Q}}_l$, which implies

$$(Rq_*\overline{\mathbb{Q}}_l)^{\mathfrak{S}_{\underline{m}}} = Rq''_*\overline{\mathbb{Q}}_l.$$

Thus we have to prove that in the case $\underline{m}' \not\geq \underline{m}$ the dimension of the fibres of q'' over $\mathrm{coh}^{\underline{m}'}$ is strictly less than $d(\underline{m}')/2$. An equivalent statement is that over

$\mathrm{coh}^{\underline{m}'}$ the map q' does not have a fibre of dimension zero. The existence of such a fibre means that a sheaf of type $\underline{m}'$ admits a filtration of type $\underline{m}$ such that all its subquotients are cyclic sheaves. This is possible precisely for $\underline{m}' \geq \underline{m}$.

10.3 Decomposition of cohomology

We now change the notation. We fix a sequence of integers $\underline{m} = (m_1, \ldots, m_r)$ with sum zero and call a sequence of dominant coweights $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r)$ definite of degree $\underline{m}$ if $\lambda_i \in P^{++}_{m_i}$ for all i. For such $\underline{\lambda}$ we consider

$$\lambda_i^+ = \begin{cases} \lambda_i & \text{if } m_i \geq 0 \\ w_0 \lambda_i^\vee & \text{if } m_i \leq 0 \end{cases}$$

as a partition of $|m_i|$ of length $\leq d$. The sequences $\underline{\lambda}^+$ which arise in this way parametrise a subset of the irreducible representations of $\mathfrak{S}_{\underline{m}} = \mathfrak{S}_{|m_1|} \times \ldots \times \mathfrak{S}_{|m_r|}$ over $\bar{\mathbb{Q}}_l$ via

$$\underline{\lambda}^+ \longmapsto \chi_{\underline{\lambda}^+} = \chi_{\lambda_1^+} \boxtimes \ldots \boxtimes \chi_{\lambda_r^+}$$

using the notation of Lemma 10.2.1. The permutation action of $\mathrm{Stab}(\underline{\lambda})$ makes $\chi_{\underline{\lambda}^+}$ into a representation of the semidirect product $\mathfrak{S}_{\underline{m}} \rtimes \mathrm{Stab}(\underline{\lambda})$.

We form the following sequence $\underline{\widetilde{\lambda}}$ of length $M = \sum |m_i|$.

$$\underline{\widetilde{\lambda}} = \underline{\widetilde{\lambda}}_{(1)} \ldots \underline{\widetilde{\lambda}}_{(r)}\,, \qquad \underline{\widetilde{\lambda}}_{(i)} = \begin{cases} (\mu^+)^{m_i} = (\mu^+, \ldots, \mu^+) & \text{if } m_i \geq 0 \\ (\mu^-)^{-m_i} = (\mu^-, \ldots, \mu^-) & \text{if } m_i \leq 0 \end{cases}$$

Then the stabiliser $\mathrm{Stab}(\underline{\widetilde{\lambda}}) \subseteq \mathfrak{S}_M$ is isomorphic to $(\mathfrak{S}_{M/2})^2$ and contains the naturally embedded subgroup $\mathfrak{S}_{\underline{m}} \rtimes \mathrm{Stab}(\underline{m})$, which respects the given decomposition of $\underline{\widetilde{\lambda}}$ into subsequences (but not necessarily the order of these subsequences).

Assumption 10.3.1. The stack $\mathrm{Sht}^{\underline{\widetilde{\lambda}}}/a^{\mathbb{Z}}$ is proper over $(X')^M$.

This implies that all $\mathrm{Sht}^{\leq \underline{\lambda}}/a^{\mathbb{Z}}$ for definite $\underline{\lambda}$ of degree $\underline{m}$ are proper over $(X')^r$ as well. The assumption holds for example if D is sufficiently ramified with respect to $M/2$.

The direct images $H^n_{I,\underline{\widetilde{\lambda}}}$ introduced in Definition 8.2.2 are smooth l-adic sheaves on $(X' \setminus I)^M$ (Lemma 8.2.3), which via the construction in Proposition 8.2.4 are made into l-adic representations of

$$\pi_1(X' \setminus I, \bar{y}\,)^M \rtimes \mathrm{Stab}(\underline{\widetilde{\lambda}}) \times \mathcal{H}_I\,.$$

Here $\bar{y} \in X(F^{\mathrm{alg}})$ is the natural geometric point.

Let Δ be the multi-diagonal $(x_1 \dots x_r) \mapsto (x_1^{|m_1|} \dots x_r^{|m_r|})$, which may be considered either as a morphism $(X' \setminus I)^r \to (X' \setminus I)^M$ or as a map $G_F^r \to G_F^M$. The equivariant action of $\mathrm{Stab}(\widetilde{\underline{\lambda}})$ on $H^n_{I,\widetilde{\underline{\lambda}}}$ restricts to an ordinary action of $\mathfrak{S}_{\underline{m}}$ on $\Delta^* H^n_{I,\widetilde{\underline{\lambda}}}$, considered either as a sheaf or as a representation. So for any irreducible representation χ of $\mathfrak{S}_{\underline{m}}$ the χ-isotypic component $\Delta^* H^n_{I,\widetilde{\underline{\lambda}}}(\chi)$ is defined. In the case $\chi = \chi_{\underline{\lambda}^+}$ this is a representation of $\pi_1(X' \setminus I, \bar{y}\,)^r \rtimes \mathrm{Stab}(\underline{\lambda}) \times \mathcal{H}_I$.

Theorem 10.3.2. *For any definite $\underline{\lambda}$ of degree $\underline{m}$ there is an isomorphism of l-adic sheaves of $(X' \setminus I)^r$*

$$(\Delta^* H^n_{I,\widetilde{\underline{\lambda}}})(\chi_{\underline{\lambda}^+}) \cong H^n_{I,\underline{\lambda}}, \tag{10.3.1}$$

which is compatible with the action of $\pi_1(X' \setminus I, \bar{y}\,)^r \rtimes \mathrm{Stab}(\underline{\lambda}) \times \mathcal{H}_I$ and with the natural inclusions for $I \subseteq J$. For any other irreducible representation χ of $\mathfrak{S}_{\underline{m}}$ the isotypic component $(\Delta^* H^n_{I,\widetilde{\underline{\lambda}}})(\chi)$ vanishes.

Corollary 10.3.3. *Let π be an irreducible automorphic representation of $D^*_{\mathbb{A}}/a^{\mathbb{Z}}$ over $\overline{\mathbb{Q}}_l$. Then for any definite $\underline{\lambda}$ of degree $\underline{m}$ there are isomorphisms*

$$\big(\Delta^* H^n_{\widetilde{\underline{\lambda}}}(\pi)\big)(\chi_{\underline{\lambda}^+}) \cong H^n_{\underline{\lambda}}(\pi)$$

of semisimple representations of $(G_F)^r \rtimes \mathrm{Stab}(\underline{\lambda})$, while $(\Delta^* H^n_{\widetilde{\underline{\lambda}}}(\pi))(\chi)$ vanishes for any other irreducible representation χ of $\mathfrak{S}_{\underline{m}}$. □

In the remainder of this section we will deduce Theorem 10.3.2 from the Springer correspondence for $\mathrm{Coh}_{\mathcal{D}}$ (Theorem 10.2.3), which is a rather formal matter. We start from the following commutative diagram with 2-cartesian left half.

$$\begin{array}{ccccccc}
\widetilde{\pi} : \mathrm{Sht}_I^{\widetilde{\underline{\lambda}}}/a^{\mathbb{Z}} & \xrightarrow{\widetilde{\alpha}} & \prod \widetilde{\mathrm{Coh}}_{\mathcal{D}}^{|m_i|} & \longrightarrow & (X' \setminus I)^M \\
\downarrow{\scriptstyle p'} & \square & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle r} \\
\pi : \mathrm{Sht}_I^{\underline{m}}/a^{\mathbb{Z}} & \xrightarrow[\alpha]{} & \prod \mathrm{Coh}_{\mathcal{D}}^{|m_i|} & \longrightarrow & \prod (X' \setminus I)^{(|m_i|)}
\end{array} \tag{10.3.2}$$

Here p is the product of the morphisms $p_i : \widetilde{\mathrm{Coh}}_{\mathcal{D}}^{|m_i|} \to \mathrm{Coh}_{\mathcal{D}}^{|m_i|}$ given by forgetting the filtration, while p' has been called a collapsing map in section 2.3. Over $\prod (X' \setminus I)^{(|m_i|)}_{\mathrm{rss}}$ the right square is 2-cartesian as well, and there the morphisms r, p, p' are $\mathfrak{S}_{\underline{m}}$-torsors. Let

$$\Delta \ : (X' \setminus I)^r \longrightarrow \prod\, (X' \setminus I)^{|m_i|} = (X' \setminus I)^M$$

and

$$\delta \ : (X' \setminus I)^r \longrightarrow \prod\, (X' \setminus I)^{(|m_i|)}$$

be the products of the respective diagonals of lengths $|m_1| \dots |m_r|$. Up to a closed immersion with a nilpotent ideal the base change of (10.3.2) by δ is the diagram

$$\begin{array}{ccccccc} \widetilde{\pi}_0 \ : \ \Delta^* \operatorname{Sht}_I^{\widetilde{\underline{\lambda}}}/a^{\mathbb{Z}} & \xrightarrow{\widetilde{\alpha}_0} & \prod \widetilde{\operatorname{coh}}_{\mathcal{D}}^{|m_i|} & \longrightarrow & (X' \setminus I)^r \\ \Big\downarrow q' & \square & \Big\downarrow q & & \Big\| \\ \pi_0 \ : \ \delta^* \operatorname{Sht}_I^{\underline{m}}/a^{\mathbb{Z}} & \xrightarrow[\alpha_0]{} & \prod \operatorname{coh}_{\mathcal{D}}^{|m_i|} & \longrightarrow & (X' \setminus I)^r \end{array} \tag{10.3.3}$$

Here q is the product of the morphisms $q_i : \widetilde{\operatorname{coh}}_{\mathcal{D}}^{|m_i|} \to \operatorname{coh}_{\mathcal{D}}^{|m_i|}$.

The $\mathfrak{S}_{|m_i|}$-action on $R(q_i)_*\overline{\mathbb{Q}}_l$ induces a $\mathfrak{S}_{\underline{m}}$-action on

$$Rq'_*\overline{\mathbb{Q}}_l = \alpha_0^* Rq_*\overline{\mathbb{Q}}_l = \alpha_0^*\big(R(q_1)_*\overline{\mathbb{Q}}_l \boxtimes \dots \boxtimes R(q_r)_*\overline{\mathbb{Q}}_l\big)$$

Using the equation $\widetilde{\pi}_0 = \pi_0\, q'$ this gives a $\mathfrak{S}_{\underline{m}}$-action on $R(\widetilde{\pi}_0)_*\overline{\mathbb{Q}}_l$ and hence on

$$\Delta^* H^n_{I,\widetilde{\underline{\lambda}}} = R^n(\widetilde{\pi}_0)_*\overline{\mathbb{Q}}_l\langle\dim(\widetilde{\underline{\lambda}})\rangle$$

with $\dim(\widetilde{\underline{\lambda}}) = M(d-1)$.

Lemma 10.3.4. *The such defined $\mathfrak{S}_{\underline{m}}$-action on $\Delta^* H^n_{I,\widetilde{\underline{\lambda}}}$ coincides with the restriction of the equivariant action of $\operatorname{Stab}(\widetilde{\underline{\lambda}})$ on $H^n_{I,\widetilde{\underline{\lambda}}}$ which has been constructed in Proposition 8.2.4.*

Proof. The $\mathfrak{S}_{\underline{m}}$-action on $Rq_*\overline{\mathbb{Q}}_l = \delta^* Rp_*\overline{\mathbb{Q}}_l$ we used above comes from an action on $Rp_*\overline{\mathbb{Q}}_l$. Therefore the action on the direct images of $\widetilde{\pi}_0$ can be obtained in the following way as well:

$$p \ \rightsquigarrow \ \alpha^* p = p' \ \rightsquigarrow \ \pi p' = r\widetilde{\pi} \ \rightsquigarrow \ \delta^*(r\widetilde{\pi}) \overset{i}{\supseteq} \Delta^*\widetilde{\pi} = \widetilde{\pi}_0$$

In the last step we use that the infinitesimal closed immersion i induces an isomorphism

$$\delta^* r_* H^n_{I,\widetilde{\underline{\lambda}}} = \Delta^* H^n_{I,\widetilde{\underline{\lambda}}}\,.$$

So we have to show that the $\mathfrak{S}_{\underline{m}}$-action on $Rp_*\overline{\mathbb{Q}}_l$ and the equivariant $\mathfrak{S}_{\underline{m}}$-action on $R^n\widetilde{\pi}_*\overline{\mathbb{Q}}_l\langle\dim(\widetilde{\underline{\lambda}})\rangle$ give the same action on $r_*R^n\widetilde{\pi}_*\overline{\mathbb{Q}}_l\langle\dim(\widetilde{\underline{\lambda}})\rangle$. Over the dense open subset

$$j \ : \ \prod (X' \setminus I)^{(|m_i|)}_{\text{rss}} \ \subseteq \ \prod (X' \setminus I)^{(|m_i|)}$$

this is true because both actions arise from the $\mathfrak{S}_{\underline{m}}$-action on the upper row of (10.3.2). Since $H^n_{I,\widetilde{\underline{\lambda}}}$ is smooth, the map $\operatorname{End}(r_*H^n_{I,\widetilde{\underline{\lambda}}}) \to \operatorname{End}(j^*r_*H^n_{I,\widetilde{\underline{\lambda}}})$ is injective. □

Proof of Theorem 10.3.2. For any definite sequence $\underline{\lambda}$ of weight $\underline{m}$ let $j_{\underline{\lambda}^+}$ be the product of the natural embeddings

$$j_{\lambda_i^+} \;:\; \mathrm{coh}_{\mathcal{D}}^{\lambda_i^+} \;\subseteq\; \mathrm{coh}_{\mathcal{D}}^{|m_i|}$$

for $i = 1 \ldots r$, let $\mathcal{F}_{\lambda_i^+} = (j_{\lambda_i^+})_{!*}\overline{\mathbb{Q}}_l\langle -d(\lambda_i^+)\rangle$ where $d(\lambda_i^+)$ is the codimension of $j_{\lambda_i^+}$ (see (10.1.3)) and let

$$\mathcal{F}_{\underline{\lambda}^+} = \mathcal{F}_{\lambda_1^+} \boxtimes \ldots \boxtimes \mathcal{F}_{\lambda_r^+}\,.$$

The base change of $j_{\underline{\lambda}^+}$ by α_0 is the natural embedding $j_{\underline{\lambda}} : \mathrm{Sht}_I^{\underline{\lambda}}/a^{\mathbb{Z}} \subseteq \delta^* \,\mathrm{Sht}_I^{\underline{m}}/a^{\mathbb{Z}}$. By smoothness of α_0 we get a canonical isomorphism

$$\alpha_0^*\,\mathcal{F}_{\underline{\lambda}^+}\langle \dim(\widetilde{\underline{\lambda}})\rangle = (j_{\underline{\lambda}})_{!*}\overline{\mathbb{Q}}_l\langle \dim(\underline{\lambda})\rangle = IC_{\underline{\lambda}}\,.$$

Thus Theorem 10.2.3 gives isomorphisms

$$\big(Rq'_*\overline{\mathbb{Q}}_l\langle \dim(\widetilde{\underline{\lambda}})\rangle\big)(\chi_{\underline{\lambda}^+}) \cong IC_{\underline{\lambda}}\,, \tag{10.3.4}$$

while the isotypic components of $Rq'_*\overline{\mathbb{Q}}_l$ for all other irreducible representations of $\mathfrak{S}_{\underline{m}}$ vanish. In view of Lemma 10.3.4 the application of $R^n(\pi_0)_*$ results in the desired isomorphisms (10.3.1) of sheaves on $(X' \setminus I)^r$:

$$\big(\Delta^* R^n\widetilde{\pi}_*\overline{\mathbb{Q}}_l\langle \dim(\widetilde{\underline{\lambda}})\rangle\big)(\chi_{\underline{\lambda}^+}) \underset{(1)}{=} \big(R^n(\pi_0)_*Rq'_*\overline{\mathbb{Q}}_l\langle \dim(\widetilde{\underline{\lambda}})\rangle\big)(\chi_{\underline{\lambda}^+}) \underset{(2)}{\cong} R^n(\pi_0)_*IC_{\underline{\lambda}} \tag{10.3.5}$$

It remains to show that these are compatible with the various actions.

To begin with, the compatibility of q' with the Hecke correspondences, with the partial Frobenii, and with the permutations in $\mathrm{Stab}(\underline{m})$ allows to construct cohomological correspondences for $Rq'_*\overline{\mathbb{Q}}_l\langle \dim(\widetilde{\underline{\lambda}})\rangle$ coming from the cohomological correspondences for $\overline{\mathbb{Q}}_l\langle \dim(\widetilde{\underline{\lambda}})\rangle$ on $\mathrm{Sht}_I^{\widetilde{\underline{\lambda}}}/a^{\mathbb{Z}}$, for which (1) is equivariant. Here the partial Frobenius F_i for $\mathrm{Sht}_I^{\underline{m}}/a^{\mathbb{Z}}$ is related to the product

$$\widetilde{F}_i = F_{|m_1|+\ldots+|m_{i-1}|+1} \circ \ldots \circ F_{|m_1|+\ldots+|m_i|}$$

on $\mathrm{Sht}_I^{\widetilde{\underline{\lambda}}}$, which is defined on an open subset of $(X' \setminus I)^M$ containing the Δ-image of the complement of all diagonals. Something similar holds for the permutations.

On the other hand, on

$$Rq_*\overline{\mathbb{Q}}_l\langle \dim(\widetilde{\underline{\lambda}})\rangle = R(q_1)_*\overline{\mathbb{Q}}_l\langle |m_1|(d-1)\rangle \boxtimes \ldots \boxtimes R(q_r)_*\overline{\mathbb{Q}}_l\langle |m_r|(d-1)\rangle$$

we have the action of F_i by the absolute Frobenius of the i-th component and the action of $\mathrm{Stab}(\underline{m})$ by permutation of the factors. Since the morphism α_0 is invariant under the Hecke correspondences and equivariant with respect to the action of the partial Frobenii and of $\mathrm{Stab}(\underline{m})$, we obtain cohomological correspondences for $Rq'_*\bar{\mathbb{Q}}_l\langle\dim(\widetilde{\underline{\lambda}})\rangle = \alpha_0^* Rq_*\bar{\mathbb{Q}}_l\langle\dim(\widetilde{\underline{\lambda}})\rangle$ over the Hecke correspondences, over the partial Frobenii, and over the permutations in $\mathrm{Stab}(\underline{\lambda}) \subseteq \mathrm{Stab}(\underline{m})$. With respect to the induced actions on the middle term of (10.3.5) the map (2) is equivariant.

It is easy to see that the two constructions of cohomological correspondences for $Rq'_*\bar{\mathbb{Q}}_l\langle\dim(\widetilde{\underline{\lambda}})\rangle$ give the same result. □

10.4 Applications

Originally the idea was to use the decomposition of the Galois representations in Corollary 10.3.3 in order to reduce the computation of all $H_{\underline{\lambda}}(\pi)$ to the case that all λ_i are μ^+ or μ^-, because then Drinfeld's case of the fundamental lemma can be applied. However, this only gives the description of these restricted $H_{\underline{\lambda}}(\pi)$ as Galois representations without the action of the symmetric groups. On the other hand we have seen that the computation of all $H_{\underline{\lambda}}(\pi)$ is possible without using the general case of the fundamental lemma. Therefore the decomposition can be used in the other direction to obtain information about the action of the symmetric groups.

The following statement without the action of $\mathfrak{S}_{\underline{m}}$ would be a direct consequence of Theorem 9.3.3. We keep the notations and assumptions of section 10.3.

Proposition 10.4.1. *There is an isomorphism of* $(G_F)^r \times \mathfrak{S}_{\underline{m}}$*-modules*

$$\Delta^* H_{\widetilde{\underline{\lambda}}}(\pi) = m_\pi \cdot \sigma(\pi)^{\otimes m_1} \boxtimes \ldots \boxtimes \sigma(\pi)^{\otimes m_r} \tag{10.4.1}$$

where $\mathfrak{S}_{|m_i|}$ *acts on* $\sigma(\pi)^{\otimes m_i}$ *by permutation of the factors.*

Proof. We have to show that for any irreducible representation of $\mathfrak{S}_{\underline{m}}$ the isotypic components of both sides of (10.4.1) are isomorphic as $(G_F)^r$-modules. In view of Corollary 10.3.3 applied to the left hand side and Lemma 10.2.1 applied to the right hand side this means

$$H_{\underline{\lambda}}(\pi) = m_\pi \cdot (\rho_{\lambda_1} \circ \sigma(\pi)) \boxtimes \ldots \boxtimes (\rho_{\lambda_r} \circ \sigma(\pi)) \tag{10.4.2}$$

as $(G_F)^r$-modules for all definite $\underline{\lambda}$ of degree $\underline{m}$, which is the statement of Theorem 9.3.3. □

Proposition 10.4.2. *In the situation of Proposition 10.4.1 there is an isomorphism* (10.4.1) *of* $(G_F^r \times \mathfrak{S}_{\underline{m}}) \rtimes \mathrm{Stab}(\underline{m})$*-modules if and only if there are isomorphisms* (10.4.2) *of* $(G_F)^r \rtimes \mathrm{Stab}(\underline{\lambda})$*-modules for all definite* $\underline{\lambda}$ *of degree* $\underline{m}$ *where the symmetric groups act on the right hand sides by the obvious permutations.*

Proof. The subgroup $\mathrm{Stab}(\underline{\lambda}) \subseteq \mathrm{Stab}(\underline{m})$ coincides with the stabiliser of the isomorphism class of the irreducible representation $\chi_{\underline{\lambda}^+}$ of $\mathfrak{S}_{\underline{m}}$. Hence using Corollary 10.3.3 there is an isomorphism of $(G_F^r \times \mathfrak{S}_{\underline{m}}) \rtimes \mathrm{Stab}(\underline{m})$-modules

$$\Delta^* H_{\widetilde{\underline{\lambda}}}(\pi) = \bigoplus_{\underline{\lambda}} \mathrm{Ind}_{(G_F)^r \rtimes \mathrm{Stab}(\underline{\lambda})}^{(G_F)^r \rtimes \mathrm{Stab}(\underline{m})} H_{\underline{\lambda}}(\pi)$$

where $\underline{\lambda}$ runs through a set of representatives of the $\mathrm{Stab}(\underline{m})$-orbits in the set of definite $\underline{\lambda}$'s of degree $\underline{m}$. There is a similar connection between the right hand sides of (10.4.1) and (10.4.2). □

For any integer $m \geq 1$ we form the sequence

$$\widetilde{\underline{\lambda}} = \underline{\lambda}(m) = (\mu^+)^m(\mu^+)^m = (\mu^+, \ldots, \mu^+, \mu^-, \ldots, \mu^-)$$

and write $H_m(\pi) = H_{\underline{\lambda}(m)}(\pi)$.

Theorem 10.4.3. *Assume that for a given integer $m \geq 1$ the stack $\mathrm{Sht}^{\leq \underline{\lambda}(m)}/a^{\mathbb{Z}}$ is proper over $(X')^{2m}$. If for some irreducible automorphic representation π of $D_{\mathbb{A}}^*/a^{\mathbb{Z}}$ the associated Galois representation $\sigma(\pi)$ contains no irreducible representation with multiplicity greater than 1 (cf. Remark 9.3.2), then there is an isomorphism*

$$H_m(\pi) = m_\pi \cdot \sigma(\pi)^{\boxtimes m} \boxtimes \sigma(\pi)^{\vee \boxtimes m} \tag{10.4.3}$$

of $(G_F)^{2m} \rtimes (\mathfrak{S}_m)^2$-modules.

Corollary 10.4.4. *Assume $\mathrm{Sht}^{(\mu^+, \mu^-)}/a^{\mathbb{Z}}$ is proper over $(X')^2$ and D is sufficiently ramified with respect to $\underline{\lambda}$. If for some irreducible automorphic representation π of $D_{\mathbb{A}}^*/a^{\mathbb{Z}}$ the associated Galois representation $\sigma(\pi)$ contains no irreducible representation with multiplicity greater than 1, then there is an isomorphism*

$$H_{\underline{\lambda}}(\pi) \cong m_\pi \cdot (\rho_{\lambda_1} \circ \sigma(\pi)) \boxtimes \ldots \boxtimes (\rho_{\lambda_r} \circ \sigma(\pi))$$

of $(G_F)^r \rtimes \mathrm{Stab}(\underline{\lambda})$-modules.

The hypothesis on $\underline{\lambda}$ in the corollary can be weakened by demanding that a certain stack $\mathrm{Sht}^{\widetilde{\underline{\lambda}}}/a^{\mathbb{Z}}$ is proper over $(X')^M$.

Proof of Corollary 10.4.4. If all λ_i are definite, the claim follows directly from Theorem 10.4.3 and Proposition 10.4.2. Lemma 3.3.4 allows to reduce the general case to this. □

Proof of Theorem 10.4.3. First we assume that $\sigma = \sigma(\pi)$ is actually irreducible. Let V_0 be the $G_F^{2m} \rtimes \mathfrak{S}_m$-module $\sigma^{\boxtimes m} \boxtimes \sigma^{\vee\boxtimes m}$. By Theorem 9.3.3 there is an isomorphism of G_F^{2m}-modules

$$H_m(\pi) \cong m_\pi \cdot V_0 \,.$$

Therefore the isomorphism class of $H_m(\pi)$ as a $G_F^{2m} \rtimes \mathfrak{S}_m^2$-module is determined by the m_π-dimensional $\mathfrak{S}_m^2$-module

$$W = \mathrm{Hom}_{G_F^{2m}}(V_0, H_m(\pi))$$

where $\mathfrak{S}_m^2$ acts on homomorphisms by conjugation; $H_m(\pi) = W \otimes V_0$.

Let $\mathbb{1}$ be the trivial one-dimensional representation of $\mathfrak{S}_m^2$. By Proposition 10.4.1 the isotypic components $H_m(\pi)(\mathbb{1})$ and $(\mathbb{1}^{m_\pi} \otimes V_0)(\mathbb{1})$ are isomorphic as G_F^2-modules, in particular they have the same dimension. In view of Lemma 10.4.5 below, for m_π-dimensional $\mathfrak{S}_m^2$-modules W' the dimension of $(W' \otimes V_0)(\mathbb{1})$ is strictly maximal in the case $W' \cong \mathbb{1}^{m_\pi}$. This implies $W \cong \mathbb{1}^{m_\pi}$ and finishes the proof if σ is irreducible.

Lemma 10.4.5. *Let W be an irreducible representation of $\mathfrak{S}_m$ and let V be the isotypic component $V = (\bar{\mathbb{Q}}_l^d)^{\otimes m}(W)$. Then we have*

$$\frac{\dim(V)}{\dim(W)} = \frac{\prod (d-i+j)}{m!}$$

with the product over all positions (i,j) of the boxes in the Young diagram of W. In the case $W = \mathbb{1}$ these are the pairs $(0,j)$ for $0 \leq j < d$.

Proof. By the Hook Length Formula [FH] 4.12, the dimension of W is $m!$ divided by a certain product of hook lengths, while by [FH] Exercise 6.4 the dimension of V is $\prod(d-i-j)$ divided by the same product of hook lengths. □

Continuation of the proof of theorem 10.4.3. For general σ we use the same approach as in the irreducible case. Again both sides of (10.4.3) are isomorphic as G_F^{2m}-modules, and the dimensions of their $\mathbb{1}$-isotypic components agree.

Let $\sigma = \sigma_1 \oplus \ldots \oplus \sigma_r$ be a decomposition into pairwise non-isomorphic irreducible factors. For any additive decomposition $m = m_1 + \ldots + m_r$ with $m_i \geq 0$ let $\mathfrak{S}_{\underline{m}} \subseteq \mathfrak{S}_m$ be the naturally embedded subgroup $\mathfrak{S}_{m_1} \times \ldots \times \mathfrak{S}_{m_r}$, and let

$$V_{\underline{m}} = \sigma_1^{\boxtimes m_1} \boxtimes \ldots \boxtimes \sigma_r^{\boxtimes m_r}$$

as $G_F^m \rtimes \mathfrak{S}_{\underline{m}}$-module. The isomorphism class of the $G_F^{2m} \rtimes \mathfrak{S}_m^2$-module $H_m(\pi)$ is uniquely determined by the family of m_π-dimensional $\mathfrak{S}_{\underline{m}} \times \mathfrak{S}_{\underline{m}'}$-modules

$$W_{\underline{m},\underline{m}'} = \mathrm{Hom}_{G_F^{2m}}\left(V_{\underline{m}} \boxtimes V_{\underline{m}'}^\vee, H_m(\pi)\right),$$

more precisely

$$H_m(\pi) = \bigoplus_{\underline{m},\,\underline{m}'} \mathrm{Ind}_{G_F^{2m} \rtimes (\mathfrak{S}_{\underline{m}} \times \mathfrak{S}_{\underline{m}'})}^{G_F^{2m} \rtimes \mathfrak{S}_m^2} W_{\underline{m},\,\underline{m}'} \otimes (V_{\underline{m}} \boxtimes V_{\underline{m}'}^{\vee}) \,. \tag{10.4.4}$$

The existence of an isomorphism (10.4.3) means that each $W_{\underline{m},\,\underline{m}'}$ is the trivial m_π-dimensional representation of $\mathfrak{S}_{\underline{m}} \times \mathfrak{S}_{\underline{m}'}$. To prove that this is true, it is sufficient to see that the dimension of the $\mathbb{1}$-isotypic component of each single summand in (10.4.4) is strictly maximal in the case of trivial $W_{\underline{m},\,\underline{m}'}$. But this dimension equals

$$\dim H^0\big(\mathfrak{S}_{\underline{m}} \times \mathfrak{S}_{\underline{m}'}, W_{\underline{m},\,\underline{m}'} \otimes (\overline{\mathbb{Q}}_l^d)^{\otimes 2m}\big)$$

and we can apply Lemma 10.4.5 again. □

Appendix

A Properness of $\mathcal{D}$-Shtukas

In section 1.5 we claimed that the natural morphism

$$\pi^m \; : \; \mathrm{Sht}^m / a^{\mathbb{Z}} \longrightarrow X'^{(m)} \times X'^{(m)}$$

is not always proper contrary to the expectation (and the assertion in [Laf97] IV.1, Theorem 1 in the case $m = 1$).

On the following pages we show that for $d = 2$ the criterion given in proposition 1.5.2 is optimal. Since in this case the local invariants of D are 0 or $1/2$, the criterion says that the above morphism is proper if D is ramified at more than $2m$ places. For the proof of non-properness in the case of less ramification it would be sufficient to find one degenerating family, but we in fact construct a partial (non-separated) compactification, based on the constructions in [Dri89] and [Laf98].

For general d there is a precise criterion for properness as well, which we will prove in a later work: for $\alpha \in \mathbb{Q}/\mathbb{Z}$ let $[\alpha] \in [0, 1)$ be the unique inverse image. The morphism π^m is proper if and only if for every integer $0 < k < d$ the following inequality holds.

$$\sum_{x \in X} [\, k \, \mathrm{inv}_x(D) \,] \; \geq \; m + 1$$

For properness of $\mathrm{Sht}^{\leq \lambda} / a^{\mathbb{Z}}$ over $(X')^r$ a similar criterion can be formulated.

A.1 Remarks on projective modules

Let $R \to R'$ be a local homomorphism of local noetherian rings, let $\mathcal{A}$ be a (not necessarily commutative) R-algebra, which is free of finite rank as an R-module, and let $\mathcal{A}' = \mathcal{A} \otimes_R R'$. We consider a finitely generated right $\mathcal{A}$-module M which is free as an R-module and write $M' = M \otimes_{\mathcal{A}} \mathcal{A}'$. Any projective $\mathcal{A}$-module is free as an R-module.

By [Laf97] I.2, Lemma 4 M is a free $\mathcal{A}$-module if and only if M' is free over $\mathcal{A}'$. (This does not depend on the noetherian assumption. Neither does the following statement, as everything can be defined over a finitely generated subring).

Lemma A.1.1. *M is a projective $\mathcal{A}$-module if and only if M' is projective over $\mathcal{A}'$.*

Proof. We have to show that if M' is projective, then so is M. If $R \to R'$ is flat (hence faithfully flat), this follows from the equations

$$\mathrm{Ext}^1_{\mathcal{A}}(M, N) \otimes_R R' = \mathrm{Ext}^1_{\mathcal{A}'}(M', N \otimes_{\mathcal{A}} \mathcal{A}')$$

for all $\mathcal{A}$-modules N. In general we may therefore replace R' by its residue field and R by its completion, which in the noetherian case is flat over R.

If M' is a free $\mathcal{A}'$-module, then any inverse image of an $\mathcal{A}'$-basis of M' is an $\mathcal{A}$-basis of M by Nakayama's lemma and flatness of M over R. If M' is only projective, there is a finitely generated projective $\mathcal{A}'$-module N' such that $N' \oplus M'$ is free. Since R is complete, this module has the form $N' = N \otimes_{\mathcal{A}} \mathcal{A}'$ with a finitely generated projective $\mathcal{A}$-module N (lifting of idempotents). Then $N \oplus M$ is free over $\mathcal{A}$, which means M is projective. □

Lemma A.1.2. *The natural map from the set of isomorphism classes of finitely generated projective $\mathcal{A}$-modules to the set of isomorphism classes of finitely generated projective $\mathcal{A}'$-modules is injective. If R is complete and R' is a quotient of R, then this map is bijective.*

Proof. We only have to consider the two cases that R' is the residue filed of R or that both R and R' are fields. In the second case the isomorphism class of a finitely generated projective $\mathcal{A}$-module M is determined by the R-dimensions of $M \otimes_{\mathcal{A}} \mathcal{A}_i$ with $\mathcal{A}_i$ running through the simple quotients of $\mathcal{A}$. These dimensions can be read off from M'.

In any case for two finitely generated projective $\mathcal{A}$-modules M and N we have

$$\mathrm{Hom}_{\mathcal{A}}(M, N) \otimes_R R' = \mathrm{Hom}_{\mathcal{A}'}(M \otimes_{\mathcal{A}} \mathcal{A}', N \otimes_{\mathcal{A}} \mathcal{A}'),$$

because this holds for free modules and this property carries over from a direct sum to each of the factors.

Thus if $R \to R'$ is surjective, any isomorphism $M \otimes_{\mathcal{A}} \mathcal{A}' \cong N \otimes_{\mathcal{A}} \mathcal{A}'$ can be lifted to homomorphisms $M \to N$ and $N \to M$, which are surjective by Nakayama's lemma. Since any surjective endomorphism of a finitely generated R-module is bijective, they are even bijective. If in addition R is complete, then the idempotent element corresponding to a direct factor of $\mathcal{A}'^n$ can be lifted to $M_n(\mathcal{A})$. □

Now let X and $\mathcal{D}$ be as defined on page xi.

Lemma A.1.3. *Any right $\mathcal{D} \boxtimes \mathcal{O}_S$-module $\mathcal{E}$ which is locally free of finite rank as an $\mathcal{O}_{X\times S}$-module is locally projective over $\mathcal{D} \boxtimes \mathcal{O}_S$.*

Proof. In view of Lemma A.1.1 it suffices to show that for an algebraically closed field k and a geometric point $\bar{y} = (\bar{x}, \bar{s}) \in (X \times S)(k)$ the geometric fibre $\mathcal{E} \otimes k(\bar{y})$ is projective over $(\mathcal{D} \boxtimes \mathcal{O}_S) \otimes k(\bar{y})$.

In the case $\bar{x} \in X'(k)$ this algebra is an Azumaya algebra over k and therefore any module is projective. Otherwise $\bar{x}$ is concentrated in a closed point $x \in X \setminus X'$. Since $\mathcal{D}_x \subset D_x$ is a maximal order, the order $(\mathcal{D} \otimes k)_{\bar{x}} \subset (D \otimes k)_{\bar{x}}$ is hereditary, which implies $(\mathcal{E} \otimes_{\mathcal{O}_S} k)_{\bar{x}}$ is projective over this algebra. □

A.2 Complete homomorphisms for $\mathcal{D}$ $(d = 2)$

The functor H which to a ring R assigns the set of

$$(u, v, \alpha) \in \mathrm{End}(R^2) \times \mathrm{End}(R) \times R$$

is representable by the 6-dimensional affine space over $\mathbb{Z}$.

Let $\Omega_1 \subset H$ be the locally closed subscheme where u, v are invertible and $\Lambda^2 u = \alpha v$, let Ω be the schematic closure of Ω_1 in the open subset of H where u, v do not vanish at any point, and let $\Omega_0 \subset \Omega$ be the fibre in $\{0\}$ of the morphism $\Omega \to \mathbb{A}^1$ which is given by α.

By [Laf98], Proposition 1.1 the subset $\Omega_0(R) \subseteq \mathrm{End}(R^2) \times \mathrm{End}(R)$ is given by the conditions that the cokernel of u is locally free of rank 1 and that v is invertible. When this holds, the natural isomorphisms $\Lambda^2 R^2 = \mathrm{Ker}(u) \otimes \mathrm{Im}(u)$ and $\Lambda^2 R^2 = \mathrm{Im}(u) \otimes \mathrm{Coker}(u)$ allow to consider v as an isomorphism $w : \mathrm{Ker}(u) \cong \mathrm{Coker}(u)$.

For a given Azumaya algebra $\mathcal{A} \,|\, R$ of rank 4 let

$$\Omega(\mathcal{A} \,|\, R) \subseteq \mathcal{A} \times R \times R$$

be the subset of elements (u, v, α) which étale locally via a trivialisation of $\mathcal{A}$ become elements of $\Omega(R)$. The condition $\alpha = 0$ defines a subset $\Omega_0(\mathcal{A} \,|\, R) \subseteq \Omega(\mathcal{A} \,|\, R)$ which is naturally bijective to the set of pairs (u, w) of the following kind: the cokernel of left multiplication by $u \in \mathcal{A}$ is locally free over R of rank 2, and $w : \mathrm{Ker}(u) \cong \mathrm{Coker}(u)$ is an isomorphism of right $\mathcal{A}$-modules.

Definition A.2.1. Let $\widetilde{\mathcal{H}}(S)$ be the groupoid of collections $(\mathscr{L}, l, \mathcal{E}, \mathcal{E}', u, v)$ with

- $\mathscr{L}$ invertible sheaf on S and $l \in \Gamma(S, \mathscr{L})$,
- $u : \mathcal{E} \to \mathcal{E}'$ homomorphism of locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank 1,

– $v : \mathscr{L} \otimes \det(\mathcal{E}) \cong \det \mathcal{E}'$ with $\det(u) = l \cdot v$,

such that (u, v, l) is locally in $X' \times S$ an element of $\Omega(\mathcal{D} \boxtimes \mathcal{O}_S \,|\, \mathcal{O}_{X' \times S})$ (after trivialising $\mathscr{L}$ and $\mathcal{E}, \mathcal{E}'$).

Since the pairs $(\mathscr{L}, l)$ are parametrised by $\mathbb{A}^1/\mathbb{G}_m$ we get a natural morphism $\widetilde{\mathcal{H}} \to \mathbb{A}^1/\mathbb{G}_m$. Let $\widetilde{\mathcal{H}}_1$ be the fibre over $\mathbb{G}_m/\mathbb{G}_m$ and $\widetilde{\mathcal{H}}_0$ be the fibre over $\{0\}/\mathbb{G}_m$. The two maps $\widetilde{\mathcal{H}}_1 \to \mathrm{Vect}^1_{\mathcal{D}}$ given by $\mathcal{E}$ or by $\mathcal{E}'$ are isomorphisms.

For any $(\mathscr{L}, 0, \mathcal{E}, \mathcal{E}', u, v) \in \widetilde{\mathcal{H}}_0(S)$ the restriction of $\mathrm{Coker}(u)$ to $X' \times S$ is locally free of rank 2 over $\mathcal{O}_{X' \times S}$.

Definition A.2.2. We denote by $\mathcal{H}_0 \subseteq \widetilde{\mathcal{H}}_0$ the open substack where $\mathrm{Coker}(u)$ is locally free of rank 2 over $\mathcal{O}_{X \times S}$. In view of Lemma A.1.3, there $\mathrm{Coker}(u)$, $\mathrm{Im}(u)$, and $\mathrm{Ker}(u)$ are locally projective over $\mathcal{D} \boxtimes \mathcal{O}_S$. Let $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$ be the maximal open substack satisfying $\mathcal{H} \cap \widetilde{\mathcal{H}}_0 = \mathcal{H}_0$.

Lemma A.2.3. *Let $u : \mathcal{E} \to \mathcal{E}'$ be a homomorphism of locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank 1 such that its cokernel is locally free of rank 2 over $\mathcal{O}_{X \times S}$. As explained above, u defines an isomorphism*

$$\mathcal{H}om_{\mathcal{D} \boxtimes \mathcal{O}_S}(\mathrm{Ker}\, u, \mathrm{Coker}\, u) \cong \mathcal{H}om_{\mathcal{O}_{X \times S}}(\det \mathcal{E}, \det \mathcal{E}') \tag{A.2.1}$$

over $X' \times S$. This extends to an isomorphism over $X \times S$, which in both directions carries isomorphisms to isomorphisms.

Proof. By Lemma A.1.3 the kernel and the cokernel of u are locally in $X \times S$ direct factors of free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules. Thus $\mathcal{H}om_{\mathcal{D} \boxtimes \mathcal{O}_S}(\mathrm{Ker}\, u, \mathrm{Coker}\, u)$ is locally a direct factor of a free $\mathcal{O}_{X \times S}$-module, i.e. both sides of (A.2.1) are locally free $\mathcal{O}_{X \times S}$-modules. Because X' and the finitely many $\mathrm{Spec}\, \mathcal{O}_{\overline{x}}$ for $x \in X \setminus X'$ form an fpqc-covering of X, the assertion must be proved only over $\mathcal{D}_{\overline{x}} \boxtimes \mathcal{O}_S$ for these x.

Let $\widetilde{\mathcal{D}}_i \supset \mathcal{D}_{\overline{x}}$ for $i = 1, 2$ be the two maximal orders, which are Azumaya algebras. For any i there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_{\overline{x}} & \longrightarrow & \mathcal{E}_{\overline{x}} \otimes_{\mathcal{D}_{\overline{x}}} \widetilde{\mathcal{D}}_i & \longrightarrow & \mathcal{E}_{\overline{x}} \otimes_{\mathcal{D}_{\overline{x}}} \widetilde{\mathcal{D}}_i/\mathcal{D}_{\overline{x}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle \overline{u}} & & \\ 0 & \longrightarrow & \mathcal{E}'_{\overline{x}} & \longrightarrow & \mathcal{E}'_{\overline{x}} \otimes_{\mathcal{D}_{\overline{x}}} \widetilde{\mathcal{D}}_i & \longrightarrow & \mathcal{E}'_{\overline{x}} \otimes_{\mathcal{D}_{\overline{x}}} \widetilde{\mathcal{D}}_i/\mathcal{D}_{\overline{x}} & \longrightarrow & 0 \end{array}$$

Therefore on the open subset $S_i \subseteq S$ where the map $\overline{u}$ is an isomorphism we have

$$\begin{aligned} \mathcal{H}om_{\mathcal{D}_{\overline{x}} \boxtimes \mathcal{O}_S}(\mathrm{Ker}\, u, \mathrm{Coker}\, u) &= \mathcal{H}om_{\mathcal{D}_{\overline{x}} \boxtimes \mathcal{O}_S}(\mathrm{Ker}\, u', \mathrm{Coker}\, u') \\ &= \mathcal{H}om_{\widetilde{\mathcal{D}}_i \boxtimes \mathcal{O}_S}(\mathrm{Ker}\, u', \mathrm{Coker}\, u') \cong \mathcal{H}om(\det \mathcal{E}, \det \mathcal{E}') \end{aligned}$$

where the last isomorphism is due to the fact that $\widetilde{\mathcal{D}}_i$ is an Azumaya algebra.

It remains to show that S is covered by S_1 and S_2. For this we may assume $S = \operatorname{Spec} k$ with an algebraically closed field k. Using an isomorphism $\mathcal{D}_{\bar{x}} \cong \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{smallmatrix}\right)$ and trivialisations of $\mathcal{E}$ and of $\mathcal{E}'$, u can be represented by a matrix $\left(\begin{smallmatrix} a & b \\ \varpi c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathcal{O}_{\bar{x}} \widehat{\otimes} k$. The defining conditions for the two S_i are then $a \not\equiv 0$ respectively $d \not\equiv 0$ modulo ϖ_x. Now $a \equiv 0$ and $d \equiv 0$ would imply $bc \not\equiv 0$, because $\operatorname{Coker}(u)$ has rank 2 in $\bar{x}$. Then over $D_{\bar{x}} \widehat{\otimes} k$ $\det(u)$ would be invertible, contradicting the hypothesis that u is not invertible in any point. □

Corollary A.2.4. *$\mathcal{H}_0$ is naturally isomorphic to the stack of $(\mathscr{L}, \mathcal{E}, \mathcal{E}', u, w)$ with*

- *$\mathscr{L}$ invertible sheaf on S,*
- *$u : \mathcal{E} \to \mathcal{E}'$ homomorphism of locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank 1,*
- *$\operatorname{Coker}(u)$ locally free of rank 2 over $\mathcal{O}_{X \times S}$,*
- *$w : \mathscr{L} \otimes \operatorname{Ker}(u) \cong \operatorname{Coker}(u)$ as $\mathcal{D} \boxtimes \mathcal{O}_S$-modules.* □

A.3 Smoothness of $\mathcal{H}$

Lemma A.3.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be locally projective $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank 2 over $\mathcal{O}_{X \times S}$. Then $\mathcal{H}om_{\mathcal{D} \boxtimes \mathcal{O}_S}(\mathcal{E}, \mathcal{F})$ is an invertible $\mathcal{O}_{X \times S}$-module, and for $s \in S$ we have*

$$0 \le \frac{1}{2}\big(\deg \mathcal{F}_s - \deg \mathcal{E}_s\big) - \deg \mathcal{H}om_{\mathcal{D} \boxtimes \mathcal{O}_S}(\mathcal{E}, \mathcal{F})_s \le \frac{1}{2} \sum_{x \in X \setminus X'} \deg(x)\,.$$

Proof. Like in the proof of Lemma A.2.3 the $\mathcal{O}_{X \times S}$-module $\mathcal{H}om_{\mathcal{D} \boxtimes \mathcal{O}_S}(\mathcal{E}, \mathcal{F})$ is locally free. Its rank can be seen étale locally over $X' \times S$, where $\mathcal{D}$ is trivialisable. To prove the inequality we may assume $S = \operatorname{Spec} k$ with an algebraically closed field k.

For $\bar{x} \in (X \setminus X')(k)$ we have $\mathcal{D}_{\bar{x}} \cong \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{smallmatrix}\right)$, from which we see that there are embeddings of the completions $\mathcal{E}_{\bar{x}} \subseteq \mathcal{F}_{\bar{x}} \subset \varpi^{-1}\mathcal{E}_{\bar{x}}$ such that the second is not an isomorphism. Over X', $\mathcal{E}$ and $\mathcal{F}$ are locally isomorphic. Hence there are embeddings

$$\mathcal{E} \subseteq \mathcal{F} \otimes \mathscr{L} \subset \mathcal{E}\Big(\sum_{x \in X \setminus X'} x\Big)$$

with some invertible $\mathcal{O}_{X \times S}$-module $\mathscr{L}$ such that the second embedding is not bijective at any point over $x \in X \setminus X'$. We conclude $\mathcal{H}om_{\mathcal{D} \boxtimes \mathcal{O}_S}(\mathcal{E}, \mathcal{F}) \otimes \mathscr{L} = \mathcal{E}nd_{\mathcal{D} \boxtimes \mathcal{O}_S}(\mathcal{E}) = \mathcal{O}_{X \times S}$ and thus

$$\frac{1}{2}\big(\deg \mathcal{F} - \deg \mathcal{E}\big) - \deg \mathcal{H}om_{\mathcal{D}}(\mathcal{E}, \mathcal{F}) = \frac{1}{2}\big(\deg(\mathcal{F} \otimes \mathscr{L}) - \deg \mathcal{E}\big)\,.$$

This number satisfies the asserted inequalities. □

Definition A.3.2. For any integer C let $\mathcal{H}_0^C \subseteq \mathcal{H}_0$ be the open substack where the inequality

$$2 \cdot \deg(\operatorname{Coker} u_s) \leq \deg \mathcal{E}_s - C \tag{A.3.1}$$

holds. Let $\mathcal{H}^C \subseteq \mathcal{H}$ be the maximal open substack satisfying $\mathcal{H}^C \cap \mathcal{H}_0 = \mathcal{H}_0^C$.

Since Corollary A.2.4 implies $\deg(\mathcal{E}_s) = \deg(\mathcal{E}'_s)$, inequality (A.3.1) is equivalent to $\deg(\operatorname{Coker} u_s) \leq \deg(\operatorname{Im} u_s) - C$.

Proposition A.3.3. *For sufficiently large C the morphism*

$$\mathcal{H}_0^C \longrightarrow \operatorname{Vect}^1_{\mathcal{D}} \times \{0\}/\mathbb{G}_m$$

given by $\mathcal{E}$ and $\mathscr{L}$ is smooth of relative dimension zero.

Proof. By Corollary A.2.4, the fibre of the morphism in question over $\mathscr{L} = \mathcal{O}_S$ and a given $\mathcal{E}$ is the groupoid of quotients $\pi : \mathcal{E} \twoheadrightarrow \mathcal{F}$ such that $\mathcal{F}$ is locally free of rank 2 over $\mathcal{O}_{X\times S}$ plus an extension $0 \to \mathcal{F} \to \mathcal{E}' \to \operatorname{Ker}(\pi) \to 0$ of $\mathcal{D} \boxtimes \mathcal{O}_S$-modules. Since $\mathcal{F}$ and $\operatorname{Ker}(\pi)$ are locally projective $\mathcal{D} \boxtimes \mathcal{O}_S$-modules, for any such extension $\mathcal{E}$ and $\mathcal{E}'$ are locally isomorphic, which means $\mathcal{E}'$ is locally free over $\mathcal{D} \boxtimes \mathcal{O}_S$.

The quotients $\mathcal{E} \twoheadrightarrow \mathcal{F}$ as above are parametrised by a closed subscheme $Q \subseteq \operatorname{Quot}_{\mathcal{E}/X\times S/S}$, which we claim is smooth over S. Let A be a local Artin ring of finite type over $\overline{\mathbb{F}}_q$ and let $\mathcal{I} \subset A$ be an ideal with $\mathcal{I} \cong \overline{\mathbb{F}}_q$. Locally in X, quotients $\mathcal{E} \twoheadrightarrow \mathcal{F}$ in Q can be lifted from $A/\mathcal{I}$ to A because $\mathcal{F}$ is locally projective over $\mathcal{D} \boxtimes \mathcal{O}_S$ (lifting of idempotents). The obstruction to a global lifting lies in

$$H^1\big(X \otimes \overline{\mathbb{F}}_q, \mathcal{H}om_{\mathcal{D}\otimes\overline{\mathbb{F}}_q}(\operatorname{Ker}(\pi) \otimes \overline{\mathbb{F}}_q, \mathcal{F} \otimes \mathcal{I})\big)\,.$$

As a consequence of Lemma A.3.1, this cohomology vanishes if $\deg \operatorname{Ker}(\pi) - \deg \mathcal{F}$ is sufficiently small. In that case the relative dimension of $Q \to S$ is

$$n = \dim H^0(X \otimes \overline{\mathbb{F}}_q, \mathcal{H}om_{\mathcal{D}\otimes\overline{\mathbb{F}}_q}(\operatorname{Ker}(\pi) \otimes \overline{\mathbb{F}}_q, \mathcal{F} \otimes \mathcal{I})).$$

Now we consider the extensions. In any case $\mathcal{E}xt^1_{\mathcal{D}\boxtimes\mathcal{O}_S}(\operatorname{Ker}(\pi), \mathcal{F})$ vanishes, and under the same condition on the degrees as before $R^1 p_{2,*}\mathcal{H}om_{\mathcal{D}\boxtimes\mathcal{O}_S}(\operatorname{Ker}(\pi), \mathcal{F})$ vanishes as well, with $p_2 : X \times S \to S$. In this case our stack of extensions is a gerbe over Q with structure group $p_{2,*}\mathcal{H}om_{\mathcal{D}\boxtimes\mathcal{O}_S}(\operatorname{Ker}(\pi), \mathcal{F})$, which is a locally free $\mathcal{O}_S$-module of rank n. $\square$

Proposition A.3.4. *For sufficiently large C the morphism*

$$\mathcal{H}^C \longrightarrow \operatorname{Vect}^1_{\mathcal{D}} \times \mathbb{A}^1/\mathbb{G}_m \tag{A.3.2}$$

given by $\mathcal{E}$ and $(\mathscr{L}, l)$ is smooth of relative dimension zero.

Proof. Over $\mathbb{G}_m/\mathbb{G}_m$ the morphism is an isomorphism. In view of Proposition A.3.3 it remains to show that the morphism (A.3.2) is flat. This is equivalent to $\mathcal{H}_1 \subseteq \mathcal{H}^C$ being dense. In fact, using smoothness of $\mathrm{Vect}^1_{\mathcal{D}}$ this density implies that $\mathcal{H}^C_0 \subset \mathcal{H}^C$ is a smooth divisor with smooth complement. Hence both sides of (A.3.2) are smooth over $\mathbb{F}_q$, and the dimension of the fibres is constantly zero.

We want to show that any given $E_0 = (\mathcal{E}_0, \mathcal{E}'_0, u_0, v_0) \in \mathcal{H}_0(\overline{\mathbb{F}}_q)$ can be extended to $(\mathscr{L} = \overline{\mathbb{F}}_q[\![t]\!], l = t, \mathcal{E}, \mathcal{E}', u, v) \in \mathcal{H}(\overline{\mathbb{F}}_q[\![t]\!])$, which means that even $\mathcal{H}_1 \subseteq \mathcal{H}$ is dense. Let $A_n = \overline{\mathbb{F}}_q[t]/t^{n+1}$. We will construct extensions

$$E_n = (\mathcal{E}_n, \mathcal{E}'_n, u_n, v_n)$$

over $X \otimes A_n$ satisfying $\det(u_n) = t \cdot v_n$ along with isomorphisms $E_n \cong E_{n+1} \otimes A_n$. Then there is an $E = (\mathcal{E}, \mathcal{E}', u, v)$ over $X \otimes \overline{\mathbb{F}}_q[\![t]\!]$ with $E_n \cong E \otimes A_n$. Together with $l = t$ this is the desired element of $\mathcal{H}(\overline{\mathbb{F}}_q[\![t]\!])$, because over the schematically dense open subset $X' \otimes \overline{\mathbb{F}}_q(\!(t)\!) \subset X' \otimes \overline{\mathbb{F}}_q[\![t]\!]$ we have locally an element of Ω_1. We also see that all E_n lie in $\mathcal{H}(A_n)$, which might not have been clear a priori.

Locally in $X \otimes \overline{\mathbb{F}}_q$ there are isomorphisms $\mathcal{E}_0 \cong \mathcal{F}_0 \oplus \mathcal{F}'_0 \cong \mathcal{E}'_0$ with projective $\mathcal{D} \boxtimes \mathcal{O}_S$-modules $\mathcal{F}_0$ and $\mathcal{F}'_0$ of rank 2 over $\mathcal{O}_{X \otimes \overline{\mathbb{F}}_q}$ such that u_0 is the map $\mathcal{E}_0 \twoheadrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{E}'_0$. Using the notation of Lemma A.3.5 below, this can be written as $u_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $v_0 = 1 \in \mathcal{O}_{X \otimes \overline{\mathbb{F}}_q} = \mathrm{End}(\det \mathcal{E}_0)$. We can lift $\mathcal{F}_0$ and $\mathcal{F}'_0$ to $\mathcal{F}_n$ and $\mathcal{F}'_n$ such that their direct sum is free over $\mathcal{D} \otimes A_n$. Then for any n, $u_n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$ and $v_n = 1$ define one local lifting E_n of E_0.

Next we show that all local liftings of this E_{n-1} to A_n are isomorphic. Inductively this implies that all local liftings of E_0 to A_n are isomorphic and that any such local lifting can be lifted further to A_{n+1}.

We write $u'_n = u_n + t^n \delta$ and $v'_n = v_n + t^n \varepsilon$ with $\delta \in \mathcal{H}om(\mathcal{E}_0, \mathcal{E}'_0)$ and $\varepsilon \in \mathcal{H}om(\det \mathcal{E}_0, \det \mathcal{E}'_0)$. Using Lemma A.3.5, the condition $\det u' = t \cdot v'_n$ means that the restriction $\bar{\delta} : \mathrm{Ker}(u_0) \to \mathrm{Coker}(u_0)$ vanishes, while no condition is imposed on ε. Since the map

$$\begin{aligned} \mathcal{E}nd(\mathcal{E}_0) \oplus \mathcal{E}nd(\mathcal{E}'_0) \longrightarrow \mathrm{Ker}\big(\mathcal{H}om(\mathcal{E}_0, \mathcal{E}'_0) \to \mathcal{H}om(\mathrm{Ker}\, u_0, \mathrm{Coker}\, u_0)\big) \\ \oplus \mathcal{H}om(\det \mathcal{E}_0, \det \mathcal{E}'_0) \end{aligned}$$

$$(\alpha, \beta) \longmapsto (u_0 \alpha + \beta u_0, v_0 \mathrm{tr}(\alpha) + \mathrm{tr}(\beta) v_0) \tag{A.3.3}$$

is surjective, the local liftings are isomorphic as claimed.

Therefore any global extension E'_{n-1} of E_0 can locally be lifted further to A_n. The sheaf $\mathcal{K}$ of the local automorphisms of these liftings is locally isomorphic to the kernel of the map (A.3.3). The obstruction to the existence of a global extension lies in $H^2(X, \mathcal{K}) = 0$. □

Lemma A.3.5. *Let $\mathcal{E}$ be a locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-module of rank 1 given over an open subset $U \subseteq X \times S$, and let $\mathcal{E} = \mathcal{F} \oplus \mathcal{F}'$ be a decomposition such that* *both factors have rank* 2 *over* $\mathcal{O}_U$. *By Lemma A.3.1* $\mathscr{L} = \mathcal{H}om_{\mathcal{D}\boxtimes\mathcal{O}_S}(\mathcal{F}', \mathcal{F})$ *and* $\mathscr{L}' = \mathcal{H}om_{\mathcal{D}\boxtimes\mathcal{O}_S}(\mathcal{F}, \mathcal{F}')$ *are invertible* $\mathcal{O}_U$*-modules.*

Then using the natural isomorphisms $\mathcal{E}nd(\mathcal{F}) \cong \mathcal{O}_{X\times S} \cong \mathcal{E}nd(\mathcal{F}')$, *the composition of endomorphisms in both directions defines the same isomorphism* $\mathscr{L} \otimes \mathscr{L}' \cong \mathcal{O}_{X\times S}(-\sum_{X\setminus X'} x)$. *In terms of the natural isomorphism*

$$\mathcal{E}nd_{\mathcal{D}\boxtimes\mathcal{O}_S}(\mathcal{E}) = \begin{pmatrix} \mathcal{O}_{X\times S} & \mathscr{L}' \\ \mathscr{L} & \mathcal{O}_{X\times S} \end{pmatrix}, \quad u \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have the formulae $\det(u) = ad - bc$ *and* $\operatorname{tr}(u) = a + d$.

Proof. We may assume S is of finite type over $\mathbb{F}_q$. Since $\mathcal{D}$ can be trivialised étale locally on X', the assertions must be proved only over $\mathcal{O}_{\bar{x}} \widehat{\otimes} \mathcal{O}_{S,s}$ for $x \in X \setminus X'$ and for a closed point $s \in S$.

There are precisely two isomorphism classes of projective $\mathcal{D}_{\bar{x}}$-modules of rank 2 over $\mathcal{O}_{\bar{x}}$. Because the direct sum of $\bar{\mathcal{F}} = \mathcal{F}/\mathfrak{m}_s\mathcal{F}$ and $\bar{\mathcal{F}}' = \mathcal{F}'/\mathfrak{m}_s\mathcal{F}'$ is isomorphic to $\mathcal{D}_{\bar{x}}$, $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}'$ are representatives of the two classes. Then by Lemma A.1.2 the decomposition $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{F}'$ is isomorphic to $\mathcal{D}_{\bar{x}} \widehat{\otimes} \mathcal{O}_{S,s} \cong \bar{\mathcal{F}} \widehat{\otimes} \mathcal{O}_{S,s} \oplus \bar{\mathcal{F}}' \widehat{\otimes} \mathcal{O}_{S,s}$. Using an isomorphism $\mathcal{D}_{\bar{x}} \cong \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{smallmatrix}\right)$, for these modules everything can be written down explicitly since $\det$ and tr are compatible with the embedding $\mathcal{D}_{\bar{x}} \subset M_2(\mathcal{O}_{\bar{x}})$. $\square$

A.4 Degenerations of $\mathcal{D}$-shtukas ($d = 2$)

For any integer $m \geq 0$ we have the stack $\operatorname{Sht}^m = \operatorname{Sht}^m_{\mathcal{D}}$ of $\mathcal{D}$-shtukas with modifications of length m and with zero and pole in X'. In the case $m = 1$ this is the restriction of Lafforgue's [Laf97] stack $\operatorname{Cht}^1_{\mathcal{D}}$ to $X' \times X'$ (here the superscript 1 denotes rank 1).

Definition A.4.1. Let $\mathcal{C}^m(S)$ be the groupoid of $\left[\mathscr{L}, l, \mathcal{E} \xrightarrow{j} \mathcal{E}' \xleftarrow{t} \mathcal{E}'', u, v\right]$ with

- $\mathscr{L}$ invertible sheaf on S and $l \in \Gamma(S, \mathscr{L})$,
- $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{E}''$ locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank 1,
- j, t modifications in $\operatorname{Inj}^{1,m}_{\mathcal{D}}(S)$, cf. definition 1.2.3,
- $u : {}^\tau\mathcal{E} \to \mathcal{E}''$ homomorphism of $\mathcal{D} \boxtimes \mathcal{O}_S$-modules,
- $v : \mathscr{L}^{q-1} \otimes \det {}^\tau\mathcal{E} \cong \det \mathcal{E}''$ with $\det(u) = l^{q-1} \cdot v$,

such that $(\mathscr{L}^{q-1}, l^{q-1}, {}^\tau\mathcal{E}, \mathcal{E}'', u, v) \in \mathcal{H}(S)$. Moreover for any geometric point $\bar{s} \in S$ we require that the σ_q-linear endomorphism $j^{-1}tu$ of the generic fibre of $\mathcal{E}_{\bar{s}}$ is not nilpotent.

There is a sequence of natural morphisms

$$\mathcal{C}^m \longrightarrow \mathcal{H} \longrightarrow \mathbb{A}^1/\mathbb{G}_m$$

defined by $(\mathscr{L}^{q-1}, l^{q-1}, {}^{\tau}\mathcal{E}, \mathcal{E}'', u, v)$ and by $(\mathscr{L}, l)$. The inverse image of $\mathbb{G}_m/\mathbb{G}_m \subset \mathbb{A}^1/\mathbb{G}_m$ in $\mathcal{C}^m$ is canonically isomorphic to Sht^m. Let $\mathcal{C}_0^m \subset \mathcal{C}^m$ be the inverse image of $\{0\}/\mathbb{G}_m$. For any integer C we denote by $\mathcal{C}^{m,C} \subseteq \mathcal{C}^m$ and by $\mathcal{C}_0^{m,C} \subseteq \mathcal{C}_0^m$ the inverse images of $\mathcal{H}^C \subseteq \mathcal{H}$.

Theorem A.4.2. *For sufficiently large C the morphism*

$$\mathcal{C}^{m,C} \longrightarrow \mathrm{Coh}_{\mathcal{D}}^m \times \mathrm{Coh}_{\mathcal{D}}^m \times \mathbb{A}^1/\mathbb{G}_m$$

given by $(\mathcal{E}'/\mathcal{E}'', \mathcal{E}'/\mathcal{E}, \mathscr{L}, l)$ is smooth of relative dimension $4m$ ($= 2dm$).

Proof. Let $\mathrm{Hecke}_{\mathcal{D}}'^m$ be the stack of $[\mathscr{L}, l, \mathcal{E} \to \mathcal{E}' \leftarrow \mathcal{E}'', \mathcal{E}''', u, v]$ with $(\mathcal{E} \to \mathcal{E}' \leftarrow \mathcal{E}'')$ like in Definition A.4.1 and $(\mathscr{L}^{q-1}, l^{q-1}, \mathcal{E}''', \mathcal{E}'', u, v) \in \mathcal{H}^{m,C}(S)$. The definition of $\mathcal{C}^{m,C}$ can be expressed by the 2-cartesian diagram:

$$\begin{array}{ccc} \mathcal{C}^{m,C} & \longrightarrow & \mathrm{Vect}_{\mathcal{D}}^1 \\ \downarrow & \square & \downarrow {\scriptstyle (\mathrm{Frob}_q, \mathrm{id})} \\ \mathrm{Hecke}_{\mathcal{D}}'^m & \xrightarrow{(\mathcal{E}''', \mathcal{E})} & \mathrm{Vect}_{\mathcal{D}}^1 \times \mathrm{Vect}_{\mathcal{D}}^1 \end{array}$$

By Proposition A.3.4 and Lemma 1.3.4 the morphism $\mathrm{Hecke}_{\mathcal{D}}'^m \to \mathrm{Vect}_{\mathcal{D}}^1 \times \mathrm{Coh}_{\mathcal{D}}^m \times \mathrm{Coh}_{\mathcal{D}}^m \times \mathbb{A}^1/\mathbb{G}_m$ given by $(\mathcal{E}''', \mathcal{E}'/\mathcal{E}'', \mathcal{E}'/\mathcal{E}, \mathscr{L}, l)$ is smooth of relative dimension $4m$. Hence the assertion follows from Lemma 1.3.6 (its hypothesis that α is representable is not essential for the conclusion of smoothness). □

For a given $[\mathscr{L}, 0, \mathcal{E} \to \mathcal{E}' \leftarrow \mathcal{E}'', u, v] \in \mathcal{C}_0(S)$ we set $\mathcal{F}'' = \mathrm{Im}(u) \subset \mathcal{E}''$ and $\bar{\mathcal{F}} = \mathrm{Ker}(u) \subset {}^{\tau}\mathcal{E}$. These are locally projective $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank 2 over $\mathcal{O}_{X \times S}$. Using Corollary A.2.4 we may view v as an isomorphism $w : \bar{\mathcal{F}} \otimes \mathscr{L}^{q-1} \cong \mathcal{E}''/\mathcal{F}''$.

Definition A.4.3. Let $\widetilde{\mathrm{Sht}}_0^m \subseteq \mathcal{C}_0^m$ and $\widetilde{\mathrm{Sht}}_0^{m,C} \subseteq \mathcal{C}_0^{m,C}$ be the open substacks where the following conditions are fulfilled.

- $\mathcal{E}'/\mathcal{F}''$ is locally free of rank 2 over $\mathcal{O}_{X \times S}$,
- the map $\mathcal{E} \to \mathcal{E}'/\mathcal{F}''$ is surjective.

Under these conditions we set $\mathcal{F}' = \mathcal{F}''$ and $\mathcal{F} = \mathcal{E} \cap \mathcal{F}'$. Let $\widetilde{\mathrm{Sht}}^m \subseteq \mathcal{C}^m$ and $\widetilde{\mathrm{Sht}}^{m,C} \subseteq \mathcal{C}^{m,C}$ be the maximal open substacks which over $\{0\}/\mathbb{G}_m$ coincide with $\widetilde{\mathrm{Sht}}_0^m$ or $\widetilde{\mathrm{Sht}}_0^{m,C}$, respectively.

Corollary A.4.4. *For sufficiently large C the morphism*

$$\widetilde{\mathrm{Sht}}^{m,C} \longrightarrow \mathrm{Coh}_{\mathcal{D}}^m \times \mathrm{Coh}_{\mathcal{D}}^m \times \mathbb{A}^1/\mathbb{G}_m$$

given by $(\mathcal{E}'/\mathcal{E}'', \mathcal{E}'/\mathcal{E}, \mathscr{L}, l)$ is smooth of relative dimension $4m$. □

In $\widetilde{\mathrm{Sht}}_0$ the situation can be described by the following diagram with short exact sequences in all columns.

$$\begin{array}{ccccccccc}
\mathcal{F} & \hookrightarrow & \mathcal{F}' & = & \mathcal{F}'' & \overset{\sim}{=} & {}^\tau\mathcal{E}/\bar{\mathcal{F}} & & \\
\downarrow & & \downarrow & & \downarrow & & \uparrow & & \\
\mathcal{E} & \overset{j}{\hookrightarrow} & \mathcal{E}' & \overset{t}{\hookleftarrow} & \mathcal{E}'' & & {}^\tau\mathcal{E} & & \\
\downarrow & & \downarrow & & \downarrow & & \uparrow & & \\
\mathcal{E}/\mathcal{F} & = & \mathcal{E}'/\mathcal{F}' & \hookleftarrow & \mathcal{E}''/\mathcal{F}'' & & \bar{\mathcal{F}} & & \mathcal{E}''/\mathcal{F}'' \cong \bar{\mathcal{F}} \otimes \mathscr{L}^{q-1}
\end{array}$$

That the semilinear endomorphism $j^{-1}tu$ of the generic fibre of $\mathcal{E}_{\bar{s}}$ is not nilpotent means $\bar{\mathcal{F}}_{\bar{s}} \cap {}^\tau\mathcal{F}_{\bar{s}} = 0$ in ${}^\tau\mathcal{E}_{\bar{s}}$. Thus there is an exact sequence of $\mathcal{D} \boxtimes \mathcal{O}_S$-modules

$$0 \longrightarrow \bar{\mathcal{F}} \oplus {}^\tau\mathcal{F} \longrightarrow {}^\tau\mathcal{E} \longrightarrow K \longrightarrow 0 \tag{A.4.1}$$

with an S-flat K with finite support over S. Considerations related to the degrees imply that $(p_2)_*K$ has rank $2m$ over $\mathcal{O}_S$. We obtain two '$\mathcal{D}$-shtukas of rank $1/2$'

$$\begin{gathered}
\big[\,\mathcal{F} \xrightarrow{j_1} \mathcal{F}' \cong {}^\tau\mathcal{E}/\bar{\mathcal{F}} \xleftarrow{t_1} {}^\tau\mathcal{F}\,\big] \\
\big[\,\mathcal{E}/\mathcal{F} \otimes \mathscr{L} \xleftarrow{t_2} \mathcal{E}''/\mathcal{F}'' \otimes \mathscr{L} \cong \bar{\mathcal{F}} \otimes \mathscr{L}^q \xrightarrow{j_2} {}^\tau(\mathcal{E}/\mathcal{F} \otimes \mathscr{L})\,\big]
\end{gathered} \tag{A.4.2}$$

along with isomorphisms $\operatorname{Coker} j \cong \operatorname{Coker} j_1$, $\operatorname{Coker} t \cong \operatorname{Coker} t_2$ and $\operatorname{Coker} t_1 \otimes \mathscr{L}^q \cong K \otimes \mathscr{L}^q \cong \operatorname{Coker} j_2$. In particular t_1 and j_2 are isomorphisms outside X'.

Remark A.4.5. In the case $m = 1$ the conditions in Definition A.4.3 are automatically satisfied, i.e. $\widetilde{\mathrm{Sht}}^1 = \mathcal{C}^1$.

Proof. We may assume $S = \operatorname{Spec} k$ with an algebraically closed field k. Let $\mathcal{F}' \subset \mathcal{E}'$ be the maximal submodule which generically coincides with $\mathcal{F}'' \subset \mathcal{E}''$, and let $\mathcal{F} = \mathcal{E} \cap \mathcal{F}'$. If at least one of the conditions in Definition A.4.3 fails, we have either $\mathcal{F}'/\mathcal{F}'' = \mathcal{E}'/\mathcal{E}''$ or $\mathcal{F} = \mathcal{F}'$, in any case $\deg \mathcal{F}'' \leq \deg \mathcal{F}$. Considering degrees we see that $\bar{\mathcal{F}} \oplus {}^\tau\mathcal{F} \cong {}^\tau\mathcal{E}$. This means that for the two $\mathcal{D}$-shtukas of rank $1/2$ defined analogously to (A.4.2) all cokernels are concentrated in X', which contradicts Lemma 1.4.4. □

Definition A.4.6. Let $\mathrm{Sht}^{(1/2,1/2),m}(S)$ be the groupoid of diagrams

$$\begin{gathered}
\big[\,\mathcal{A} \xrightarrow{j_1} \mathcal{A}' \xleftarrow{t_1} {}^\tau\mathcal{A}\,\big] \\
\big[\,\mathcal{B} \otimes \mathscr{L} \xleftarrow{t_2} \mathcal{B}' \otimes {}^\tau\mathscr{L} \xrightarrow{j_2} {}^\tau(\mathcal{B} \otimes \mathscr{L})\,\big] \\
\beta \,:\, \mathcal{A}'/{}^\tau\mathcal{A} \cong {}^\tau\mathcal{B}/\mathcal{B}'
\end{gathered}$$

on $X \times S$ with

– $\mathscr{L}$ invertible $\mathcal{O}_S$-module,
– $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$, $\mathcal{B}'$ locally projective $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank 2 over $\mathcal{O}_{X\times S}$,
– j_1, t_1, j_2, t_2 injective $\mathcal{D} \boxtimes \mathcal{O}_S$-homomorphisms with S-flat quotient,
– β isomorphism of $\mathcal{D} \boxtimes \mathcal{O}_S$-modules,

such that j_1 and t_2 are isomorphisms outside X' and $p_{2,*}(\mathcal{A}'/\mathcal{A})$ has rank 2 over $\mathcal{O}_S$. (Then the same holds for the other three quotients.)

Proposition A.4.7. *The morphism*

$$\widetilde{\mathrm{Sht}}_0^m \to \mathrm{Sht}^{(1/2,1/2),m}$$

given by the construction (A.4.2) *is finite, surjective and radicial.* $\widetilde{\mathrm{Sht}}_0^C$ *is the inverse image of the open substack of* $\mathrm{Sht}^{(1/2,1/2),m}$ *which is defined by the condition* $\deg \mathcal{A}' \geq \deg \mathcal{B}' + C$.

Proof. For an object of $\widetilde{\mathrm{Sht}}_0^m$ the exact sequence (A.4.1) implies another exact sequence

$$0 \longrightarrow {}^{\tau}\mathcal{E} \longrightarrow {}^{\tau}\mathcal{E}/{}^{\tau}\mathcal{F} \oplus {}^{\tau}\mathcal{E}/\bar{\mathcal{F}} \xrightarrow{(1,-1)} K \longrightarrow 0\,.$$

Let an object of $\mathrm{Sht}^{(1/2,1/2),m}(S)$ be given. For any inverse image in $\widetilde{\mathrm{Sht}}_0^m$ we have $K = \mathcal{A}'/{}^{\tau}\mathcal{A} = {}^{\tau}\mathcal{B}/\mathcal{B}'$ and

$$\begin{aligned} {}^{\tau}\mathcal{E} &= \mathrm{Ker}(\mathcal{A}' \oplus {}^{\tau}\mathcal{B} \xrightarrow{(1,-1)} K) \\ \bar{\mathcal{F}} &= \mathcal{B}' \subseteq {}^{\tau}\mathcal{B} \subseteq {}^{\tau}\mathcal{E} \\ {}^{\tau}\mathcal{F} &= {}^{\tau}\mathcal{A} \subseteq \mathcal{A}' \subseteq {}^{\tau}\mathcal{E} \end{aligned} \tag{A.4.3}$$

Now we define $\bar{\mathcal{F}}$ and ${}^{\tau}\mathcal{F} \subseteq {}^{\tau}\mathcal{E}$ by (A.4.3) and claim that ${}^{\tau}\mathcal{E}$ is locally free over $\mathcal{D} \boxtimes \mathcal{O}_S$. Over $X' \times S$ any $\mathcal{D} \boxtimes \mathcal{O}_S$-module which over $\mathcal{O}_{X\times S}$ is locally free of rank 4 is locally free. By [Laf97] I.2, Lemma 4 (or Lemma A.1.2 above) the claim remains to be proved over $\mathcal{D}_x \widehat{\otimes} k(\bar{s})$ for $x \in X \setminus X'$ and for a geometric point $\bar{s} \in S$. We may assume $k(\bar{s}) = \overline{\mathbb{F}}_q$. Since ${}^{\tau}\mathcal{E}$ is an extension of $\mathcal{A}'$ and $\mathrm{Ker}({}^{\tau}\mathcal{B} \to K) = \mathcal{B}'$, it is isomorphic to the direct sum of these projective modules and the claim follows from Lemma A.4.8 below.

Let $\mathrm{Vect}_{\mathcal{D}}^{1/2}$ be the stack of locally projective $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank 2 over $\mathcal{O}_{X\times S}$ and let $\widetilde{\mathrm{Vect}}_{\mathcal{D}}^1$ be the stack of locally free $\mathcal{D} \boxtimes \mathcal{O}_S$-modules of rank 1 equipped with a quotient in $\mathrm{Vect}_{\mathcal{D}}^{1/2}$. The possible choices of $\mathcal{F} \subseteq \mathcal{E}$ are given by the fibre of the finite surjective and radicial morphism

$$(\mathrm{Frob}_q, \mathrm{id}) \,:\, \widetilde{\mathrm{Vect}}_{\mathcal{D}}^1 \longrightarrow \widetilde{\mathrm{Vect}}_{\mathcal{D}}^1 \times_{\mathrm{Vect}_{\mathcal{D}}^{1/2} \times \mathrm{Vect}_{\mathcal{D}}^{1/2},\, \mathrm{Frob}_q \times \mathrm{Frob}_q} \mathrm{Vect}_{\mathcal{D}}^{1/2} \times \mathrm{Vect}_{\mathcal{D}}^{1/2}$$

over $({}^\tau\mathcal{E} \twoheadrightarrow {}^\tau\mathcal{E}/{}^\tau\mathcal{F}, \mathcal{A}, \mathcal{B})$. After this choice the remaining data of an object of $\widetilde{\mathrm{Sht}}_0^m$ can be reconstructed uniquely and without obstructions: we have $\mathcal{E}' = (\mathcal{E} \oplus \mathcal{A}')/\mathcal{A}$ with the submodule $\mathcal{F}' = \mathcal{A}'$ and $\mathcal{E}'' = \mathrm{Ker}(\mathcal{E}' \to \mathcal{B}/(\mathcal{B}' \otimes \mathscr{L}^{q-1}))$ with the submodule $\mathcal{F}'' = \mathcal{F}'$. Moreover the definitions give natural isomorphisms ${}^\tau\mathcal{E}/\bar{\mathcal{F}} = \mathcal{A}' = \mathcal{F}''$ and $\mathcal{E}''/\mathcal{F}'' = \mathcal{B}' \otimes \mathscr{L}^{q-1} = \bar{\mathcal{F}} \otimes \mathscr{L}^{q-1}$. □

Lemma A.4.8. *For some $x \in X \setminus X'$ let $(V_\nu, \varphi_\nu, i_\nu)$ for $\nu = 1, 2$ be Dieudonné-D_x-modules of rank 2 over $\bar{F}_x$, and let $M_\nu \subset V_\nu$ be two $\bar{\mathcal{D}}_x$-lattices with $\varphi_1 M_1 \subseteq M_1$ and $M_2 \subseteq \varphi_2 M_2$, along with an isomorphism of right $\bar{\mathcal{D}}_x$-modules*

$$M_1/\varphi_1 M_1 \cong \varphi_2 M_2/M_2\,. \tag{A.4.4}$$

Then the $\bar{\mathcal{D}}_x$-module $M_1 \oplus M_2$ is free of rank 1.

Proof. We begin with a few general remarks. For any geometric point $\bar{x}$ over x there are precisely two isomorphism classes of projective $\mathcal{D}_{\bar{x}}$-modules M of $F_{\bar{x}}$-dimension 2. For every integer $k \geq 0$, any such module has a unique quotient $M \twoheadrightarrow M_k$ of $\bar{\mathbb{F}}_q$-dimension k. In the case $k > 0$ the isomorphism class of M_k determines the class of M. We have $M \cong \mathrm{Ker}(M \to M_k)$ if and only if k is even. The direct sum of two projective $\mathcal{D}_{\bar{x}}$-modules with $F_{\bar{x}}$-Dimension 2 is free if and only if the two factors are not isomorphic.

Let $r = \deg(x)$. According to the various embeddings $k(x) \subset \bar{\mathbb{F}}_q$ over $\mathbb{F}_q$ we have decompositions $V_\nu = V_\nu^{(1)} \oplus \ldots \oplus V_\nu^{(r)}$ and $M_\nu = M_\nu^{(1)} \oplus \ldots \oplus M_\nu^{(r)}$ with $\varphi_\nu^{(i)} : V_\nu^{(i)} \to V_\nu^{(i+1)}$. The isomorphism (A.4.4) decomposes into the direct sum of

$$M_1^{(i)}/\varphi_1 M_1^{(i-1)} \cong \varphi_2 M_2^{(i-1)}/M_2^{(i)} \tag{A.4.5}$$

for $i = 1 \ldots r$.

The quotient (A.4.4) cannot be zero because in that case both Dieudonné-D_x-modules would be trivial, i.e. their dimension would be divisible by 4, contradicting the assumption. Hence at least one of the quotients $M_1^{(i)}/\varphi_1 M_1^{(i-1)}$ is nonzero. Using the initial remarks the corresponding isomorphism (A.4.5) implies $M_1^{(i)} \not\cong M_2^{(i)}$. This implies $\varphi_1 M_1^{(i)} \not\cong \varphi_2 M_2^{(i)}$, and in view of the next isomorphism (A.4.5) we get $M_1^{(i+1)} \not\cong M_2^{(i+1)}$ etc. Hence all $M_1^{(i)} \oplus M_2^{(i)}$ are free of rank 1 over $\mathcal{D}_{\bar{x}}$. □

A.5 Existence of degenerate $\mathcal{D}$-shtukas ($d = 2$)

Proposition A.5.1. *For a fixed integer $m \geq 0$ the following conditions are equivalent:*

(1) The groupoid $\widetilde{\mathrm{Sht}}_0^m(\bar{\mathbb{F}}_q)$ is nonempty.

(2) For all C the groupoid $\widetilde{\mathrm{Sht}}_0^{m,C}(\bar{\mathbb{F}}_q)$ is nonempty.

(3) The division algebra D is ramified in at most $2m$ places.

Proof. Because of Proposition A.4.7 we can replace (1) and (2) by the following conditions.

(1′) The groupoid $\mathrm{Sht}^{(1/2,1/2),m}(\overline{\mathbb{F}}_q)$ is nonempty.

(2′) For all C there are objects in $\mathrm{Sht}^{(1/2,1/2),m}(\overline{\mathbb{F}}_q)$ with $\deg \mathcal{A}' \geq \deg \mathcal{B}' + C$.

Since for any $x \in X \setminus X'$ the $\bar{F}_x$-rank of any trivial Dieudonné-D_x-module is a multiple of 4, for any diagram $[\,\mathcal{A} \to \mathcal{A}' \leftarrow {}^{\tau}\mathcal{A}\,]$ on $X \otimes \overline{\mathbb{F}}_q$ satisfying the conditions in Definition A.4.6 the support of $\mathcal{A}'/{}^{\tau}\mathcal{A}$ must cover all points of $X \setminus X'$. This means (1′)⇒(3), and it remains to show (3)⇒(2′).

We will first construct the generic fibres of the desired $\mathcal{D}$-shtukas with rank $1/2$ and then show that there are appropriate lattices in their localisations.

We choose $\Pi \in F^* \otimes \mathbb{Q} = \mathrm{Div}^0(F) \otimes \mathbb{Q}$ such that

$$\begin{aligned} \deg_x(\Pi) &= 1/2 \qquad \text{for } x \in X \setminus X', \\ \deg_x(\Pi) &\in \mathbb{Z}_{\leq 0} \qquad \text{for } x \in X'. \end{aligned}$$

Let (V, φ, i) be the simple (D, φ)-space with associated F-pair (F, Π) and let (V', φ', i') be the simple (D, φ)-space with associated F-pair (F, Π^{-1}). By Corollary 6.2.3 both of them have rank 2 over $\bar{F}$. We want to find two adelic families of $\overline{\mathcal{D}}_x$-lattices $M_x \subset V_x$ and $M'_x \subset V'_x$ with the following properties.

In the case $\deg_x(\Pi) = 0$ (which implies $x \in X'$) we demand $M_x = \varphi_x M_x$ and $M'_x = \varphi'_x M'_x$. Such lattices exist because (V_x, φ_x) and (V'_x, φ'_x) are trivial Dieudonné modules by Proposition 6.2.7.

For $x \in X'$ with $\deg_x(\Pi) < 0$ we demand $M_x \subset \varphi_x M_x$ and $M'_x \supset \varphi'_x M'_x$. This is possible, because (V_x, φ_x, i_x) (respectively (V'_x, φ'_x, i'_x)) is Morita equivalent to a Dieudonné-F_x-module of rank 1 with negative (respectively positive) slope.

For $x \in X \setminus X'$, thus $\deg_x(\Pi) = 1/2$, we consider only lattices such that M_x and $\varphi_x M_x$ (respectively M'_x and $\varphi'_x M'_x$) differ only in one fixed geometric point $\bar{x}$ over x. For any two such lattices in V_x (or in V'_x) always one of them contains the other. Since (V_x, φ_x) (respectively (V'_x, φ'_x)) is the simple Dieudonné-F_x-module with slope $1/2$ (respectively $-1/2$), it follows $M_x \supset \varphi_x M_x$ and $M'_x \subset \varphi'_x M'_x$ with quotients of dimension 1 over $\overline{\mathbb{F}}_q$. Changing M_x we can obtain for $M_x/\varphi_x M_x$ any of the two isomorphism classes of $\mathcal{D}_{\bar{x}}$-modules with dimension 1 over $\overline{\mathbb{F}}_q$. In particular, $M_x/\varphi_x M_x \cong \varphi'_x M'_x / M'_x$ is possible.

Let $\mathcal{A}$ and $\mathcal{B}$ be the locally projective $\mathcal{D} \otimes \overline{\mathbb{F}}_q$-modules which are defined by the lattices $M_x \subset V_x$ and $M'_x \subset V'_x$. We fix a point $x \in X'$. Outside x we define $\mathcal{A}' = \mathcal{A} + {}^{\tau}\mathcal{A}$ and $\mathcal{B}' = \mathcal{B} \cap {}^{\tau}\mathcal{B}$, and we choose

$$\mathcal{A}'_x \supseteq {}^{\tau}\mathcal{A}_x (\supseteq \mathcal{A}_x)\,, \qquad \mathcal{B}'_x \subseteq {}^{\tau}\mathcal{B}_x (\subseteq \mathcal{B}_x)$$

with $\mathcal{A}'_x/{}^\tau\!\mathcal{A}_x \cong {}^\tau\mathcal{B}_x/\mathcal{B}'_x$ such that $\mathcal{A}'/\mathcal{A}$ has dimension $2m$ over $\overline{\mathbb{F}}_q$. This is possible because of condition (3). Twisting with a line bundle makes $\deg \mathcal{A}'$ arbitrary large. □

Theorem A.5.2. *In the case $d = 2$ the natural morphism*

$$\mathrm{Sht}^m\big/a^{\mathbb{Z}} \longrightarrow X'^{(m)} \times X'^{(m)}$$

is proper if and only if D is ramified in more than $2m$ places.

Proof. It has been proved in Proposition 1.5.2 that ramification in more than $2m$ places implies properness. Our morphism factors as

$$\pi \;:\; \mathrm{Sht}^m\big/a^{\mathbb{Z}} \overset{i}{\subseteq} \widetilde{\mathrm{Sht}}^{m,C}\big/a^{\mathbb{Z}} \xrightarrow{\widetilde{\pi}} X'^{(m)} \times X'^{(m)}$$

where i is an open immersion which is dense for sufficiently large C by Corollary A.4.4. By Proposition A.5.1, i is not an isomorphism if D is ramified in at most $2m$ places. However, this does not yet complete the proof because $\widetilde{\pi}$ will in general not be separated.

We want to show that the valuative criterion for properness is not satisfied for the morphism π and freely use the notation of the proof of Proposition 1.5.2. In particular, A is a complete discrete valuation ring with quotient field K and residue field k. The properties of i imply that for a suitable such A there is a $\mathcal{D}$-shtuka over K with a degenerate extension to A. This corresponds to a φ-stable lattice M_1 in the generic fibre V of the given $\mathcal{D}$-shtuka such that the map $\bar{\varphi} : {}^\tau\!M_1 \otimes k \to M_1 \otimes k$ is neither nilpotent nor an isomorphism.

Suppose the valuative criterion for properness holds for π. This means that after a finite extension of A there is a non-degenerate $\mathcal{D}$-shtuka $[\,\mathcal{E} \to \mathcal{E}' \leftarrow {}^\tau\mathcal{E}\,]$ on $X \otimes A$ extending the one given over K. This corresponds to a φ-stable lattice $M_0 \subset V$ such that $\bar{\varphi} : {}^\tau\!M_0 \otimes k \to M_0 \otimes k$ is an isomorphism. The image of the natural map $M_1 \to M_0 \otimes k$ is a $\bar{\varphi}$-stable subspace N with $0 \neq N \neq M_0 \otimes k$, which determines a diagram of maximal $\mathcal{D} \otimes k$-submodules

$$\big[\,\mathcal{F} \to \mathcal{F}' \leftarrow {}^\tau\mathcal{F}\,\big] \;\subset\; \big[\,\mathcal{E} \otimes k \to \mathcal{E}' \otimes k \leftarrow {}^\tau\mathcal{E} \otimes k\,\big]$$

The quotients $\mathcal{F}'/\mathcal{F} \subseteq (\mathcal{E}'/\mathcal{E}) \otimes k$ and $\mathcal{F}'/{}^\tau\mathcal{F} \subseteq (\mathcal{E}'/{}^\tau\mathcal{E}) \otimes k$ are then supported in $X' \otimes k$, which is impossible in view of Lemma 1.4.4. □

Remark A.5.3. Similarly one can prove that the natural morphism

$$\mathrm{Sht}^{\underline{m}}\big/a^{\mathbb{Z}} \longrightarrow \prod X'^{(|m_i|)}$$

is proper if and only if D is ramified in more than $\sum |m_i|$ places and that $\mathrm{Sht}^{\leq \underline{\lambda}}/a^{\mathbb{Z}}$ is proper over X'^r if and only if D is ramified in more than $2 \cdot \sum (\lambda_i, \rho)$ places (we are still in the case $d = 2$).

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