ARithmetische Geometrie OberSeminar

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Finiteness of integral Hecke algebras

Of principal importance in the study of automorphic forms, and in particular in the local Langlands correspondence, is the representation theory of p-adic groups like $\mathrm{GL}_n(\mathbb{Q}_p)$. More generally, we fix a p-adic field E and a connected reductive group $\mathbb{G}$ over E, and consider the group $G = \mathbb{G}(E)$, which is a totally disconnected topological group. An important structural property of G is that it has a neighborhood basis of 1 consisting of open compact subgroups $K \subset G$ (that one can, and we sometimes tacitly will, assume to be pro-p).

For a coefficient field k (assumed to be algebraically closed for simplicity), a smooth representation of G is a k-vector space V equipped with a map of groups

$$\pi : G \to \mathrm{Aut}_k(V)$$

such that for all $v \in V$, the stabilizer K_v of v is an open subgroup of G. (In other words, π is continuous when V is equipped with the discrete topology.) Traditionally, the case where k is of characteristic 0 has been considered. If the representation is finitely generated, it is generated by its K-fixed vectors V^K for some compact open subgroup $K \subset G$, in which case V is completely determined by

$$V^K = \mathrm{Hom}_G(c\text{-}\mathrm{Ind}_K^G k, V)$$

together with the action of the Hecke algebra

$$k[K\backslash G/K] = \mathrm{End}_G(c\text{-}\mathrm{Ind}_K^G k).$$

A central finiteness result is the following theorem of Bernstein.

Theorem 1 ([1])**.** *The algebra $k[K\backslash G/K]$ is noetherian. More precisely, it is a finite module over its center, which is a finitely generated k-algebra.*

In particular, the category of smooth G-representations (over k) is noetherian, i.e. any subrepresentation of a finitely generated representation is finitely generated.

The center $\mathcal{Z}_k(G,K)$ of $k[K\backslash G/K]$ is known as the Bernstein center, and has an explicit description in terms of "supercuspidal supports". More precisely, if V is an irreducible smooth G-representation generated by its K-fixed vectors, corresponding to a $k[K\backslash G/K]$-module M, Schur's lemma ensures that any element of $\mathcal{Z}_k(G,K)$ acts on M by a scalar, and one gets a resulting map

$$\mathcal{Z}_k(G,K) \times \mathrm{Irr}_k(G,K) \to k.$$

The induced map

$$\mathcal{Z}_k(G,K) \to \mathrm{Map}(\mathrm{Irr}_k(G,K), k)$$

is injective, and the image can be described explicitly as those maps that depend only on the supercuspidal support, and define algebraic functions with respect to a natural structure of algebraic variety on the set of possible supercuspidal supports.

It has long been conjectured that the same theorem is true in any characteristic $\neq p$, but this has remained open until very recently.[1]

[1] The case of characteristic p representations is very different, in many respects; for example, there is no k-valued Haar measure, the functor $V \mapsto V^K$ is not exact, and $c\text{-}\mathrm{Ind}_K^G k$ is not a projective representation.

Theorem 2 (Dat–Helm–Kurinczuk–Moss [6])**.** *For any field k of characteristic $\neq p$, the algebra $k[K\backslash G/K]$ is noetherian. More precisely, it is a finite module over its center, which is a finitely generated k-algebra.*

In particular, the category of smooth G-representations (over k) is noetherian, i.e. any subrepresentation of a finitely generated representation is finitely generated.

In fact, [6] establish a more precise result with integral coefficients (they basically prove the result with $\mathbb{Z}[\frac{1}{p}]$-coefficients).

Surprisingly, while the theorem seems like a basic result in representation theory, of a class of reasonably concrete non-commutative algebras, its proof makes use of very heavy machinery. Basically, the problem is that it is very hard to construct elements in the center of $k[K\backslash G/K]$. Bernstein's proof in characteristic 0 makes critical use of certain semisimplicity phenomena (for cuspidal representations) that fail in positive characteristic. Relatedly, the theory of "supercuspidal supports" does not (naively) extend to positive characteristic, and no explicit description of the Bernstein center is known in positive characteristic.

Instead, the proof makes use of the local Langlands correspondence. This gives a map, constructed in [7],

$$\pi \mapsto \varphi_\pi : \mathrm{Irr}_k(G) \to \{L\text{-parameters}\},$$

where the set of L-parameters has a natural structure as an algebraic variety. Moreover, it is proved in [7] that for any algebraic function f on the set of L-parameters, the composite map

$$\pi \mapsto f(\varphi_\pi) : \mathrm{Irr}_k(G,K) \to \{L\text{-parameters}\} \xrightarrow{f} k$$

agrees with the action of some central element of $k[K\backslash G/K]$. This gives a natural map from some finitely generated k-algebra towards $\mathcal{Z}_k(G,K)$, and then [6] prove that $k[K\backslash G/K]$ is already a finite module over this part of the center. However, the argument is rather indirect, and proceeds by reduction from characteristic 0, using Bernstein's results as a starting point, and that the results of [7] apply even with coefficients in $W(k)$.

Remark 3. *In the case $G = \mathrm{GL}_n$, the algebraic variety of L-parameters agrees with the algebraic variety of supercuspidal supports. In this case, the Bernstein center admits a description in terms of supercuspidal supports like in characteristic* 0*, see the work of Helm* [9]*. In general, the spectrum of the Bernstein center is expected to lie strictly between the variety of L-parameters and the variety of supercuspidal supports.*

Besides the intrinsic interest, Dat [4] shows that the noetherianity of Hecke algebras has important representation-theoretic consequences, in particular Bernstein's "second adjunction" showing that parabolic induction is also a left adjoint functor.

Talks

Talk 1: Smooth representation theory, cuspidal representations

Following [3, Chapter 1], recall the basics on smooth representation theory in case $k = \mathbb{C}$, and discuss the notion of cuspidal representations. The results covered should include [3, 2.10, 2.11, 2.12, 2.28, 2.41, 2.42, 2.44]. Indicate which of these results/proofs make essential use of the choice of coefficients.

Talk 2: Parabolic induction, finiteness results

Following [3, Chapter 2], discuss parabolic induction, and prove the basic finiteness results: Admissibility of irreducible representations, finiteness of number of cuspidal representations of $\mathbb{C}[K\backslash G/K]$. The results covered should include [3, 3.13, 3.14, 3.21, 3.25, 4.1, 4.7, 4.14, 4.17, 4.19]. Again, indicate which of these results/proofs make essential use of the choice of coefficients.

Talk 3: Bernstein decomposition and Bernstein center

Following [1], explain the notion of supercuspidal supports, show existence and uniqueness of supercuspidal supports (with characteristic 0 coefficients) and the structure of an algebraic variety on the set of supercuspidal supports. Prove Bernstein's theorem [1, Proposition 2.10, Théor'eme 2.13], and discuss [1, 3.3, 3.4, 3.5.2, 3.9, 3.10].

Talk 4: Second Adjunction

State Bernstein's second adjunction, and prove it following Bernstein's original argument, by first proving "stability" and Jacquet's lemma. This is explained in [2, Section III.3], see also [9, Section 11].

Talk 5: Depth

Discuss the decomposition of the category $\mathrm{Rep}_k(G)$ into a product $\prod_{r\in\mathbb{R}_{\geq 0}} \mathrm{Rep}_k(G)_r$ according to depth, cf. [10], [11, II.5], [12]. In particular, discuss the relevant background on Bruhat–Tits theory and Moy–Prasad filtrations, see also [8, Section 4]. Show that this decomposition exists already with $\mathbb{Z}[\frac{1}{p}]$-coefficients, cf. [4, Appendix]. As an application, explain the equivalence [6, Lemma 3.2] between several formulations of Theorem 2.

Talk 6: L-parameters

Discuss the Langlands dual group and the Weil group, and introduce the notion of (semisimple) L-parameters. Construct the stack of L-parameters following [5], [7, Section VIII.1]. (Almost) identify the functions on the GIT quotient with the algebra of excursion operators [7, Section VIII.3.2], and points of the GIT quotient with semisimple L-parameters [7, Section VIII.3.1], [5, Proposition 4.13].

Talk 7: Finiteness on the side of L-parameters

Prove that certain maps of stacks of L-parameters are finite, following [6, Section 2].

Talk 8: Reduction to [7]

Discuss some expected properties of the local Langlands correspondence as a map

$$\mathrm{Irr}_k(G) \to \{L\text{-parameters}\}.$$

Then state the properties of the correspondence constructed in [7] and deduce Theorem 2 following [6, p.3, Section 3.1].

Talk 9: The construction of [7]

Discuss the construction of excursion operators given some abstract categorical data (a category C together with a functor $\mathrm{Rep}\,\check{G} \times C \to C^{W_E}$ satisfying various natural properties), [7, Section VIII.4]. Indicate how [7] constructs the required categorical data, by taking $C = D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbb{Z}_\ell[\sqrt{q}])$. In particular, give a brief overview of the stack Bun_G, of Hecke operators, and of the geometric Satake equivalence, and thereby show how the a priori very separate objects $G(E)$, $\hat{G}$ and W_E come together.

Talk 10: Representation-theoretic consequences

Deduce from Theorem 2 some representation-theoretic consequences, in particular second adjointness (with $\mathbb{Z}[\frac{1}{p}]$-coefficients), and the characterization of ℓ-adically integral irreducible representations, following [6, Section 4].

References

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