Non-correlation between Fourier coefficients of automorphic forms and trace functions

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for a Maass cusp form of weight 0.
This normalization is chosen so that the terms are almost bounded on average. Here $e(nz) = e^{2\pi i n z}$ and W_{it_f} is a Whittaker function.

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The trivial bound is

$$S(f, K, p) \ll_{f,V} \|K\|_\infty p$$

Trace functions vs. modular forms

For trace functions, we can do much better! Here is the theorem of É. Fouvry, E. Kowalski and Ph. Michel, [GAFA15]

Theorem

Let f be a Hecke eigenform. Let K be an isotypic trace function of conductor cond(K).
There exists $s \geq 1$ absolute constant such that:

$$S_V(f, K; p) \ll_{f,V,\delta} cond(K)^s p^{1-\delta}$$

holds for any $\delta < 1/8$.

What is a Trace function?

Trace function

A function $K : \mathbb{F}_p \to \mathbb{C}$ is called a trace function if there exists a constructible ℓ-adic sheaf $\mathcal{F}$ on $\mathbb{A}^1_{\mathbb{F}_p}$ (satisfying some technical conditions) s.t.

$$K(x) = \iota(tr\mathcal{F}(\mathbb{F}_p, x))$$

Examples

$$K(n) = \begin{cases} e(\frac{\phi_1(n)}{p})\chi(\phi_2(n)) & S_1(n)S_2(n) \not\equiv 0 \mod p \\ 0 & \text{otherwise} \end{cases}$$

for $\phi_i(X) \in \mathbb{F}_p(X)$ and $S_i(x) \in \mathbb{F}_p[X]$ the denominator of $\phi_i(X)$. We exclude the case

$$K(x) = e\left(\frac{ax+b}{p}\right), a, b \in \mathbb{F}_p$$

.

Trace functions

Example (due to Deligne, studied extensively by Katz)

Define the Hyper-kloostermann sum as the multiplicative convolution of additive characters

$$Kl_m(a;p) = \frac{1}{p^{(m-1)/2}} \sum_{x_1 x_2 \ldots x_m = a} e\left(\frac{x_1 + \ldots + x_m}{p}\right)$$

where $x_1, \ldots, x_m \in \mathbb{F}_p^\times$.

$$K(n) = \begin{cases} Kl_m(\phi(n);p) & S(n) \not\equiv 0 \mod p \\ 0 & \text{otherwise} \end{cases}$$

for $\phi(X) \in \mathbb{F}_p(X)$ and $S(X) \in \mathbb{F}_p[X]$ its denominator.

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ϕ – cuspidal GL_2-automorphic form over F of level $\mathfrak{N}$.

$\mathfrak{N}$ coprime to $\mathfrak{p}$ fixed.

The question that we consider is to bound

$$\sum_{m \in F^{\times}} K(m_{\mathfrak{p}}) W_{\phi}\left(\begin{pmatrix} m\varpi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} \right)$$

where W_{ϕ} is the global Whittaker function of ϕ.

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where W_ϕ is the global Whittaker function of ϕ. In fact $m \in \mathfrak{p}^{-1}$. Here $m_\mathfrak{p}$ the congruence class of $m\varpi_\mathfrak{p}$ at $\mathfrak{p}$.

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Assume ϕ is an automorphic form that is spherical at $\mathfrak{p}$ (i.e. $K_{\mathfrak{p}}$ invariant) and $||\phi||_2 = 1$. We have the following trivial bound:

$$\left| \sum_{m \in F^\times} K(m_{\mathfrak{p}}) W_\phi\left(\begin{pmatrix} m\pi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} \right) \right| \ll_{\phi,F,K} \mathrm{Nm}(\mathfrak{p})^{\frac{1}{2}+\vartheta}$$

where $\vartheta > \frac{7}{64}$ is the known approximation to the Ramanujan-Petersson conjecture.

Our result

Theorem[N. 2022+]

Assume that F is a totally real field. If K a trace function s.t. its Fourier transform $\widehat{K}$ has trivial automorphism group, then

$$\left| \sum_{m \in F^\times} K(m_{\mathfrak{p}}) W_\phi\left(\begin{pmatrix} m\pi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} \right) \right| \ll_{\phi,F,K,\delta} \mathrm{Nm}(\mathfrak{p})^{\frac{1}{2}-\delta}$$

for any $\delta < \frac{1}{12}$.

Strategy of the proof

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Define a factorizable function $h \in C_c^\infty(\mathrm{GL}_2(\mathbb{A}_F))$ i.e. a smooth function that is compactly supported modulo the center. We will consider then a spectral average whose cuspidal part looks as follows and apply to it the relative trace formula:

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$$\sum_{\pi} \sum_{\varphi \in \mathscr{B}(\pi, \mathfrak{N}\mathfrak{p})} \left| \sum_{m \in F^\times} W_{R(h)\varphi, f}\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \right) \right|^2$$

where the π varies over cuspidal representations of level $\mathfrak{N}\mathfrak{p}$ and $\mathscr{B}(\pi, \mathfrak{N}\mathfrak{p})$ is an orthonormal basis of $\pi^{K_0(\mathfrak{N}\mathfrak{p})}$

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- ϕ a pure tensor i.e. $W_\phi = \prod_v W_{\phi,v}$.

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$$= |w(\pi_\infty)|^2 \left| \sum_{\mathfrak{l} \in \Lambda} x_{\mathfrak{l}} \lambda_\pi(\mathfrak{l}) \right|^2 \left| \sum_{m \in F^\times} W_\phi \begin{pmatrix} m\varpi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} K(m_p) \right|^2$$

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$w(\pi_\infty) \in \mathbb{C}$ is the spectral weight and Λ is a set of prime ideals whose norm is of size L.

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By applying the relative trace formula to the operator $R(h)$ and using positivity the cuspidal contribution satisfies

$$\sum_{\pi} \sum_{\varphi \in \mathscr{B}(\pi, \mathfrak{N}\mathfrak{p})} \left| \sum_{m \in F^\times} W_{R(h)\varphi, f} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \right|^2$$

$$\ll_{f_\infty, F, K} (\mathrm{Nm}(\mathfrak{p}))^{1+\epsilon}.L^{1+\epsilon} + \sqrt{\mathrm{Nm}(\mathfrak{p})}.L^{4+\epsilon}$$

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Recall that L is the length of the amplifier.

Strategy of the proof- suite

Now for ϕ a cusp form that is $K_0(\mathfrak{N})$ invariant in a representation π, with ϕ a pure tensor, by positivity:

$$\left| \sum_{m \in F^\times} W_{R(h)\phi, f} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \right|^2 \ll_{f_\infty, F, K} (\mathrm{Nm}(\mathfrak{p}))^{1+\epsilon} . L^{1+\epsilon} + \sqrt{\mathrm{Nm}(\mathfrak{p})} . L^{4+\epsilon}$$

Using our previous calculation,

$$|w(\pi_\infty)|^2 \left|\sum_{\mathfrak{l}\in\Lambda} x_{\mathfrak{l}}\lambda_\pi(\mathfrak{l})\right|^2 \left|\sum_{m\in F^\times} W_\phi \begin{pmatrix} m\varpi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} K(m_p)\right|^2$$

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Strategy of proof- fin

With ϕ and π as above, we will choose h s.t. $|w(\pi_\infty)| > 0$ and using the amplifier due to A.Venkatesh, we choose:

$$x_{\mathfrak{l}} = \begin{cases} \text{sign}(\lambda_\pi(\mathfrak{l})) & \text{if } \mathfrak{l} \in \Lambda \text{ and } \lambda_\pi(\mathfrak{l}) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

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This gives

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$$\ll_{f_\infty,F,K} (\text{Nm}(\mathfrak{p}))^{1+\epsilon}.L^{1+\epsilon} + \sqrt{\text{Nm}(\mathfrak{p})}.L^{4+\epsilon}$$

Since

$$\sum_{\mathfrak{l}\in\Lambda} |\lambda_\pi(\mathfrak{l})| \gg_\pi L^{1-\epsilon}$$

we may conclude by setting $L = (\text{Nm}(\mathfrak{p}))^{\frac{1}{6}}$.

Thank you for your kind attention!