ARithmetische Geometrie OberSeminar

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The B_{dR}^+-affine Grassmannian and the Geometric Satake equivalence

Talks

1. Talk: Diamonds and the v-topology

Introduce the v-topology on the category of perfectoid spaces. Recall the definition of the category of diamonds and show that diamonds are v-sheaves, see [4, Proposition 17.1.6], also see [3, Section 11]. Discuss small v-sheaves as in Section 17.2 of [4]. Define spatial v-sheaves and give the criterion for a spatial v-sheaf to be a diamond, see [4, Section 17.3]. Moreover, give the definitions and results concerning morphisms of v-sheaves from Section 17.4 of [4].

2. Talk: Adic spaces as v-sheaves

Explain how to associate a diamond to an analytic pre-adic space over $\operatorname{Spa}\mathbb{Z}_p$ and discuss how this preserves topological information (see Section 10, in particular Proposition 10.3.7 and Theorem 10.4.2, in [4]). Generalize this construction to not necessarily analytic pre-adic spaces as in Section 18 of [4] and show the fully faithfulness results for perfect and formal schemes.

3. Talk: The B_{dR}^+-affine Grassmannian

Define the B_{dR}^+-affine Grassmannian Gr_G as in [4, Chapter 19] and show the alternative characterization in terms of torsors on "$\operatorname{Spa} C^\flat \times \operatorname{Spa}\mathbb{Z}_p$". Introduce Schubert varieties and prove [4, Proposition 19.2.3]. Finally show [4, Theorem 19.2.4], the main result of this talk, stating that the Schubert varieties $\mathrm{Gr}_{G,\leq\mu} \subseteq \mathrm{Gr}_G$ are spatial diamond.

4. Talk: Families of affine Grassmannians

Introduce the Beilinson-Drinfeld B_{dR}^+-Grassmannian, fibered over (self-products of) $\operatorname{Spd}\mathbb{Q}_p$, following the exposition in Sections 20.1, 20.2 and 20.4 of [4]. In particular, define the Schubert varieties in this setup and discuss the relation with the convolution affine Grassmannian when base points collide. Extend these definitions and statements over $\operatorname{Spd}\mathbb{Z}_p$ as in Sections 20.3 and 20.5 of [4]. In particular, define Schubert varieties for reductive models $\mathcal{G}$ over $\mathbb{Z}_p$.

5. Talk: Affine flag varieties and local models

In this talk, models $\mathcal{G}/\operatorname{Spec}\mathbb{Z}_p$ of G with parahoric identity component $\mathcal{G}^o$ are considered. Show that in this case, the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G},\operatorname{Spd}\mathbb{Z}_p}$ is indproper, see [4, Theorem 21.2.1]. Discuss the relation with local model theory as in Section 21.4 of [4]. Specialize to the situation of the local models considered in [2], following the Appendix to Section 21 in [4]. In particular, recall the definition of an EL and PEL datum, of the associated group $\mathcal{G}$ and of the local model $\mathbb{M}^{loc}_{\mathcal{G},\mu}$. Show that the associated diamond embeds into the Beilinson-Drinfeld Grassmannian, see [4, Corollary 21.5.10]. If time permits, discuss one of the examples in Sections 21.6 and 21.7 of [4].

6. Talk: Perverse sheaves on the B_{dR}^+-affine Grassmannian

Study L^+G-equivariant sheaves on the B_{dR}^+-affine Grassmannian and in particular dualizing complexes on Schubert varieties $\mathrm{Gr}_{G,\leq\mu}$. Define a perverse t-structure and the Satake category.

7. Talk: Universally locally acyclic sheaves

Introduce the notion of ULA (universally locally acyclic) sheaves in the setting of diamonds, and verify analogues of the basic results on ULA sheaves. Use ULA sheaves in the setting of the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\,\mathbb{Z}_p}/\,\mathrm{Spd}\,\mathbb{Z}_p$ for reductive $\mathcal{G}$ to identify the Satake category of the B_{dR}^+-affine Grassmannian with the Satake category of the Witt vector affine Grassmannian.

8. Talk: Monoidal structures

Show that the equivalence of Satake categories from the previous talk is compatible with the convolution product and deduce semismallness of the convolution morphism for the B_{dR}^+-affine Grassmannian. Moreover, identify the convolution product with the fusion product, and deduce that the Satake category has a symmetric monoidal structure.

9. Talk: Fibre functors

Show that the total cohomology functor defines a symmetric monoidal fibre functor on the Satake category.

10. Talk: Identification of the dual group

Identify the Tannakian group defined by the Satake category with the Langlands dual group, with $\mathbb{Z}_\ell$-coefficients and for $\mathrm{Gr}_{G,\mathrm{Spd}\,\mathbb{Q}_p}$ including the Galois action.

References

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