ARITHMETISCHE **G**EOMETRIE **O**BER**S**EMINAR

The Witt vector affine Grassmannian

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In this ARGOS we want to study the Witt vector affine Grassmannian, often also called mixed-characteristic affine Grassmannian, $\mathrm{Gr}^W_{\mathrm{GL}_n}$. For any perfect field k, its k-valued points are given by $W(k)$-lattices in $W(k)[1/p]^n$, where $W(k)$ are the Witt vectors of k. For a long time, the Witt vector affine Grassmannian was only known through this *set* of points. This is in contrast to the case of equal characteristics, where there is an affine Grassmannian (an ind-scheme) parametrizing $R[[t]]$-lattices in $R((t))^n$.

M. Kreidl, [K], investigated to what extent $\mathrm{Gr}^W_{\mathrm{GL}_n}$, when considered as a functor on *all* rings of characteristic p, was ind-representable. This seems to be not the case, but Kreidl already observed that the functor had better properties when restricted to *perfect* rings, where it parametrizes $W(R)$-lattices in $W(R)[1/p]^n$. This led to the question whether this functor on perfect rings might be representable by an ind-(perfect scheme). The aim of the seminar is to go through the paper [BS] which proves this result. Importantly, X. Zhu, [Z], had previously proved that the functor is representable by an ind-(perfect algebraic space).[1] The methods used in the two papers are independent.

As the functor $X \mapsto X_{\mathrm{perf}}$ from schemes of characteristic p to perfect schemes preserves the étale topology and underlying topological space, all topological questions such as connected components, dimensions, or étale cohomology, can thus be defined for objects like affine Deligne-Lusztig varieties.

Let us give a brief overview of the construction. The affine Grassmannian $\mathrm{Gr}^W_{\mathrm{GL}_n}$ is the functor taking a perfect $\mathbb{F}_p$-algebra R to the set of finite projective $W(R)$-submodules $M \subset W(R)[1/p]^n$ such that $M[1/p] = W(R)[1/p]^n$. This is covered by the subfunctors where $p^N W(R)^n \subset M \subset p^{-N} W(R)^n$; up to translation, we may replace this by $p^{2N} W(R)^n \subset M \subset W(R)^n$.

In the analogous equal characteristic situation, the $R[[t]]$-module $R[[t]]^n/M$ is actually an R-module, and one can define a line bundle

$$\mathcal{L} = \det{}_R(R[[t]]^n/M)\ .$$

This line bundle turns out to be ample, and gives a projective embedding.

However, in mixed characteristic, $W(R)^n/M$ is in general not an R-module. Still, one can often (e.g., if R is a field) filter $W(R)^n/M$ such that all associated gradeds M_i are finite projective R-modules. Then, one can define

$$\mathcal{L} = \bigotimes_i \det{}_R(M_i)\ .$$

One would like to know that this defines a canonical line bundle, independent of the choice of the filtration. We give two proofs of this fact. Both make use of h-descent for line bundles on perfect schemes: It turns out that various descent properties hold true over perfect schemes without any flatness assumptions.[2]

The first uses K-theory. In K-theoretic language, we want to construct a map

$$\widetilde{\det} : K(W(R)\,\mathrm{on}\,R) \to \mathrm{Pic}^{\mathbb{Z}}(R)$$

[1] Moreover, Zhu proves the geometric Satake equivalence in this setup.

[2] This is reminiscent of the theory of perfectoid spaces, but more elementary.

from a K-theory spectrum to the groupoid of graded line bundles on R.[3] It is well-known that there is a map

$$\det : K(R) \to \mathrm{Pic}^{\mathbb{Z}}(R) \ ,$$

and the task is to extend this along the map $\alpha : K(R) \to K(W(R) \,\mathrm{on}\, R)$. But α is known to be an equivalence if R is the perfection of a regular $\mathbb{F}_p$-algebra. To deal with the general case, it remains to do a descent from the regular case; this is possible by h-descent of line bundles on perfect schemes.

Our second proof is more elementary. Here, we observe that $\mathcal{L}$ is naturally a line bundle on the Demazure resolution of a closed Schubert cell, which parametrizes filtrations of $W(R)^n/M$ whose associated gradeds are finite projective R-modules. The question becomes that of descending a line bundle along a map $f : Y \to X$ of perfect schemes. Here, we prove that if f (is proper perfectly finitely presented and) satisfies $Rf_*\mathcal{O}_Y = \mathcal{O}_X$ (a condition that can be checked on fibres), then a line bundle $\mathcal{L}$ on Y descends to X if and only if it is trivial on all geometric fibres.

After the line bundle $\mathcal{L}$ is constructed, we prove that it is ample by using a theorem of Keel on semiample line bundles in characteristic p.

1) Affine Grassmannians in equal characteristics

Define the affine Grassmannian in equal characteristics, and prove that it is representable by a strict ind-projective ind-scheme, cf. [BL, Proposition 2.3, Proposition 2.4].

2) The h-topology

Define Voevodsky's h-topology, following [BS, Section 2], and characterize the class of h-sheaves, [BS, Proposition 2.7, Theorem 2.8], at least in the case of sheaves of groupoids (i.e., stacks).

3) Perfect schemes

Prove various basic properties on the comparison between schemes and perfect schemes as in [BS, Section 3].

4) The v-topology on perfect schemes

Prove that the v-topology on perfect schemes is subcanonical (i.e., representable presheaves are sheaves), and that the v-cohomology of the structure sheaf vanishes on affines, cf. [BST, Section 3]. Moreover, prove v-descent for vector bundles, [BS, Section 4].

5) The geometric construction of line bundles

Prove the geometric criterion [BS, Theorem 6.7, Lemma 6.8] for descent of vector bundles. Moreover, prove [BS, Lemma 6.10].

6) The K-theoretic construction of line bundles: Recollection on $\det$

Recall the definition of the map $\det : K(R) \to \mathrm{Pic}^{\mathbb{Z}}(R)$ from the K-theory spectrum of a commutative ring R to the groupoid of graded line bundles on R, cf. [BS, Appendix].

7) The K-theoretic construction of line bundles

Extend det to a map $\widetilde{\det} : K(W(R) \,\mathrm{on}\, R) \to \mathrm{Pic}^{\mathbb{Z}}(R)$ by using v-descent of line

[3] The quotient $W(R)^n/M$ defines a point of $K(W(R) \,\mathrm{on}\, R)$, and its image is the desired line bundle.

bundles, cf. [BS, Section 5].

8) Families of torsion $W(k)$-modules
Prove the basic lemmas on torsion $W(R)$-modules, [BS, Section 7]. In particular, prove that the fibres of the Demazure resolution are geometrically connected, [BS, Lemma 7.12].

9) The Witt vector affine Grassmannian
Define the closed Schubert cells $\mathrm{Gr}_{\leq\lambda}$ in the affine Grassmannian, and the Demazure resolution $\widetilde{\mathrm{Gr}_\lambda} \to \mathrm{Gr}_{\leq\lambda}$, cf. [BS, Section 8]. Prove that the natural line bundle $\widetilde{\mathcal{L}}$ on $\widetilde{\mathrm{Gr}_\lambda}$ descends to a line bundle $\mathcal{L}$ on $\mathrm{Gr}_{\leq\lambda}$, cf. [BS, Theorem 8.8].

10) Projectivity of the Witt vector affine Grassmannian
Finish the proof by showing that the line bundle $\mathcal{L}$ on $\mathrm{Gr}_{\leq\lambda}$ is ample, so that $\mathrm{Gr}_{\leq\lambda}$ is the perfection of a projective algebraic variety, cf. [BS, Section 8.4].

References

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