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WS 2015/16

Seminar on Algebraic Surfaces

Tuesdays 16-18h, MZ Room 0.006
Organizational meeting: Monday 13.07.2015, 16h(c.t.), MZ Room 0.008.

1) Divisors:
Recall (without proof) the relation between Weil- and Cartierdivisors and line bundles, see [3, II.6], especially explain when the equalities $\mathrm{Pic}(X) = \mathrm{CaCl}(X)$ resp. $\mathrm{CaCl}(X) = \mathrm{Cl}(X)$ hold. Explain how divisors can be pulled back or pushed forward [2, I.1]. Introduce linear systems and rational morphisms [2, II.4-6], see also [3, II, 7.1]. Recall the notions of ample and very ample line bundles and discuss their relations [4, Def. 1.2.1; Theorem 1.2.6]. Prove Bertini's theorem [3, II, Theorem 8.18] and deduce that any divisor on a smooth projective variety (over an algebraically closed field) is linearly equivalent to the difference of two smooth hypersurface sections. Recall the degree of a line bundle on a smooth projective curve and the notion of the canonical divisor on a smooth, projective variety. Then state the Riemann-Roch theorem for curves [3, IV.1].

2) Intersection theory and the Riemann-Roch theorem on surfaces:
Prove [3, V, Theorem 1.1] see also [2, I.1-I.7] and the Riemann-Roch theorem for surfaces [3, V, Theorem 1.6] and [2, I.12]. Then discuss the formulas [2, I.8, I.15] and give examples, see e.g. [2, I.9].

3) Blow-ups:
Define the blow-up of a scheme in a closed subscheme [3, p. 162 ff.] and discuss the universal property of blow-ups [3, II Prop. 7.14]. See also [2, II.1-3; II. 8]. Finally discuss how blow-ups can be used for the elimination of points of indeterminacy of a rational map [2, II.7].

4) Birational maps between surfaces:
Describe the structure of a birational morphism between surfaces [2, II.11,12]. Discuss the exponential sequence over the complex numbers and the Néron-Severi group [2, I.10]. Compute the Néron-Severi group of the blow-up of a surface X in terms of the Néron-Severi group of X [2, II.13]. Introduce the notion of minimal surfaces [2, II.15-16].

5) Birational invariants:
Prove Castelnuovo's contractibility criterion [2, II.17]. Introduce the Kodairadimension and the plurigenera, see [1, 5.4-5.8] and [2, VII], and the irregularity [2, III.19,20]. If time permits give a reformulation in the language of the minimal model program of Mori [5, Prop. 3.13, Ref.3.14].

6) Ruled surfaces:
Introduce Ruled surfaces and geometrically ruled surfaces [2, III.1-3]. Prove the theorem of Noether-Enriques [2, III.4] and discuss the structure of geometrically

ruled surfaces [2, III.7]. Prove that ruled surfaces are the minimal models for surfaces birational to $C \times \mathbb{P}^1$, where C is a non-rational curve, see [2, III.10].

7) Rational surfaces:

Introduce the Hirzebruch surfaces $\mathbb{F}_n$ [2, IV.1] and the Del-Pezzo surfaces [2, Iv.9]. Show that a cubic surface in $\mathbb{P}^3$ contains exactly 27 lines [2, IV.9-15], see also [3, V, Thm. 4.9].

8) Castelnuovo's criterion for rationality:

Present [2, V.1-10]. Especially show that minimal surfaces with $q = 0$ and $p_2 = 0$ are rational. Show that the minimal rational surfaces are $\mathbb{P}^2$ and $\mathbb{F}_n$, for $n \neq 1$.

9) The Albanese variety and minimal models for surfaces of Kodaira dimension ≥ 0:

Introduce the Picard variety [1, p.70-72] and the Albanese variety [1, p. 73, Def. 5.2-p. 75, 5.4]. Prove that there is a unique minimal model for surfaces of non-negative Kodaira-dimension [2, V.15-19]. Finally state the theorem about the characterization of ruled surfaces [1, Thm. 13.2].

10) Surfaces of Kodaira dimension 0:

Present [2, VIII.1-6]. Especially define bielliptic surfaces.

11) K3-surfaces and Enriques surfaces

K3 surfaces are minimal surfaces with $q = 0, p_g = 1$ and trivial canonical divisor. Show that K3 surfaces are simply connected, [1, Thm. 10.3 (ii)] and give examples of K3-surfaces, see [2, VIII.8-11]. Study linear systems on K3 surfaces [2, VIII.13]. Show that a K3 surface is elliptic if and only if it contains a non-trivial divisor D with $D^2 = 0$, see [6, Thm. 2.2]. Show that Enriques surfaces can be written as quotients of K3 surfaces (and vice versa) [2, VIII.17] and give examples of Enriques surfaces.

References

[1] L. Bădescu, *Algebraic Surfaces*, Universitext, Springer, 2001.

[2] A. Beauville, *Complex Algebraic Surfaces*, 2nd edition, LMS Student Texts, vol. **34**, Cambridge University Press, 1996.

[3] R. Hartshorne, *Algebraic Geometry*, GTM, vol. **52**, Springer, 1977.

[4] R. Lazarsfeld, *Positivity in Algebraic Geometry*, Erg. Math., 3. Folge, vol. **48**, Springer, 2004.

[5] Peters, *Complex Surfaces* (long version), lecture notes, available at `http://www-fourier.ujf-grenoble.fr/~peters/Courses.html`.

[6] J. Sawon, *Abelian fibred holomorphic symplectic manifolds*, Turkish J. Math. **27** (2003), no. 1, pp. 197-230.